

# Pacific Journal of Mathematics

**TWO-SIDED  $L_p$  ESTIMATES OF CONVOLUTION  
TRANSFORMS**

GARY SAMPSON

## TWO-SIDED $L_p$ ESTIMATES OF CONVOLUTION TRANSFORMS

GARY SAMPSON

Let  $f$  and  $g$  be two Lebesgue measurable functions on the real line. Then the equation

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x - t)dt$$

defines the convolution transform of  $f$  and  $g$ . In an earlier paper [4] we obtained sharp upper and lower estimates for the expression

$$(A) \quad \sup_{\substack{|E| \leq u \\ f_i \sim g_i^*}} \int_E |(f_1 * \dots * f_n)(x)|^p d(x)$$

where  $p = 1, 2$  and  $4$ , with applications to Fourier transform inequalities. This paper contains estimates of (A) for all values of  $p(p \geq 1)$  in the case where  $E = (-\infty, +\infty)$ . For example, one of our theorems implies the following:

“If  $g_i^*$  is bounded and has compact support for all  $i$ , then there exists a constant  $K$ ,  $1/(p + 1)^p \leq K \leq (p + 1)^p(2^{n-1})^p$ , such that

$$(A) = K \int_0^\infty |x^{n-1}(g_1^{**} - g_1^*) \dots (g_n^{**} - g_n^*)|^p d(x) .”$$

Here  $g_i^*$  are preassigned decreasing functions and the symbol  $f_i \sim g_i^*$  means

$$|\{x: |f_i(x)| > y\}| = |\{x: g_i^*(x) > y\}| \quad \text{for all } y > 0 .$$

**Introduction.** In an earlier paper [4], we obtained sharp upper and lower estimates for the expression

$$(A) \quad \sup_{\substack{f_i \sim g_i^* \\ |E| \leq u}} \int_E \psi(f_1 * f_2 * \dots * f_n)(x) d(x)$$

where  $\psi(u) = u, u^2$ . R. O’Neil obtained sharp upper and lower estimates when  $\psi(u) = u$  and  $n = 2$ , [3, Lemma 1.5]. Our results coincide with his for this case.

We were able to apply our estimates for the case  $\psi(u) = u^2$  ( $n$ -arbitrary) to classical Fourier transform inequalities of Hardy and Littlewood.

The main problem of this paper is to determine whether or not one can obtain the same types of upper and lower estimates for the functions  $\psi(u) = u^p, p > 1$ . We have, in fact, obtained such estimates for a class of functions  $\psi$  containing the class  $\psi(u) = u^p$ . We are

able to prove [see Theorem 2.12 and Corollary 2.18 of this paper] that there exist  $p, q > 1$  such that

$$\begin{aligned} & \frac{1}{(p' + 1)^p} \cdot \frac{1}{(2^{n-1})^p} \int_0^\infty \psi[x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)(x)] d(x) \\ & \leq \sup_{f_i \sim g_i^*} \int_{-\infty}^{+\infty} \psi((f_1 * \cdots * f_n)(x)) d(x) \\ & \leq (q + 1)^q \int_0^\infty \psi[x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)(x)] d(x). \end{aligned}$$

In proving our estimates, we use as a major tool Lemma 2.4 which contains the following inequality:

$$\left( \frac{1}{p + 1} \right) \|g\|_p \leq \left\| \frac{1}{x} \int_0^x g - g(x) \right\|_p \leq (p' + 1) \|g\|_p.$$

All the functions  $f, g, \dots$  which appear in this paper will be non-negative, Lebesgue measurable functions for which  $|\{x: f(x) > y\}| < \infty$  for every  $y > 0$ . By the statement  $f(x) \doteq g(x)$  we mean that  $|\{x: f(x) \neq g(x)\}| = 0$ .

