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**WALLMAN COMPACTIFICATIONS ON *E*-COMPLETELY  
REGULAR SPACES**

NORMA MARY PIACUN AND LI PI SU

## WALLMAN COMPACTIFICATIONS ON $E$ -COMPLETELY REGULAR SPACES

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**The Wallman space on  $E$ -completely regular spaces is considered. Let  $\mathcal{F}$  be the family of all  $E$ -closed subsets of an  $E$ -completely regular space  $X$ . Then the Wallman space  $\mathcal{W}(X, \mathcal{F})$  is a compactification of  $X$ . In particular, if  $E$  is such that  $I = [0, 1]$  is  $E$ -completely regular, then  $\mathcal{W}(X, \mathcal{F})$  is an  $E$ -compactification. An example is given to show that  $I$  being  $E$ -completely regular is necessary.**

Recently, the relations between Stone-Čech compactifications and Wallman compactifications, and those between realcompactifications and Wallman compactifications have been studied by Frink [7], Njåstad [11], the Steiners [12], [13], Alo and Shapiro [1], [2], [3], [4], and some others.

Frink [7] introduced the concept of a normal base. (A normal base in a  $T_1$ -space  $X$  is a base,  $\mathcal{F}$ , for the closed subsets of  $X$  such that (i)  $\mathcal{F}$  is disjunctive, i.e., given any closed set  $F$  in  $X$  and any point  $x$  in  $X \setminus F$ , there is a member  $A$  of  $\mathcal{F}$  which contains  $x$  and is disjoint from  $F$ ; (ii)  $\mathcal{F}$  is a ring, i.e.,  $\mathcal{F}$  contains all finite unions and intersections of its members; and (iii) any two disjoint members  $A$  and  $B$  of  $\mathcal{F}$  are separated by disjoint complements of two members of  $\mathcal{F}$ , i.e., there exist elements  $C$  and  $D$  of  $\mathcal{F}$  such that  $A \subset X \setminus C$ ,  $B \subset X \setminus D$ , and  $(X \setminus C) \cap (X \setminus D) = \emptyset$ .) Frink showed that if  $X$  has a normal base  $\mathcal{F}$ , equivalently  $X$  is Tychonoff, then the Wallman space  $\mathcal{W}(X, \mathcal{F})$ , consisting of the  $\mathcal{F}$ -ultrafilters, is a Hausdorff compactification of  $X$ . Hence, the Stone-Čech compactification is always such a Wallman compactification. Njåstad [11] came along and gave a condition for a Hausdorff compactification to be of the Wallman type as defined by Frink. The condition is that the corresponding proximity admits a productive base consisting of closed subsets. Alo and Shapiro [2]\* used another approach for the results. While Alo and Shapiro in [1]\* imposed some conditions on the normal base  $\mathcal{F}$  (see Theorem 2, [1]), and gave similar results for a wider class of compactifications, Njåstad showed that Alexandroff, Stone-Čech, Freudenthal [6], Fan-Gottesman [5], and Gould [9] compactifications satisfy the conditions in his theorem.

In [3]\*, Alo and Shapiro used a delta normal base on a Tychonoff

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\* The authors wish to express their thanks to the referee for calling these articles to their attention.

space  $X$ . (A delta normal base  $\mathcal{F}$  is a normal base which is closed under countable intersections, and such that for each  $A \in \mathcal{F}$  there exist  $B_1, B_2, \dots \in \mathcal{F}$  with  $Z = X \setminus \bigcup_{i=1}^{\infty} B_i$ ). They show that the subspace  $\rho(X, \mathcal{F})$  of  $\mathcal{W}(X, \mathcal{F})$  which consists of all  $\mathcal{F}$ -ultrafilters with the countable intersection property assigned is realcompact. They [4] also used the notion of  $\mathcal{F}$ -ultrafilters in a countably productive normal base  $\mathcal{F}$  to introduce a new space  $\eta(X, \mathcal{F})$  consisting of all those  $\mathcal{F}$ -ultrafilters with the countable intersection property. They showed that if  $\mathcal{F}$  is the collection of all zero-sets, then  $\eta(X, \mathcal{F})$  is precisely the Hewitt realcompactification. However, the Steiners [13] provided an example to show that not every realcompactification can be obtained as an  $\eta(X, \mathcal{F})$ . They also gave an example of a space which is an  $\eta(X, \mathcal{F})$  but not realcompact.

