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VARIETIES OF IMPLICATIVE SEMI-LATTICES. II

WILLIAM CHARLES NEMITZ AND THOMAS PAUL WHALEY

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W. NEMITZ AND T. WHALEY

This paper is concerned with a process of coordinatization of the lattice of varieties of implicative semilattices. Equational descriptions of some elements in each coordinate class, and a complete equational description of one coordinate class are given.

1. **Introduction.** This paper is a continuation of [8]. Familiarity with [8] and [6] is assumed. After stating some of the consequences of the local finiteness of the variety of implicative semilattices, we describe a system for partitioning the lattice of varieties of implicative semi-lattices into coordinate intervals, and give some results that can be obtained from a study of this coordinatization. Finally, we give equational descriptions for the largest and smallest varieties in each coordinate class, the covers of the smallest variety in each coordinate class and a complete equational description of the coordinate class 4,2.

Recall that an implicative semi-lattice is subdirectly irreducible if and only if it has a single dual atom. In accordance with the usage of [8], this dual atom will be denoted by u . If in a subdirectly irreducible implicative semi-lattice, the dual atom is deleted, the remaining structure is both a subalgebra and a homomorphic image of the original. Thus every subdirectly irreducible implicative semi-lattice may be thought of as obtained by appending a single dual atom to some already given implicative semi-lattice. If L is an implicative semilattice, the subdirectly irreducible implicative semi-lattice obtained in this manner will be denoted by \hat{L} .

2. **Local finiteness.** The following theorem was proven first by A. Diego [2] in a slightly different context. McKay [4] extended the result to implicative semi-lattices. We present a much simpler proof here.

THEOREM 2.1. *The variety of implicative semi-lattices is locally finite.*

Proof. Let F_n denote the free implicative semi-lattice on n generators. The proof proceeds by induction. F_1 has two elements. Assume that F_n is finite. $F_{n+1} \leq_s \coprod \hat{L}_i$, where each \hat{L}_i is $n+1$ generated. Hence each L_i is n generated. It follows from the induction assumption that there are only a finite number of distinct L_i each

of which is finite. Therefore the same statement applies to the \hat{L}_i , and hence F_{n+1} is finite.

COROLLARY 2.2. *Every variety of implicative semi-lattices is generated by its finite sub-directly irreducible members.*

COROLLARY 2.3. *If f is a homomorphism of an implicative semi-lattice L onto a finite implicative semi-lattice M , then there exists $L' \leq_s L$ such that $f|_{L'}$ is an isomorphism.*

COROLLARY 2.4. *The lattice of all varieties of implicative semi-lattices is itself implicative.*

COROLLARY 2.5. *If L is a finite subdirectly irreducible implicative semi-lattice, then the class of all those implicative semi-lattices which do not contain a sub-implicative semi-lattice isomorphic to L is a variety.*

3. Coordinates of varieties. In this section, A will denote a sub-directly irreducible implicative semi-lattice. Also the term "algebra" will be used in place of "implicative semi-lattice". Let \mathcal{C}_n denote the variety generated by C_n , the n chain, and \mathcal{B}_n denote the variety generated by \hat{B}_n , where B_n is the Boolean algebra with n atoms. Let $\overline{\mathcal{C}}_n$ denote the variety of all algebras which do not have $n + 1$ chains as sub-algebras, and similarly let $\overline{\mathcal{B}}_n$ denote the variety of all algebras which do not have sub-algebras isomorphic to \hat{B}_{n+1} . (Throughout n and m will denote natural numbers.) Let $W_{n,m} = \mathcal{C}_n \vee \mathcal{B}_m$, and $V_{n,m} = \overline{\mathcal{C}}_n \cap \overline{\mathcal{B}}_m$. We say that a variety has coordinates n, m if it is in the interval $[W_{n,m}, V_{n,m}]$.