**I. Preliminaries.** The idea of considering the decreasing rearrangement  $f^*$  and symmetrically decreasing rearrangement  $\bar{f}$  of a function  $f$  for finding sharp inequalities of convolution transforms was first noticed by Hardy, Littlewood and Pólya [1, Chapter X]. Since all our estimates are in terms of  $f^*$  and  $\bar{f}$ , we shall start by defining these concepts.

**DEFINITION 1.1** We say the functions  $f$  and  $g$  are equimeasurable; we write  $f \sim g$ , if

$$|\{x: f(x) > y\}| = |\{x: g(x) > y\}| \quad \text{for all } y > 0.$$

**DEFINITION 1.2** By  $f^*(x)$ , we denote a function such that

$$(i) \quad f^*(x) \text{ decreases for } x > 0$$

and

$$(ii) \quad f^* \sim f.$$

Further, for  $x > 0$  we set,

$$f^{**}(x) = \frac{1}{x} \int_0^x f^*(t) dt,$$

and finally, we set  $\bar{f}(x) = f^*(2|x|)$ .

In a similar manner, we can discuss the decreasing rearrangement of a sequence of nonnegative numbers  $a_1, a_2, \dots, a_n$ . That is,

we rearrange this sequence into a decreasing sequence. The new sequence is denoted by  $a_1^*, a_2^*, \dots, a_n^*$ . This sequence is characterized by the following two properties:

$$(i) \quad a_1^* \geq a_2^* \geq \dots \geq a_n^*,$$

and

$$(ii) \quad N(\{a_i \mid a_i > y\}) = N(\{a_i^* \mid a_i^* > y\}) \quad \text{for all } y > 0.$$

For a given set  $A$ ,  $N(A)$  stands for the number of points in  $A$ . Therefore, if  $\psi \in C(-\infty, +\infty)$ ,  $\psi$  is increasing, and  $\psi(0) = 0$ , we have

$$(1.3).^1 \quad \sum_{k=1}^n \psi(a_k) = \sum_{k=1}^n \psi(a_k^*).$$

Also, if  $f(\geq 0)$  is a step function with compact support, then  $\int_{-\infty}^{+\infty} \psi(f(x))d(x) = -\int_0^{+\infty} \psi(y)d(m(f, y))$ . Where the second term is a Riemann-Stieltjes integral with  $m(f, y) = |\{x: f(x) > y\}|$ . Now by a limiting argument, we see that

$$(1.4).^2 \quad \int_{-\infty}^{+\infty} \psi(f(x))dx = \int_0^{+\infty} \psi(f^*(x))d(x)$$

for all functions  $f$  such that  $m(f, y) < \infty$  for  $y > 0$ .

A nonnegative sequence  $\langle \bar{a}_i \rangle_{-\infty}^{+\infty}$  is said to be symmetrically decreasing if  $\bar{a}_0 \geq \bar{a}_1 = \bar{a}_{-1} \geq \dots \geq \bar{a}_n = \bar{a}_{-n} \geq \dots$ . It is well-known [1, Theorem 375, p. 273] that the convolution of symmetrically decreasing sequences is again a symmetrically decreasing sequence. The previous statement also holds if the term "sequence" is replaced with the term "function".

LEMMA 1.5.<sup>3</sup> *The function  $h_n(x) = \bar{g}_1 * \bar{g}_2 * \dots * \bar{g}_n(x)$  is a symmetrically decreasing function.*

Let  $(\bar{a}_i)$ ,  $(\bar{b}_i)$  and  $(\bar{c}_i)$  be given symmetrically decreasing sequences. If a nonnegative sequence  $(a_i)$  can be rearranged to equal  $(\bar{a}_i)$  term-for-term, then we write  $(a_i) \sim (\bar{a}_i)$ .