E. F. Steiner [12] generalized Frink's results and established the necessary and sufficient conditions for a Wallman space to be a compactification. The Steiners [13] used the notion of separating (see Definition 3) nest generated intersection rings (see (1.1), [13]) and studied the Wallman compactification  $\mathcal{W}(X, \mathcal{F})$  and the Wallman realcompactification  $\nu(X, \mathcal{F})$ . Incidentally, the concept of a delta normal base, introduced by Alo and Shapiro [3], is equivalent to that of separating nest generated intersection rings for collections  $\mathcal{F}$  of closed sets.\*\*

This note is to consider the Wallman compactification of an  $E$ -completely regular space. (See [10].) We have found a class of Hausdorff spaces,  $E$ , for which the Wallman compactification arising out of the ring of all  $E$ -closed subsets of  $X$  is an  $E$ -compactification. In light of the examples in [13], we know that not every  $E$ -compactification can be obtained as a Wallman compactification.

We first recall some terminologies from [10].

**DEFINITION 1.** Let  $E$  be any Hausdorff space. A  $T_1$ -space  $X$  is said to be  $E$ -completely regular if  $\bigcup_{n=1}^{\infty} C(X, E^n)$  separates the closed subsets and points in  $X$ . Here,  $C(X, E^n)$  is the set of all continuous functions from  $X$  into the Cartesian product  $E^n$ .

Note that this is equivalent to saying that for each closed subset  $A$  of  $X$  and for each  $p \in (X \setminus A)$ , there is a positive integer  $n$  and a continuous function  $f \in C(X, E^n)$  such that  $f(p) \notin \text{cl } f[A]$ . This is also equivalent to saying that  $X$  is homeomorphic to a subset of  $E^\alpha$  for some cardinal  $\alpha$ . (See [10].)

We will always assume that  $E$  is a Hausdorff space.

**DEFINITION 2.** A subset  $A$  in a space  $X$  is called an  $E$ -closed

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\*\* The authors wish to thank the referee for pointing out this fact.

subset of  $X$  if there is a positive integer  $n$  and a continuous function  $f \in C(X, E^n)$  such that  $A = f^{-1}[F]$  for some closed subset  $F$  of  $E^n$ .

One can easily show that a finite union and a finite intersection of  $E$ -closed subsets of  $X$  is  $E$ -closed. (See 3.18 [10].) That is, the family of all  $E$ -closed subsets of  $X$  forms a ring.

Combining these two definitions, we have:

**LEMMA 1.** *A  $T_1$ -space  $X$  is  $E$ -completely regular if and only if each closed subset  $F$  of  $X$  and each point  $x \in X \setminus F$  are separated by disjoint  $E$ -closed sets; i.e., there are disjoint  $E$ -closed subsets  $A$  and  $B$  of  $X$  such that  $x \in A$  and  $F \subset B$ .*

*Proof. Necessity.* By definition of  $E$ -complete regularity, there is a positive integer  $n$  and a continuous function  $f \in C(X, E^n)$  such that  $f(x) \notin \text{cl}_{E^n} f[F]$ . Let  $A = f^{-1}[f(x)]$  and  $B = f^{-1}[\text{cl}_{E^n} f[F]]$ . Then  $A \cap B = \emptyset$  and  $A$  and  $B$  are  $E$ -closed subsets of  $X$ .

*Sufficiency.* Let  $F$  be a closed subset in  $X$  and  $x \notin F$ . By assumption, there are disjoint  $E$ -closed sets  $A$  and  $B$  such that  $x \in A$ ,  $F \subset B$  and  $A \cap B = \emptyset$ . Since  $B$  is  $E$ -closed, there exist a positive integer  $n$ , and an  $f \in C(X, E^n)$  such that  $B = f^{-1}[D]$ , for some closed subset  $D$  in  $E^n$ . Now, since  $x \notin B = f^{-1}[D]$ ,  $f(x) \notin D$ . This implies that  $f(x) \notin \text{cl}_{E^n} f[B]$  as  $\text{cl}_{E^n} f[B] \subset D$ . Hence,  $X$  is  $E$ -completely regular.