LEMMA 3.1. *If $A \in V_{n,m}$, and if A is finite, then $|A| \leq 2^{m(n-3)}(2^m + 1)$, where $|A|$ denotes the number of elements in A .*

Proof. Since A is subdirectly irreducible and does not contain \hat{B}_{m+1} as a subalgebra, A cannot contain B_{m+1} . Thus the closed algebra of A has at most m atoms. The proof now proceeds by induction. The case $n = 3$ holds since $A \in V_{3,m}$ implies $A = \hat{B}_l$ for some $l \leq m$. Assume that the proposition holds for some n , and let $A \in V_{n+1,m}$. Then the dense filter D of A is an element of $V_{n,m}$. Thus $|D| \leq 2^{m(n-3)}(2^m + 1)$. The proposition follows for the $n + 1$ case since every element of A is the meet of a closed element and a dense element.

COROLLARY 3.2. *$V_{n,m}$ contains only a finite number of distinct finite subdirectly irreducible algebras.*

THEOREM 3.3. $V_{n,m}$ contains no infinite subdirectly irreducible algebras.

Proof. Assume the contrary, and let n be the least integer for which there is an m such that $V_{n,m}$ has an infinite subdirectly irreducible algebra, A . Now A is unbounded, since if A were bounded, the dense filter of A would be an infinite subdirectly irreducible algebra in $V_{n-1,m}$. This reasoning also shows that any principal filter of A is bounded in size by the bound of Lemma 3.1, and this in turn implies that A is bounded, which establishes a contradiction.

COROLLARY 3.4. If V is a variety of implicative semi-lattices, then the following are equivalent:

- (i) V has only finitely many subvarieties.
- (ii) V is generated by a finite algebra.
- (iii) V has coordinates n, m for some natural numbers n and m .

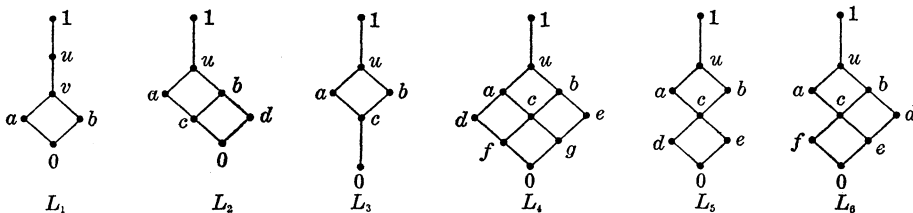


FIGURE 1

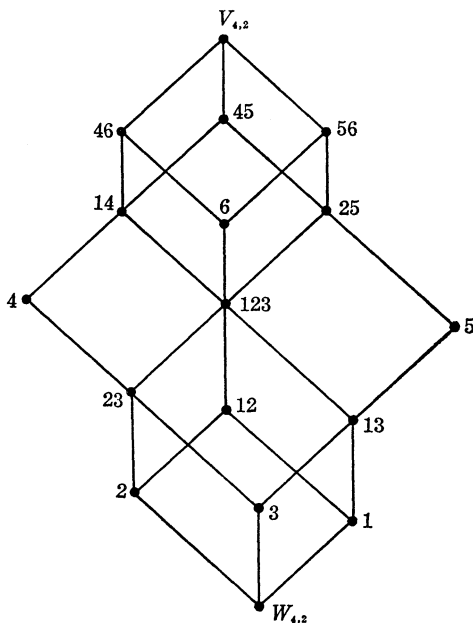


FIGURE 2

In order for A to be in $V_{4,2}$, the closed algebra of A must be B_1 or B_2 , and the dense filter of A must be \hat{B}_2 , C_2 or C_3 . In [6] a method is given for constructing all algebras having a given closed algebra and a given dense filter. We omit the details, but using this process one finds that the subdirectly irreducible members of $V_{4,2} - W_{4,2}$ are those shown in Figure 1. We have $L_1 \leq_s L_5$, $L_2 \leq_s L_5$; $L_2, L_3 \leq_s L_4$; $L_1, L_2, L_3 \leq_s L_6$; and these are the only subalgebra relations holding among these six algebras. Thus the interval $[W_{4,2}, V_{4,2}]$ is as pictured in Figure 2, where the numbers beside a point in the lattice correspond to the indices of the algebras which generate that variety.

