WALLMAN COMPACTIFICATIONS ON E-COMpletely REGULAR SPACES

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The Wallman space on E-completely regular spaces is considered. Let $\tau^E$ be the family of all $E$-closed subsets of an $E$-completely regular space $X$. Then the Wallman space $\mathcal{W}(X, \tau^E)$ is a compactification of $X$. In particular, if $E$ is such that $I = [0, 1]$ is $E$-completely regular, then $\mathcal{W}(X, \tau^E)$ is an E-compactification. An example is given to show that $I$ being $E$-completely regular is necessary.

Recently, the relations between Stone-Čech compactifications and Wallman compactifications, and those between realcompactifications and Wallman compactifications have been studied by Frink [7], Njåstad [11], the Steiners [12], [13], Alo and Shapiro [1], [2], [3], [4], and some others.

Frink [7] introduced the concept of a normal base. (A normal base in a $T_1$-space $X$ is a base, $\mathcal{F}$, for the closed subsets of $X$ such that (i) $\mathcal{F}$ is disjunctive, i.e., given any closed set $F$ in $X$ and any point $x$ in $X\setminus F$, there is a member $A$ of $\mathcal{F}$ which contains $x$ and is disjoint from $F$; (ii) $\mathcal{F}$ is a ring, i.e., $\mathcal{F}$ contains all finite unions and intersections of its members; and (iii) any two disjoint members $A$ and $B$ of $\mathcal{F}$ are separated by disjoint complements of two members of $\mathcal{F}$, i.e., there exist elements $C$ and $D$ of $\mathcal{F}$ such that $A \subset X\setminus C$, $B \subset X\setminus D$, and $(X\setminus C) \cap (X\setminus D) = \emptyset$.) Frink showed that if $X$ has a normal base $\mathcal{F}$, equivalently $X$ is Tychonoff, then the Wallman space $\mathcal{W}(X, \mathcal{F})$, consisting of the $\mathcal{F}$-ultrafilters, is a Hausdorff compactification of $X$. Hence, the Stone-Čech compactification is always such a Wallman compactification. Njåstad [11] came along and gave a condition for a Hausdorff compactification to be of the Wallman type as defined by Frink. The condition is that the corresponding proximity admits a productive base consisting of closed subsets. Alo and Shapiro [2] used another approach for the results. While Alo and Shapiro in [1] imposed some conditions on the normal base $\mathcal{F}$ (see Theorem 2, [1]), and gave similar results for a wider class of compactifications, Njåstad showed that Alexandroff, Stone-Čech, Freudenthal [6], Fan-Gottesman [5], and Gould [9] compactifications satisfy the conditions in his theorem.

In [3], Alo and Shapiro used a delta normal base on a Tychonoff
space $X$. (A delta normal base $\mathcal{F}$ is a normal base which is closed under countable intersections, and such that for each $A \in \mathcal{F}$ there exist $B_1, B_2, \ldots \in \mathcal{F}$ with $Z = X \setminus \bigcup_{i=1}^{n} B_i$). They show that the subspace $\rho(X, \mathcal{F})$ of $W(X, \mathcal{F})$ which consists of all $\mathcal{F}$-ultrafilters with the countable intersection property assigned is realcompact. They [4] also used the notion of $\mathcal{F}$-ultrafilters in a countably productive normal base $\mathcal{F}$ to introduce a new space $\eta(X, \mathcal{F})$ consisting of all those $\mathcal{F}$-ultrafilters with the countable intersection property. They showed that if $\mathcal{F}$ is the collection of all zero-sets, then $\eta(X, \mathcal{F})$ is precisely the Hewitt realcompactification. However, the Steiner [13] provided an example to show that not every realcompactification can be obtained as an $\eta(X, \mathcal{F})$. They also gave an example of a space which is an $\eta(X, \mathcal{F})$ but not realcompact.

E. F. Steiner [12] generalized Frink's results and established the necessary and sufficient conditions for a Wallman space to be a compactification. The Steiner [13] used the notion of separating (see Definition 3) nest generated intersection rings (see (1.1), [13]) and studied the Wallman compactification $W(X, \mathcal{F})$ and the Wallman realcompactification $\nu(X, \mathcal{F})$. Incidentally, the concept of a delta normal base, introduced by Alo and Shapiro [3], is equivalent to that of separating nest generated intersection rings for collections $\mathcal{F}$ of closed sets.**

This note is to consider the Wallman compactification of an incompletely regular space. (See [10].) We have found a class of Hausdorff spaces, $E$, for which the Wallman compactification arising out of the ring of all $E$-closed subsets of $X$ is an $E$-compactification. In light of the examples in [13], we know that not every $E$-compactification can be obtained as a Wallman compactification.

We first recall some terminologies from [10].