Pólya [2] had asked: When is the sum  $S = \sum_{r+s+t=0} a_r b_s c_t$ , for all rearrangements of the  $a_r$ 's,  $b_s$ 's,  $c_t$ 's (where  $(a_i) \sim (\bar{a}_i)$ ,  $(b_i) \sim (\bar{b}_i)$ ,  $(c_i) \sim (\bar{c}_i)$ ) the greatest? Hardy and Littlewood answered this question [2] by proving the general statement

<sup>1</sup> This equation holds without any restriction on  $\psi$ ; however, we need these restrictions in order for equation (1.4) to hold.

<sup>2</sup> This is the counterpart to (1.3) for functions.

<sup>3</sup> The proof can be found in [4].

$$\sum_{r+s+t+\dots=0} \alpha_r b_s c_t \dots \leq \sum_{r+s+t+\dots=0} \bar{\alpha}_r \bar{b}_s \bar{c}_t \dots$$

They were later able to prove the theorem for functions [1, Theorem 379; p. 279] which we now state.

**THEOREM A** (Hardy and Littlewood).<sup>4</sup>

$$(1.6) \quad \sup_{\substack{r_1 \sim r^* \\ f_i \sim g_i^*}} \int_{-\infty}^{+\infty} r_1(t) (f_1 * \dots * f_n)(t) dt = \int_{-\infty}^{+\infty} \bar{r}(t) h_n(t) dt .$$

In (1.6) if we set  $r^*(x) = \chi_{[0,u]}^{(x)}$ , then the estimate on the right reduces to  $\int_{-u/2}^{u/2} h_n(x) d(x)$ . In Theorem B, we give our estimate of (1.6) which has many advantages over the above estimate.

**THEOREM B.**<sup>3</sup> *If  $g_i^{**}$  is finite for each  $x$ , then there exists a constant  $K$ ,  $1/2^{n-1} \leq K \leq 1$ , such that*

$$(1.7) \quad \sup_{\substack{|E| \leq u \\ f_i \sim g_i^*}} \int_E (f_1 * f_2 * \dots * f_n)(x) d(x) \\ = Ku \int_u^\infty x^{n-2} (g_1^{**} - g_1^*) \dots (g_n^{**} - g_n^*) d(x) .$$

Here,  $K$  depends on  $u$  and  $g_1^*, g_2^* \dots, g_n^*$ .

If we set  $R_n(x) = d/(d(x)) \{x \int_x^\infty dv (v^{n-2} (g_1^{**} - g_1^*) \dots (g_n^{**} - g_n^*))\}$ , then by combining Theorem A with Theorem B we see that there exists a constant  $K$  ( $1/2^{n-1} \leq K \leq 1$ ) such that

$$(1.8)^5 \quad \int_{-u/2}^{u/2} h_n(x) d(x) = K \int_0^u d(x) R_n(x) .$$

In general, the right side of (1.8) (our estimate) is easier than the left side to determine. For example, take  $g_i^*(x) = 1/x^{\lambda_i}$ , with  $0 < \lambda_i < 1$ . Thus, we see that  $R_n(x)$  plays an important role in the solution of (1.7). In the next lemma, we state some important properties of this function.

**LEMMA 1.9.**<sup>6</sup> *If  $t^{n-2} (g_1^{**} - g_1^*) \dots (g_n^{**} - g_n^*) \in L(a, \infty)$  for each  $a > 0$ , then the function  $R_n(x)$  has the following properties:*

<sup>4</sup> The case  $n = 2$  appears in the cited reference; however, the general case is easily derivable from it.

<sup>5</sup> One of the properties of  $R_n(x)$  is that

$$\int_0^u R_n(x) d(x) = u \int_u^\infty (x^{n-2} (g_1^{**} - g_1^*) \dots (g_n^{**} - g_n^*) d(x) .$$

<sup>6</sup> This lemma can easily be verified.