Before stating our next result, we give the following:

**DEFINITION 3.** A family  $\mathcal{F}$  of closed subsets of a space  $X$  is called *separating* if for each closed subset  $F$  of  $X$  and each point  $x \in X \setminus F$ , there are disjoint elements  $A$  and  $B$  of  $\mathcal{F}$  such that  $x \in A$  and  $F \subset B$ . (See [12].)

E. F. Steiner in [12] proved:

**THEOREM 2.** *If  $X$  is a  $T_1$ -space and  $\mathcal{F}$  is a separating family, then the Wallman space  $\mathcal{W}(X, \mathcal{F})$  is a compactification. If the Wallman space  $W(X, \mathcal{F})$  is a compactification, then  $X$  is  $T_1$  and the ring generated from  $\mathcal{F}$  is separating.*

Now, suppose  $X$  is  $E$ -completely regular. Then there is a cardinal  $\alpha$ , and a homeomorphism,  $h$ , from  $X$  into  $E^\alpha$ . Let  $\mathcal{S}$  denote the family of all  $E$ -closed subsets of  $E^\alpha$ , and  $\mathcal{F} = \{F \subset X: F = h^{-1}(F'), \text{ for some } F' \in \mathcal{S}\}$ . Then we have:

**THEOREM 3.** *The Wallman space  $\mathcal{W}(X, \mathcal{F})$  is a compactification of  $X$ .*

*Proof.* By Theorem 2, we only have to show that  $\mathcal{F}$  is a separating ring. However, by remark of Definition 2,  $\mathcal{S}$  is a ring, so that  $\mathcal{F}$  is a ring. Now, let  $F$  be any closed of  $X$ , and  $x \in X \setminus F$ . Then that  $h(x) \notin \text{cl}_{E^\alpha} h[F]$  is clear. Since  $E^\alpha$  is  $E$ -completely regular,  $h(x)$  and  $\text{cl}_{E^\alpha} h[F]$  are separated by two disjoint  $E$ -closed sets, say  $A_1$  and  $A_2$ , where  $h(x) \in A_1$  and  $\text{cl}_{E^\alpha} h[F] \subset A_2$ . Then  $B_i = h^{-1}[A_i]$ ,  $i = 1, 2$  are in  $\mathcal{F}$  and  $x \in B_1$  and  $F \subset B_2$ .

**THEOREM 4.** *Let  $X$  be a  $T_1$  space and  $\mathcal{F}$  be the family of all  $E$ -closed subset of  $X$ . Then the Wallman space  $\mathcal{W}(X, \mathcal{F})$  is a compactification of  $X$  if and only if  $X$  is  $E$ -completely regular.*

*Proof. Sufficiency.* We know that  $\mathcal{F}$  is a ring, and by Lemma 1,  $\mathcal{F}$  is separating. Hence,  $\mathcal{W}(X, \mathcal{F})$  is a compactification.

*Necessity.* If  $\mathcal{W}(X, \mathcal{F})$  is a compactification of  $X$ , then the ring  $\mathcal{F}$  is separating by Theorem 2, and, and by Lemma 1,  $X$  is  $E$ -completely regular.

In general, we do not know if  $\mathcal{W}(X, \mathcal{F})$  is  $E$ -completely regular.

Next, we would like to determine under what conditions the Wallman compactification defined by the ring of all  $E$ -closed subsets of an  $E$ -completely regular space is an  $E$ -compactification.