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subset of $X$ if there is a positive integer $n$ and a continuous function $f \in C(X, E^n)$ such that $A = f^{-1}[F]$ for some closed subset $F$ of $E^n$.

One can easily show that a finite union and a finite intersection of $E$-closed subsets of $X$ is $E$-closed. (See 3.18 [10].) That is, the family of all $E$-closed subsets of $X$ forms a ring.

Combining these two definitions, we have:

**Lemma 1.** A $T_1$-space $X$ is $E$-completely regular if and only if each closed subset $F$ of $X$ and each point $x \in X \setminus F$ are separated by disjoint $E$-closed sets; i.e., there are disjoint $E$-closed subsets $A$ and $B$ of $X$ such that $x \in A$ and $F \subset B$.

**Proof.** Necessity. By definition of $E$-complete regularity, there is a positive integer $n$ and a continuous function $f \in C(X, E^n)$ such that $f(x) \notin \text{cl}_{E^n} f[F]$. Let $A = f^{-1}[f(x)]$ and $B = f^{-1}[\text{cl}_{E^n} f[F]]$. Then $A \cap B = \emptyset$ and $A$ and $B$ are $E$-closed subsets of $X$.

Sufficiency. Let $F$ be a closed subset in $X$ and $x \notin F$. By assumption, there are disjoint $E$-closed sets $A$ and $B$ such that $x \in A$, $F \subset B$ and $A \cap B = \emptyset$. Since $B$ is $E$-closed, there exist a positive integer $n$, and an $f \in C(X, E^n)$ such that $B = f^{-1}[D]$, for some closed subset $D$ in $E^n$. Now, since $x \notin B = f^{-1}[D]$, $f(x) \notin D$. This implies that $f(x) \notin \text{cl}_{E^n} f[B]$ as $\text{cl}_{E^n} f[B] \subset D$. Hence, $X$ is $E$-completely regular.

Before stating our next result, we give the following:

**Definition 3.** A family $\mathcal{F}$ of closed subsets of a space $X$ is called *separating* if for each closed subset $F$ of $X$ and each point $x \in X \setminus F$, there are disjoint elements $A$ and $B$ of $\mathcal{F}$ such that $x \in A$ and $F \subset B$. (See [12].)

E. F. Steiner in [12] proved:

**Theorem 2.** If $X$ is a $T_1$-space and $\mathcal{F}$ is a separating family, then the Wallman space $\mathcal{W}(X, \mathcal{F})$ is a compactification. If the Wallman space $W(X, \mathcal{F})$ is a compactification, then $X$ is $T_1$ and the ring generated from $\mathcal{F}$ is separating.

Now, suppose $X$ is $E$-completely regular. Then there is a cardinal $\alpha$, and a homeomorphism, $h$, from $X$ into $E^\alpha$. Let $\mathcal{F}$ denote the family of all $E$-closed subsets of $E^\alpha$, and $\mathcal{F} = \{ F \subset X: F = h^{-1}(F')$, for some $F' \in \mathcal{F} \}$. Then we have:

**Theorem 3.** The Wallman space $\mathcal{W}(X, \mathcal{F})$ is a compactification of $X$. 
Proof. By Theorem 2, we only have to show that $\mathcal{F}$ is a separating ring. However, by remark of Definition 2, $\mathcal{F}$ is a ring, so that $\mathcal{F}$ is a ring. Now, let $F$ be any closed of $X$, and $x \in X \setminus F$. Then that $h(x) \in \text{cl}_{E^a} h[F]$ is clear. Since $E^a$ is $E$-completely regular, $h(x)$ and $\text{cl}_{E^a} h[F]$ are separated by two disjoint $E$-closed sets, say $A_1$ and $A_2$, where $h(x) \in A_1$ and $\text{cl}_{E^a} h[F] \subset A_2$. Then $B_i = h^{-1}[A_i]$, $i = 1, 2$ are in $\mathcal{F}$ and $x \in B_i$ and $F \subset B_i$.

**THEOREM 4.** Let $X$ be a $T_i$ space and $\mathcal{F}$ be the family of all $E$-closed subset of $X$. Then the Wallman space $\mathcal{W}(X, \mathcal{F})$ is a compactification of $X$ if and only if $X$ is $E$-completely regular.

**Proof.** Sufficiency. We know that $\mathcal{F}$ is a ring, and by Lemma 1, $\mathcal{F}$ is separating. Hence, $\mathcal{W}(X, \mathcal{F})$ is a compactification.

Necessity. If $\mathcal{W}(X, \mathcal{F})$ is a compactification of $X$, then the ring $\mathcal{F}$ is separating by Theorem 2, and, and by Lemma 1, $X$ is $E$-completely regular.