- (a)  $R_n(x) \geq 0, x \geq 0,$
- (b)  $R_n(x)$  decreases,  $x \geq 0.$
- (c)  $\int_0^x R_n(t)dt = x \int_x^\infty dt(t^{n-2}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)).$
- (d)  $\int_0^x R_n(t)dt - xR_n(x) \doteq x^n(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*).$

II. Two-sided estimates. As has been stated earlier, our main problem here is to show that there is a constant  $K > 0$  such that

$$\begin{aligned} & \sup_{f_i \sim g_i^*} \int_{-\infty}^{+\infty} \psi((f_1 * \cdots * ) (x))d(x) \\ &= K \cdot \int_0^\infty \psi(x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*))d(x) \end{aligned}$$

where  $g_1^*, \dots, g_n^*$  are preassigned decreasing functions and the supremum is taken over all  $f_i$ 's such that  $f_i \sim g_i^*$ . We obtain this result for  $\psi$ 's that are a proper subclass of the convex functions (see Definition 2.7). Let us begin by developing properties of this class of  $\psi$ 's.

DEFINITION 2.1. We say that a function  $\psi \in \mathbf{V}(\Lambda)$  on  $[a, b]$ , if

- 1.  $\psi(a) = 0$
- 2.  $\psi'(t)$  exists for  $t \in [a, b]$
- 3.  $0 \leq \psi'(t)$  and increases (decreases) for  $t \in [a, b]$ .

LEMMA 2.2. If

- 1.  $\psi \in \mathbf{V}$  on  $[0, \infty)$
- 2.  $0 \leq f, g \in L_1[0, u]$  for each  $u > 0$
- 3.  $\int_0^u f(t)dt \leq \int_0^u g(t)dt$  for  $u > 0$
- 4.  $f(t)$  decreases for  $t \geq 0$ , then  $\int_0^u \psi(f(t))dt \leq \int_0^u \psi(g(t))dt$  for  $u \geq 0$ .

Proof. First let us assume that  $f$  and  $g$  are continuous on  $[0, \infty)$ . Since  $\psi \in \mathbf{V}$  we get by the Mean-Value-Theorem

$$\psi(g(t)) - \psi(f(t)) \geq \psi'(f(t))(g(t) - f(t)) .$$

Therefore,

$$\int_0^u \psi(g(t)) - \psi(f(t))dt \geq \int_0^u \psi'(f(t))(g(t) - f(t))dt$$

and by the second Mean-Value-Theorem for integrals there is a  $0 \leq \xi \leq u$  such that,

$$= \psi'(f(0)) \int_0^\epsilon [g(t) - f(t)]dt \geq 0 .$$

Now we apply a limiting argument to get the general case.

LEMMA 2.3. *If  $\psi \in \mathbf{V}$  on  $[0, \infty)$ , then*

$$\sup_{f_i \sim g_i^*} \int_{-\infty}^{+\infty} \psi(f_1 * \dots * f_n)(x)d(x) = \int_{-\infty}^{+\infty} \psi(h_n(x)d(x)) .$$

*Proof.* In Theorem A set  $r^*(t) = \chi_{[0, u]}(t)$ , then we find that

$$\int_{-\infty}^{+\infty} \chi_E(x)(f_1 * \dots * f_n)(x)d(x) \leq \int_{-u/2}^{u/2} h_n(x)d(x)$$

where  $|E| \leq u$ .

Therefore,

$$\int_0^u (f_1 * \dots * f_n)^*(x)d(x) \leq \int_0^u h_n^*(x)d(x) \text{ for } u \geq 0 .$$

By Lemma 2.2 this implies

$$\int_0^u \psi((f_1 * \dots * f_n)^*(x)d(x) \leq \int_0^u \psi(h_n^*(x)d(x) \text{ for } u \geq 0 .$$

Hence,

$$\int_0^\infty \psi[(f_1 * \dots * f_n)^*(x)]d(x) \leq \int_{-\infty}^{+\infty} \psi(h_n(x)d(x)) .$$

Thus by (1.4),

$$\int_{-\infty}^{+\infty} \psi[(f_1 * \dots * f_n)(x)]d(x) \leq \int_{-\infty}^{+\infty} \psi(h_n(x)d(x)) .$$

To say that  $g \in L_p(0, \infty)$ , means

$$\|g\|_p = \left\{ \int_0^\infty |g(x)|^p d(x) \right\}^{1/p} < \infty .$$

We shall henceforth use the symbol  $\|\cdot\|_p$  to mean the  $p$ th norm over  $L_p(0, \infty)$ .