For $n \leq 4$ and $m \leq 2$, it is clear that the varieties $W_{n+1,m}$, $W_{n,m+1}$, and $W_{n,m} \vee \{L_i\}^e$ for $i = 1, 2, 3$ cover $W_{n,m}$. ($\{L_i\}^e$ is the variety generated by L_i .) It is also clear that any other cover of $W_{n,m}$ would have to be a subvariety of $V_{n,m}$. We now show that there are no additional covers of $W_{n,m}$.

DEFINITION 3.5. For $B, D \leq_s L$, we say B is fixed with respect to D if $d*b = b$ for $b \in B$, and $d \in D$. We say that D is total with respect to B if $b*d \in D$ for $b \in B, d \in D$. Let $B \nabla D = \{b \wedge d \mid b \in B, d \in D\}$.

It was shown in [5] that $B \nabla D$ is a subalgebra of L if B is fixed with respect to D and D is total with respect to B .

THEOREM 3.6. *If L is a subdirectly irreducible implicative semi-lattice, and if $C_4 \leq_s L$, then either L is a chain or $L_i \leq_s L$ for some $i = 1, 2, 3$.*

Proof. First, consider the case where L is bounded. If the dense filter of L is not a chain, then it contains \hat{B}_2 as a subalgebra, and thus $L_3 \leq_s L$. Hence, we may assume that the dense filter of L is a chain. If the closed algebra of L is simple, then L is also a chain. Therefore we may assume that the closed algebra of L contains a subalgebra $\{1, b, b', 0\}$, where b' is the complement of b in the closed algebra. Now either $b*d = 1$ for every dense element d , or there is a dense element $d < 1$ such that $b*d = d$. If $b*d = d$, then $b'*d = 1$. Thus in either case, we have a subalgebra $D = \{1, u, d\}$ of the dense filter of L such that B is fixed with respect to D and D is total with respect to B . Hence $B \nabla D \leq_s L$. We may assume that $b' \leq d$. If $b \leq d$, then $B \nabla D = L_1$. If $b \not\leq d$, then $B \nabla D = L_2$. Now suppose that L is not bounded and that $L_i \not\leq_s L$ for any $i = 1, 2$, or 3 . Let $a, b \in L$, and let d be the least element of some example of C_4 in L . Then from consideration of the bounded case, it follows that the principal filter generated by $a \wedge b \wedge d$ is a chain. Thus a and b are comparable and so L is a chain.

COROLLARY 3.7. For $n \geq 4$ and $m \geq 2$, $W_{n,m}$ has exactly five covers.

COROLLARY 3.8. $\mathcal{E}_n \vee \overline{\mathcal{E}_3}$ and $\mathcal{B}_m \vee \overline{\mathcal{B}_2}$ have exactly three covers.

4. Identities. If $g(x_1, \dots, x_n)$ is an implicative semi-lattice term and if L is an implicative semi-lattice, then we say that $g(x_1, \dots, x_n)$ holds in L , or simply that g holds in L , provided the equation $g(x_1, \dots, x_n) = 1$ holds in L . If this is not the case we say that g fails in L . We let $V(g)$ denote the variety of all implicative semi-lattices in which g holds. We are interested here only in subdirectly irreducible implicative semi-lattices, and we let u denote the dual atom in any such algebra. If there exist elements $a_1, \dots, a_n \in L$ such that $g(a_1, \dots, a_n) = u$, then we say that g u -fails in L . If g u -fails in every subdirectly irreducible algebra in which it fails, then we say that g has property U .