In general, we do not know if $\mathcal{W}(X, \mathcal{F})$ is $E$-completely regular.

Next, we would like to determine under what conditions the Wallman compactification defined by the ring of all $E$-closed subsets of an $E$-completely regular space is an $E$-compactification.

We recall that an $E$-completely regular regular space $X$ is $E$-compact if and only if $X$ is homeomorphic to a closed subset of $E^a$ for some cardinal $\alpha$. Hence, each compact $E$-completely regular space is $E$-compact. Then we have:

**THEOREM 5.** If $E$, a Hausdorff space, is such that $I = [0, 1]$ with the usual topology is $E$-completely regular, then if $X$ is an $E$-completely regular space, the Wallman space $\mathcal{W}(X, \mathcal{F})$ generated by the ring $\mathcal{F}$ of all $E$-closed subsets of $X$ is an $E$-compactification of $X$.

**Proof.** By Theorem 2, $\mathcal{W}(X, \mathcal{F})$ is $T_i$-compact. Since $I$ is $E$-completely regular and compact, $I$ is $E$-compact and $\mathcal{W}(X, \mathcal{F})$ is $I$-compact. Thus, $\mathcal{W}(X, \mathcal{F})$ is $E$-compact by (4.6) [10]. Hence, it is an $E$-compactification of $X$.

**Remark.** (1) We know that there exists a space $E$ such that $I$ is $E$-completely regular. For example, let $E_i$ be any Hausdorff space. Define $E$ to be the topological sum of $I$ and $E_i$. Then $I$ is clearly $E$-completely regular, as $I$ is homeomorphic with a subspace (namely $I$) of $E$. Note that as long as $E_i$ is Hausdorff and not completely regular, $E$ is not completely regular.
(2) Next we point out that the condition that $I$ be $E$-completely regular cannot be omitted, for consider $E = X$, where $X$ is the space of Knaster and Kuratowski. We still recall it here (see p. 210 of [14]). Let $C$ denote the Cantor middle third set, and $Q$ the end points in $C$. Let $p = (1/2, 1/2) \in \mathbb{R}^2$, and for each $x \in C$, denote by $L_x$ the straight line segment joining $p$ and $x$.

Define

$$L_x^* = \begin{cases} \{(x_1, x_2) \in L_x : x_2 \text{ is rational}\}, & \text{if } x \in Q \\ \{(x_1, x_2) \in L_x : x_2 \text{ is irrational}\}, & \text{if } x \in C \setminus Q. \end{cases}$$

Then $E = X = \bigcup_{x \in C} L_x^* \setminus \{p\}$. Here $\bigcup_{x \in C} L_x^*$ is connected, while $E = X$ is $T_1$, totally disconnected, and $\dim X = \dim E \neq 0$ (see 29.8 [14]). It is then clear that $I$ is not $E$-completely regular, since $E^*$ is totally disconnected and so is any subset of $E^*$. (See 29.3 [14].)

Now, $X$ is $E$-completely regular, a metric space (see 29.8 [14]), and is hence normal. Consider $\mathcal{F}$, the family of all $E$-closed subsets of $X$. $\mathcal{F}$, in fact, consists of all closed subsets of $X$. Thus, the Wallman compactification $\mathcal{W}(X, \mathcal{F})$ is $\beta X$, the Stone-Cech compactification (see [8], p. 269).

Finally, $\beta X$ is $T_1$ compact space, but $\beta X$ is not totally disconnected, for otherwise by Theorem 16.17 in [8], we would have $\dim \beta X = 0$. But Theorem 16.11 [8] says that $\dim \beta X = \dim X$, and we know that $\dim X \neq 0$.

Therefore, $X = \mathcal{W}(X, \mathcal{F})$ cannot be $E$-completely regular, and is thus not an $E$-compact space.

In view of Remark (2), we have:

**Corollary 6.** For a Hausdorff space $E$, if $X$ is a $T_1$, zero-dimensional normal space having more than one point and such that every closed subset of $X$ is $E$-closed, then the Wallman space $\mathcal{W}(X, \mathcal{F})$ generated by the ring of all closed subsets of $X$ is an $E$-compactification of $X$.

**Proof.** Since $\dim X = 0$, $\dim \beta X = 0$. Also, $\beta X = \mathcal{W}(X, \mathcal{F})$ since $X$ is normal. Now, $\mathcal{W}(X, \mathcal{F})$ is $T_1$ and zero-dimensional. One can easily show that it is $E$-completely regular. Hence, $\mathcal{W}(X, \mathcal{F})$ is $E$-compact.

**Corollary 7.** If $X$ is discrete, then $\mathcal{W}(X, \mathcal{F})$ is an $E$-compactification of $X$, where $\mathcal{F}$ is the family of all closed subsets of $X$. 
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