LEMMA 2.4. *If  $g \in L_p(0, \infty)$  for  $p > 1$ , then*

$$\frac{1}{p+1} \|g\|_p \leq \left\| \frac{1}{x} \int_0^x g - g(x) \right\|_p \leq (p'+1) \|g\|_p$$

where  $1/p + 1/p' = 1$ . In particular, for the case  $p = 2$  we get that

$$\|g\|_2 = \left\| \frac{1}{x} \int_0^x g - g(x) \right\|_2 .$$

*Proof.* Let  $g_n(x) = \begin{cases} g(x) & g(x) \leq n, |x| \leq n \\ 0 & \text{elsewhere} \end{cases}$

and suppose that  $f \in L_1 \cap L_\infty$ . We see that

$$\int_0^u f(t)g_n(t)dt = \frac{1}{u} \int_0^u f \int_0^u g_n + \int_0^u \frac{d(x)}{x^2} \left( \int_0^x f - xf \right) \left( \int_0^x g_n - xg_n \right)$$

by simply differentiating both sides for  $u \in [\eta, m]$  and then letting  $\eta \rightarrow 0$ . Therefore,

$$(2.5) \quad \int_0^\infty f(t)g_n(t)dt = \int_0^\infty \frac{d(x)}{x^2} \left( \int_0^x f - xf \right) \left( \int_0^x g_n - xg_n \right).$$

Since  $1/p + 1/p' = 1$  we have,

$$\begin{aligned} \|g_n\|_p &= \sup_{\substack{\|f\|_{p'} \leq 1 \\ f \in L_1 \cap L_\infty}} \int_0^\infty f(t)g_n(t)dt \leq \sup_{\|f\|_{p'} \leq 1} \left\| \frac{1}{x} \int_0^x f - f(x) \right\|_{p'} \cdot \left\| \frac{1}{x} \int_0^x g_n - g_n \right\|_p \\ &\leq (p+1) \left\| \frac{1}{x} \int_0^x g_n - g_n \right\|_p. \end{aligned}$$

We also have that

$$\begin{aligned} \left\| \frac{1}{x} \int_0^x g_n - g_n(x) \right\|_p &\leq \left\| \frac{1}{x} \int_0^x g_n \right\|_p + \|g_n\|_p \\ &\leq (p'+1) \|g_n\|_p. \end{aligned}$$

Therefore,

$$\frac{1}{(p+1)} \|g_n\|_p \leq \left\| \frac{1}{x} \int_0^x g_n - g_n(x) \right\|_p \leq (p'+1) \|g_n\|_p \quad \text{for each } n.$$

Since  $g \in L_p(0, \infty)$ , we finally see that

$$\frac{1}{(p+1)} \|g\|_p \leq \left\| \frac{1}{x} \int_0^x g - g(x) \right\|_p \leq (p'+1) \|g\|_p \quad (p > 1).$$

The case  $p = 2$  follows immediately from (2.5).

**DEFINITION 2.6.** We say that a function  $\psi \in \mathbf{V}(p, q)$  on  $[a, b]$  if there exist  $1 < p$  and  $1 < q$  such that  $\psi^{1/p} \in \mathbf{V}$  on  $[a, b]$  and  $\psi^{1/q} \in \mathbf{\Lambda}$  on  $[a, b]$ .