We let $a + b$ denote the psuedo-join (see [7]) of the elements a and b (i.e. $a + b = ((a*b)*b) \wedge ((b*a)*a)$). In general this is not an associative operation, and when not indicated otherwise, we intend for the grouping to be to the left (i.e. $a + b + c = (a + b) + c$). If a and b are comparable elements, then $a + b$ is the larger of the two.

LEMMA 4.1. If $a_1 \geq a_i$ for $i = 2, \dots, n$, then

$$a_1 + a_2 + \dots + a_n = a_1 .$$

We should note that this lemma depends on our convention of association.

DEFINITION 4.2. If $g_1(x_1, \dots, x_n)$ and $g_2(x_1, \dots, x_m)$ are terms, then we let

$$(g_1 \oplus g_2)(x_1, \dots, x_{n+m}) = g_1(x_1, \dots, x_n) + g_2(x_{n+1}, \dots, x_{n+m})$$

and

$$(g_1 \wedge g_2)(x_1, \dots, x_{n+m}) = g_1(x_1, \dots, x_n) \wedge g_2(x_{n+1}, \dots, x_{n+m}) .$$

LEMMA 4.3. If g_1 u -fails in L and if g_2 fails in L , then $g_1 \oplus g_2$ u -fails in L . Thus if g_1 has property U , then so does $g_1 \oplus g_2$.

LEMMA 4.4. If g_1 has property U , then $V(g_1) \vee V(g_2) = V(g_1 \oplus g_2)$.

Proof. By [2, Lemma 4.1] any subdirectly irreducible member, L , of $V(g_1) \vee V(g_2)$ is in $V(g_1) \cup V(g_2)$. Thus g_1 holds in L or g_2 holds in L . Hence $g_1 \oplus g_2$ holds in L .

On the other hand, if L is any subdirectly irreducible not in

$V(g_1) \vee V(g_2)$, then g_1 and g_2 both fail in L . Thus $g_1 u$ -fails in L ; so $g_1 \oplus g_2$ fails in L .

LEMMA 4.5. $V(g_1) \wedge V(g_2) = V(g_1 \wedge g_2)$. Furthermore, if g_1 and g_2 both have property U , then so does $g_1 \wedge g_2$.

The main idea in the following theorem is present in a similar theorem for Heyting algebras due to Alan Day [1].

THEOREM 4.6. Letting t^* denote $t^*(x_1 \wedge \dots \wedge x_{n+1})$ and l_{ij} denote $x_i^{**} * x_j^{**}$, we have

$$\overline{\mathcal{B}}_n = V(P_n)$$

where

$$P_n(x_1, \dots, x_{n+2}) = x_{n+2} + l_{12} + l_{21} + \dots + l_{n+1,n}$$

where each l_{ij} with $i \neq j$ and $i, j \leq n + 1$ occurs exactly once. Also, P_n has property U .

Proof. Let a_1, \dots, a_{n+1} be the atoms of \hat{B}_{n+1} . Then $a_i^{**} = a_i$ and $a_i^{**} * a_j^{**} < 1$ if $i \neq j$. Thus $P_n(a_1, \dots, a_{n+1}, u) = u$. Hence $V(P_n) \subseteq \overline{\mathcal{B}}_n$.

Suppose now that L is any subdirectly irreducible member of $\overline{\mathcal{B}}_n$ and that $P_n(a_1, \dots, a_{n+2}) < 1$ in L . Then $a_1^{**}, \dots, a_{n+1}^{**}$ are pairwise incomparable closed elements in the principal filter generated by $a_1 \wedge \dots \wedge a_{n+1}$. Thus $\hat{B}_{n+1} \leq_s L$, a contradiction. Hence P_n holds in L .

In [8] terms were given which characterize the varieties \mathcal{E}_n and $\overline{\mathcal{E}}_n$. Denote these terms by q_n and r_n , respectively. It is easy to see that q_n and r_n have property U .