We recall that an  $E$ -completely regular space  $X$  is  $E$ -compact if and only if  $X$  is homeomorphic to a closed subset of  $E^\alpha$  for some cardinal  $\alpha$ . Hence, each compact  $E$ -completely regular space is  $E$ -compact. Then we have:

**THEOREM 5.** *If  $E$ , a Hausdorff space, is such that  $I = [0, 1]$  with the usual topology is  $E$ -completely regular, then if  $X$  is an  $E$ -completely regular space, the Wallman space  $\mathcal{W}(X, \mathcal{F})$  generated by the ring  $\mathcal{F}$  of all  $E$ -closed subsets of  $X$  is an  $E$ -compactification of  $X$ .*

*Proof.* By Theorem 2,  $\mathcal{W}(X, \mathcal{F})$  is  $T_2$ -compact. Since  $I$  is  $E$ -completely regular and compact,  $I$  is  $E$ -compact and  $\mathcal{W}(X, \mathcal{F})$  is  $I$ -compact. Thus,  $\mathcal{W}(X, \mathcal{F})$  is  $E$ -compact by (4.6) [10]. Hence, it is an  $E$ -compactification of  $X$ .

**REMARK.** (1) We know that there exists a space  $E$  such that  $I$  is  $E$ -completely regular. For example, let  $E_1$  be any Hausdorff space. Define  $E$  to be the topological sum of  $I$  and  $E_1$ . Then  $I$  is clearly  $E$ -completely regular, as  $I$  is homeomorphic with a subspace (namely  $I$ ) of  $E$ . Note that as long as  $E_1$  is Hausdorff and not completely regular,  $E$  is not completely regular.

(2) Next we point out that the condition that  $I$  be  $E$ -completely regular cannot be omitted, for consider  $E = X$ , where  $X$  is the space of Knaster and Kuratowski. We still recall it here (see p. 210 of [14]). Let  $C$  denote the Cantor middle third set, and  $Q$  the end points in  $C$ . Let  $p = (1/2, 1/2) \in R^2$ , and for each  $x \in C$ , denote by  $L_x$  the straight line segment joining  $p$  and  $x$ .

Define

$$L_x^* = \begin{cases} \{(x_1, x_2) \in L_x : x_2 \text{ is rational}\}, & \text{if } x \in Q \\ \{(x_1, x_2) \in L_x : x_2 \text{ is irrational}\}, & \text{if } x \in C \setminus Q. \end{cases}$$

Then  $E = X = \bigcup_{x \in C} L_x^* \setminus \{p\}$ . Here  $\bigcup_{x \in C} L_x^*$  is connected, while  $E = X$  is  $T_2$ , totally disconnected, and  $\dim X = \dim E \neq 0$  (see 29.8 [14]). It is then clear that  $I$  is not  $E$ -completely regular, since  $E^\alpha$  is totally disconnected and so is any subset of  $E^\alpha$ . (See 29.3 [14].)

Now,  $X$  is  $E$ -completely regular, a metric space (see 29.8 [14]), and is hence normal. Consider  $\mathcal{F}$ , the family of all  $E$ -closed subsets of  $X$ .  $\mathcal{F}$ , in fact, consists of all closed subsets of  $X$ . Thus, the Wallman compactification  $\mathcal{W}(X, \mathcal{F})$  is  $\beta X$ , the Stone-Ćech compactification (see [8], p. 269).

Finally,  $\beta X$  is  $T_2$  compact space, but  $\beta X$  is not totally disconnected, for otherwise by Theorem 16.17 in [8], we would have  $\dim \beta X = 0$ . But Theorem 16.11 [8] says that  $\dim \beta X = \dim X$ , and we know that  $\dim X \neq 0$ .

Therefore,  $X = \mathcal{W}(X, \mathcal{F})$  cannot be  $E$ -completely regular, and is thus not an  $E$ -compact space.

In view of Remark (2), we have:

**COROLLARY 6.** *For a Hausdorff space  $E$ , if  $X$  is a  $T_1$  zero-dimensional normal space having more than one point and such that every closed subset of  $X$  is  $E$ -closed, then the Wallman space  $\mathcal{W}(X, \mathcal{F})$  generated by the ring of all closed subsets of  $X$  is an  $E$ -compactification of  $X$ .*

*Proof.* Since  $\dim X = 0$ ,  $\dim \beta X = 0$ . Also,  $\beta X = \mathcal{W}(X, \mathcal{F})$  since  $X$  is normal. Now,  $\mathcal{W}(X, \mathcal{F})$  is  $T_1$  and zero-dimensional. One can easily show that it is  $E$ -completely regular. Hence,  $\mathcal{W}(X, \mathcal{F})$  is  $E$ -compact.