For example, the functions  $\psi(x) = x^r$  with  $r > 1$  belong to  $\mathbf{V}(p, q)$  on  $[0, \infty]$  where  $p = r - \varepsilon$  and  $q = r + \varepsilon$  for some suitably chosen  $\varepsilon > 0$ .

**LEMMA 2.7.** If  $\psi \in \mathbf{V}(p, q)$  on  $[0, \infty)$  and  $\int_0^\infty \psi(f^*(t))dt < \infty$ , then

$$\begin{aligned} \frac{1}{(q+1)^q} \int_0^\infty \psi(f^*(x))d(x) &\leq \int_0^\infty \psi\left(\frac{1}{x} \int_0^x f^* - f^*(x)\right)d(x) \\ &\leq (p'+1)^p \int_0^\infty \psi(f^*(x))d(x), \end{aligned}$$

where  $1/p + 1/p' = 1$ .

*Proof.* First, since  $\psi \in \mathbf{V}(p, q)$  on  $[0, \infty)$ , this implies there exist  $p, q > 1$  such that  $\psi^{1/p} \in \mathbf{V}$  and  $\psi^{1/q} \in \mathbf{\Lambda}$ . Thus,

$$\psi^{1/p}\left(\frac{1}{x} \int_0^x f^* - f^*(x)\right) \leq \psi^{1/p}\left(\frac{1}{x} \int_0^x f^*\right) - \psi^{1/p}(f^*(x)).$$

Now by applying Jensen's inequality we see

$$(2.8) \quad \psi^{1/p}\left(\frac{1}{x} \int_0^x f^* - f^*(x)\right) \leq \frac{1}{x} \int_0^x \psi^{1/p}(f^*) - \psi^{1/p}(f^*(x)).$$

Now by (2.8) we get that

$$\left\| \psi^{1/p}\left(\frac{1}{x} \int_0^x f^* - f^*(x)\right) \right\|_p \leq \left\| \frac{1}{x} \int_0^x \psi^{1/p}(f^*) - \psi^{1/p}(f^*(x)) \right\|_p$$

and by applying Lemma 2.4 we find

$$\leq (p'+1) \|\psi^{1/p}(f^*)\|_p.$$

Since  $\psi^{1/q} \in \mathbf{\Lambda}$  we have that

$$\begin{aligned} \psi^{1/q}\left(\frac{1}{x} \int_0^x f^* - f^*(x)\right) &\geq \psi^{1/q}\left(\frac{1}{x} \int_0^x f^*\right) - \psi^{1/q}(f^*(x)) \\ &\geq \frac{1}{x} \int_0^x \psi^{1/q}(f^*) - \psi^{1/q}(f^*(x)). \end{aligned}$$

Therefore by Lemma 2.4 we have that

$$\begin{aligned} \left\| \psi^{1/q}\left(\frac{1}{x} \int_0^x f^* - f^*(x)\right) \right\|_q &\geq \left\| \frac{1}{x} \int_0^x \psi^{1/q}(f^*) - \psi^{1/q}(f^*(x)) \right\|_q \\ &\geq \left(\frac{1}{q+1}\right) \|\psi^{1/q}(f^*)\|_q. \end{aligned}$$

**THEOREM 2.9.** If  $\psi \in \mathbf{V}(p, q)$  on  $[0, \infty)$  and  $\int_0^\infty \psi(R_n(x))d(x) < \infty$ , then

$$\begin{aligned} &\frac{1}{(p'+1)} \cdot \frac{1}{(2^{n-1})q} \int_0^\infty \psi[x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)]d(x) \\ &\leq \sup_{f_i \sim g_i^*} \int_{-\infty}^{+\infty} \psi((f_1^* \cdots f_n^*))(x)d(x) \\ &\leq (q+1)^q \int_0^\infty \psi[x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)]d(x) \end{aligned}$$

where  $1/p' + 1/p = 1$ .