COROLLARY 4.7. $V_{n,m} = V(P_m \wedge r_n)$. In particular, $\mathcal{B}_m = V(P_m \wedge r_3)$.

COROLLARY 4.8. $W_{n,m} = V(q_n \oplus (P_m \wedge r_3))$.

We now turn our attention to the varieties of the interval $[W_{4,2}, V_{4,2}]$. First we shall give an indexed list of identities which can be used to describe these varieties. Note that for a term t , t^* is as defined in Theorem 4.6.

$$\begin{aligned}
 g_1 &= x_4 + ((x_1 \wedge x_2) * (x_1 \wedge x_2 \wedge x_3)) + (x_1 * x_2) + (x_2 * x_1) \\
 g_{12} &= x_4 + (x_1 * x_2) + (x_2 * x_1) + (x_1 \wedge x_2)^* + (x_1^{**} * x_1) + (x_2^{**} * x_2) \\
 g_{23} &= x_4 + (x_1 * x_3) + (x_1 * x_2) + (x_2 * x_1) + (x_3 + (x_3 * x_1)) + (x_3 + (x_3 * x_2)) \\
 g_2 &= g_{12} \wedge g_{23} \\
 g_3 &= x_4 + (x_1 * x_3) + (x_1 * x_2) + (x_2 * x_1) + (x_3 + (x_3 * (x_1 \wedge x_2))) \\
 g_{13} &= g_1 \oplus g_3 \\
 g_{123} &= g_{12} \oplus g_3 \\
 g_4 &= x_4 + (x_1 * x_3) + ((x_3 \wedge x_1) * (x_3 \wedge x_2)) + ((x_3 \wedge x_2) * (x_3 \wedge x_1)) \\
 &\quad + ((x_3 + (x_3 * (x_3 \wedge x_1))) + (x_3 + (x_3 * (x_3 \wedge x_2)))) \\
 g_{14} &= g_1 \oplus g_4 \\
 g_5 &= x_4 + (x_1 * x_2) + (x_2 * x_1) + (x_1 * x_3) + (x_3 * x_1) + (x_2 * x_3) \\
 g_{25} &= g_2 \oplus g_5 \\
 g_{45} &= g_4 \oplus g_5 \\
 g_{56} &= x_4 + (x_1 * x_2) + (x_2 * x_1) + (x_1 * x_3) + (x_3 * x_1) + (x_2 * x_3) + (x_3 * x_2) \\
 g_{46} &= x_5 + (x_1 * x_2) + (x_2 * x_1) + (x_1 \wedge x_2 \wedge x_3) * (x_1 \wedge x_2 \wedge x_4) \\
 &\quad + (x_1 \wedge x_2 \wedge x_4) * (x_1 \wedge x_2 \wedge x_3) \\
 &\quad + (x_1 + ((x_1 \wedge x_2) + (x_1 \wedge x_2 \wedge x_3))) \\
 &\quad + (x_1 + ((x_1 \wedge x_2) * (x_1 \wedge x_2 \wedge x_4))) \\
 &\quad + (x_2 + ((x_1 \wedge x_2) * (x_1 \wedge x_2 \wedge x_3))) \\
 &\quad + (x_2 + ((x_1 \wedge x_2) * (x_1 \wedge x_2 \wedge x_4))) .
 \end{aligned}$$

THEOREM 4.9. For $i, j = 1, \dots, 6$ let $h_i = g_i \wedge P_4 \wedge r_3$, $h_{ij} = g_{ij} \wedge P_4 \wedge r_3$, $h_{123} = g_{123} \wedge P_4 \wedge r_3$.

Then

- (i) $\{L_i\}^e = V(h_i)$
- (ii) $\{L_i, L_j\}^e = V(h_{ij})$ for $\{i, j\} = \{1, 3\}, \{1, 2\}, \{2, 3\}, \{1, 4\}, \{2, 5\}, \{4, 5\}, \{4, 6\}$, and $\{5, 6\}$.
- (iii) $\{L_1, L_2, L_3\}^e = V(h_{123})$.