**COROLLARY 7.** *If  $X$  is discrete, then  $\mathcal{W}(X, \mathcal{F})$  is an  $E$ -compactification of  $X$ , where  $\mathcal{F}$  is the family of all closed subsets of  $X$ .*

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William George Bade, <i>Complementation problems for the Baire classes</i> .....	1
Ian Douglas Brown, <i>Representation of finitely generated nilpotent groups</i> .....	13
Hans-Heinrich Brungs, <i>Left Euclidean rings</i> .....	27
Victor P. Camillo and John Cozzens, <i>A theorem on Noetherian hereditary rings</i> .....	35
James Cecil Cantrell, <i>Codimension one embeddings of manifolds with locally flat triangulations</i> .....	43
L. Carlitz, <i>Enumeration of up-down permutations by number of rises</i> .....	49
Thomas Ashland Chapman, <i>Surgery and handle straightening in Hilbert cube manifolds</i> .....	59
Roger Cook, <i>On the fractional parts of a set of points. II</i> .....	81
Samuel Harry Cox, Jr., <i>Commutative endomorphism rings</i> .....	87
Michael A. Engber, <i>A criterion for divisoriality</i> .....	93
Carl Clifton Faith, <i>When are proper cyclics injective</i> .....	97
David Finkel, <i>Local control and factorization of the focal subgroup</i> .....	113
Theodore William Gamelin and John Brady Garnett, <i>Bounded approximation by rational functions</i> .....	129
Kazimierz Goebel, <i>On the minimal displacement of points under Lipschitzian mappings</i> .....	151
Frederick Paul Greenleaf and Martin Allen Moskowitz, <i>Cyclic vectors for representations associated with positive definite measures: nonseparable groups</i> .....	165
Thomas Guy Hallam and Nelson Onuchic, <i>Asymptotic relations between perturbed linear systems of ordinary differential equations</i> .....	187
David Kent Harrison and Hoyt D. Warner, <i>Infinite primes of fields and completions</i> .....	201
James Michael Hornell, <i>Divisorial complete intersections</i> .....	217
Jan W. Jaworowski, <i>Equivariant extensions of maps</i> .....	229
John Jobe, <i>Dendrites, dimension, and the inverse arc function</i> .....	245
Gerald William Johnson and David Lee Skoug, <i>Feynman integrals of non-factorable finite-dimensional functionals</i> .....	257
Dong S. Kim, <i>A boundary for the algebras of bounded holomorphic functions</i> .....	269
Abel Klein, <i>Renormalized products of the generalized free field and its derivatives</i> ...	275
Joseph Michael Lambert, <i>Simultaneous approximation and interpolation in <math>L_1</math> and <math>C(T)</math></i> .....	293
Kelly Denis McKennon, <i>Multipliers of type <math>(p, p)</math> and multipliers of the group <math>L_p</math>-algebras</i> .....	297
William Charles Nemitz and Thomas Paul Whaley, <i>Varieties of implicative semi-lattices. II</i> .....	303
Donald Steven Passman, <i>Some isolated subsets of infinite solvable groups</i> .....	313
Norma Mary Piacun and Li Pi Su, <i>Wallman compactifications on <math>E</math>-completely regular spaces</i> .....	321
Jack Ray Porter and Charles I. Votaw, <i><math>S(\alpha)</math> spaces and regular Hausdorff extensions</i> .....	327
Gary Sampson, <i>Two-sided <math>L_p</math> estimates of convolution transforms</i> .....	347
Ralph Edwin Showalter, <i>Equations with operators forming a right angle</i> .....	357
Raymond Earl Smithson, <i>Fixed points in partially ordered sets</i> .....	363
Victor Snaith and John James Ucci, <i>Three remarks on symmetric products and symmetric maps</i> .....	369
Thomas Rolf Turner, <i>Double commutants of weighted shifts</i> .....	379
George Kenneth Williams, <i>Mappings and decompositions</i> .....	387