*Proof.* By (1.8), where  $1/2^{n-1} \leq K \leq 1$ , we see that

$$\int_0^u h_n^*(x) d(x) \leq \int_0^u R_n(x) d(x) \leq 2^{n-1} \int_0^u h_n^*(x) d(x) \quad \text{for } u \geq 0.$$

Since both  $R_n$  and  $h_n^*$  are decreasing, this implies by Lemma 2.2

$$(2.10) \quad \int_0^\infty \psi(h_n^*(x)) d(x) \leq \int_0^\infty \psi(R_n(x)) d(x) \\ \leq \int_0^\infty \psi(2^{n-1} h_n^*(x)) d(x).$$

Now applying Lemma 2.7 to (2.10) we see that there exist a  $p > 1$  and  $q > 1$  such that

$$(2.11) \quad \frac{1}{(q+1)^q} \int_0^\infty \psi(h_n^*(x)) d(x) \leq \int_0^\infty \psi\left(\frac{1}{x} \int_0^x R_n - R_n(x)\right)$$

and

$$(2.12) \quad \int_0^\infty \psi\left(\frac{1}{x} \int_0^x R_n - R_n(x)\right) \leq (p'+1)^p \int_0^\infty \psi(2^{n-1} h_n^*(x)) d(x).$$

Since  $\psi^{1/q} \in \mathbf{A}$ , this implies

$$\psi^{1/q}(t) = \int_0^t \varphi(v) dv$$

where  $0 < \varphi$  and is decreasing on  $[0, \infty)$ .

Therefore for  $a \geq 1$ ,

$$(2.13) \quad \psi(at) \leq a^q \psi(t).$$

Now applying (2.13) to the right side of (2.12) we conclude

$$(2.14) \quad \int_0^\infty \psi\left(\frac{1}{x} \int_0^x R_n - R_n(x)\right) \leq (p'+1)^p (2^{n-1})^q \int_0^\infty \psi(h_n^*(x)) d(x).$$

We then apply Lemma 1.9(d) and Lemma 2.3 to (2.11) and (2.14) to obtain our result.

**COROLLARY 2.15.** *If*

$$\int_0^\infty (R_n(x))^r d(x) < \infty,$$

*then*

$$\begin{aligned} & \frac{1}{(r' + 1)} \frac{1}{(2^{n-1})^r} \int_0^\infty [x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)]^r d(x) \\ & \leq \sup_{f_i \sim g_i^*} \int_{-\infty}^{+\infty} [(f_1 * f_2 * \cdots * f_n)(x)]^r d(x) \\ & \leq (r + 1)^r \int_0^\infty [x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)]^r d(x) \end{aligned}$$

where  $1/r + 1/r' = 1$  and  $r \geq 1$ .

*Proof.* Apply Theorem 2.9 with  $\psi(u) = u^r$  and  $q = r + \varepsilon$ ,  $p = r - \varepsilon$  with  $\varepsilon > 0$ . Then, let  $\varepsilon$  go to zero.

#### REFERENCES

1. Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge at the University Press, London, 1934.
2. Hardy and Littlewood, *Notes on the theory of series (VIII). An inequality*, J. London Math. Soc., **3** (1928), 105-110.
3. R. O'Neil, *Convolution operators and  $L(p, q)$  spaces*, Duke Math. J. **30** (1963).
4. G. Sampson, *Sharp estimates of convolution transforms in terms of decreasing functions*, Pacific J. Math., **38** (1971), 213-231.

Received July 26, 1971. I would like to thank W. B. Jurkat for bringing this problem to my attention.