COROLLARY 4.10. For i, j as in the previous theorem and $n > 4$, $m > 2$ we have

- (i) $\{L_i\}^e \vee W_{n,m} = V(h_i \oplus (q_n \oplus (P_m \wedge r_3)))$,
- (ii) $\{L_i, L_j\}^e \vee W_{n,m} = V(h_{ij} \oplus (q_n \oplus (P_m \wedge r_3)))$,
- (iii) $\{L_1, L_2, L_3\}^e \vee W_{n,m} = V(h_{123} \oplus (q_n \oplus (P_m \wedge r_3)))$.

In some cases the identities given can be simplified somewhat, but these were chosen for convenience in presentation.

Proof. The proof amounts to showing that each of the indexed polynomials g is valid in the corresponding variety of the diagram

of figure 2 and its subvarieties, and that it fails elsewhere in the diagram. Note that each of these identities has property U . We shall establish the validity of three of the more complicated identities only.

(1) g_{12} holds in L_1 and L_2 , but fails in L_3 : If $g_{12}(a_1, \dots, a_4) < 1$ in L_1 , then a_1 and a_2 are incomparable and $(a_1 \wedge a_2)^* = 1$, a contradiction.

If $g_{12}(a_1, \dots, a_4) < 1$ in L_2 , then we must have $\{a_1, a_2\} = \{a, b\}$ and $a_1 \wedge a_2 \wedge a_3 = 0$. However, $a^{***}a = 1$ then yields a contradiction.

In L_3 we have

$$g_{12}(a, b, 0, u) = u + b + a + (c*0) + (1*a) + (1*b) = u .$$

(2) g_4 holds in L_4 but fails in L_1 : In L_1 we have $g_4(a, b, v, u) = u + v + b + a + ((u + a) + (v + b)) = u$.

If $g_4(a_1, \dots, a_4) < 1$ in L_4 , then $a_3 < u$. In fact $a_3 = a, b$, or c since there must be a pair of incomparable elements below a_3 . If $a_3 = a$ we have $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, c\}, \{d, g\}$, or $\{f, g\}$. If $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, c\}$, then $a_3 + (a_3*c) = a + b = 1$. If $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{d, g\}$, then $a_3 + (a_3*g) = a + e = 1$. If $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{f, g\}$, then we get the same contradiction as in the preceding case. The case $a_3 = b$ is completely analogous. If $a_3 = c$, then $\{a_3 \wedge a_1, a_3 \wedge a_2\} = \{f, g\}$. Then $(a_3 + (a_3*f)) + (a_3 + (a_3*g)) = (c + d) + (c + e) = a + b = 1$, a contradiction.

(3) g_{46} holds in L_4 and L_6 , but fails in L_5 : If $g_{46}(a_1, \dots, a_5) < 1$, then a_1 and a_2 are incomparable and there must be a pair of incomparable elements, $a_1 \wedge a_2 \wedge a_3$ and $a_1 \wedge a_2 \wedge a_4$, which are less than $a_1 \wedge a_2$. Thus in L_4 we would have to have $\{a_1, a_2\} = \{a, b\}$ and $\{a_1 \wedge a_2 \wedge a_3, a_1 \wedge a_2 \wedge a_4\} = \{f, g\}$. However, we have $a + ((a \wedge b)*g) = 1$ which would give a contradiction. In L_6 we would have to have $\{a_1, a_2\} = \{a, b\}$ and $\{a_1 \wedge a_2 \wedge a_3, a_1 \wedge a_2 \wedge a_4\} = \{f, e\}$. This would lead to a contradiction, however, since $a + ((a \wedge b)*e) = 1$.

In L_5 we have

$$g_{46}(a, b, d, e, u) = u + b + a + e + d \\ + (a + d) + (a + e) + (b + d) + (b + e) = u .$$

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