CALIFORNIA INSTITUTE OF TECHNOLOGY  
AND  
STATE UNIVERSITY OF NEW YORK AT BUFFALO

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON  
Stanford University  
Stanford, California 94305

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

C. R. HOBBY  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

# Pacific Journal of Mathematics

Vol. 45, No. 1

September, 1973

William George Bade, <i>Complementation problems for the Baire classes</i> . . . . .	1
Ian Douglas Brown, <i>Representation of finitely generated nilpotent groups</i> . . . . .	13
Hans-Heinrich Brungs, <i>Left Euclidean rings</i> . . . . .	27
Victor P. Camillo and John Cozzens, <i>A theorem on Noetherian hereditary rings</i> . . . . .	35
James Cecil Cantrell, <i>Codimension one embeddings of manifolds with locally flat triangulations</i> . . . . .	43
L. Carlitz, <i>Enumeration of up-down permutations by number of rises</i> . . . . .	49
Thomas Ashland Chapman, <i>Surgery and handle straightening in Hilbert cube manifolds</i> . . . . .	59
Roger Cook, <i>On the fractional parts of a set of points. II</i> . . . . .	81
Samuel Harry Cox, Jr., <i>Commutative endomorphism rings</i> . . . . .	87
Michael A. Engber, <i>A criterion for divisoriality</i> . . . . .	93
Carl Clifton Faith, <i>When are proper cyclics injective</i> . . . . .	97
David Finkel, <i>Local control and factorization of the focal subgroup</i> . . . . .	113
Theodore William Gamelin and John Brady Garnett, <i>Bounded approximation by rational functions</i> . . . . .	129
Kazimierz Goebel, <i>On the minimal displacement of points under Lipschitzian mappings</i> . . . . .	151
Frederick Paul Greenleaf and Martin Allen Moskowitz, <i>Cyclic vectors for representations associated with positive definite measures: nonseparable groups</i> . . . . .	165
Thomas Guy Hallam and Nelson Onuchic, <i>Asymptotic relations between perturbed linear systems of ordinary differential equations</i> . . . . .	187
David Kent Harrison and Hoyt D. Warner, <i>Infinite primes of fields and completions</i> . . . . .	201
James Michael Hornell, <i>Divisorial complete intersections</i> . . . . .	217
Jan W. Jaworowski, <i>Equivariant extensions of maps</i> . . . . .	229
John Jobe, <i>Dendrites, dimension, and the inverse arc function</i> . . . . .	245
Gerald William Johnson and David Lee Skoug, <i>Feynman integrals of non-factorable finite-dimensional functionals</i> . . . . .	257
Dong S. Kim, <i>A boundary for the algebras of bounded holomorphic functions</i> . . . . .	269
Abel Klein, <i>Renormalized products of the generalized free field and its derivatives</i> . . . . .	275
Joseph Michael Lambert, <i>Simultaneous approximation and interpolation in <math>L_1</math> and <math>C(T)</math></i> . . . . .	293
Kelly Denis McKennon, <i>Multipliers of type <math>(p, p)</math> and multipliers of the group <math>L_p</math>-algebras</i> . . . . .	297
William Charles Nemitz and Thomas Paul Whaley, <i>Varieties of implicative semi-lattices. II</i> . . . . .	303
Donald Steven Passman, <i>Some isolated subsets of infinite solvable groups</i> . . . . .	313
Norma Mary Piacun and Li Pi Su, <i>Wallman compactifications on <math>E</math>-completely regular spaces</i> . . . . .	321
Jack Ray Porter and Charles I. Votaw, <i><math>S(\alpha)</math> spaces and regular Hausdorff extensions</i> . . . . .	327
Gary Sampson, <i>Two-sided <math>L_p</math> estimates of convolution transforms</i> . . . . .	347
Ralph Edwin Showalter, <i>Equations with operators forming a right angle</i> . . . . .	357
Raymond Earl Smithson, <i>Fixed points in partially ordered sets</i> . . . . .	363
Victor Snaith and John James Ucci, <i>Three remarks on symmetric products and symmetric maps</i> . . . . .	369
Thomas Rolf Turner, <i>Double commutants of weighted shifts</i> . . . . .	379
George Kenneth Williams, <i>Mappings and decompositions</i> . . . . .	387