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## **THREE REMARKS ON SYMMETRIC PRODUCTS AND SYMMETRIC MAPS**

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## THREE REMARKS ON SYMMETRIC PRODUCTS AND SYMMETRIC MAPS

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The first remark establishes that the homotopy type of a certain space related to the  $m$ -fold symmetric product  $SP^m S^n$  of the  $n$ -sphere is that of an  $n^{\text{th}}$  suspension space. Remark two generalizes a well-known adjunction formula for  $SP^2 S^n$  due to Steenrod to a filtration of length  $m$  of  $SP^m S^n$ . The final remark provides a group-theoretic construction of  $G$ -maps  $f: (S^n)^m \rightarrow S^n$  where  $G \subset S(m)$  acts on  $(S^n)^m$  by permutation of its factors.

1. Joins. The join  $X * Y$  of  $X$  and  $Y$  is the quotient space  $X \times Y \times I / \sim$  where  $(x, y, 0) \sim (x, y', 0)$  and  $(x, y, 1) \sim (x', y, 1)$  for all  $x, x' \in X$  and all  $y, y' \in Y$ . Let  $(D, S)$  denote the unit disc and sphere in euclidean  $n$ -space  $R^n$  with its usual inner product

$$\langle x, y \rangle = \sum x_i y_i .$$

For any decomposition  $R^n = W_1 \oplus W_2$  of  $R^n$  into the direct sum of a  $k$ -dimensional subspace  $W_1$  and its orthogonal complement  $W_2 = W_1^\perp$ , let  $D_i = D \cap W_i$  and  $S_i = S \cap W_i$ ,  $i = 1, 2$ , be the associated discs and spheres. As well known the map  $f: D_1 * S_2 \rightarrow D$  given by

$$f [sx, y, t] = s\sqrt{1-t}x + \sqrt{t}y$$

defines a homeomorphism of pairs

$$(1) \quad (D_1 * S_2, S_1 * S_2) \cong (D, S) .$$

Give  $V_i = R^n$ ,  $i = 1, \dots, n$ , its usual inner product  $\langle , \rangle_i$ . Then the formula  $\langle x, y \rangle = \sum \langle x_i, y_i \rangle_i$  for  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  in  $V = V_1 \times \dots \times V_m \cong R^{nm}$  coincides with the usual inner product on  $R^{nm}$ . So we may apply the preceding remarks to the diagonal subspace  $W_1 = \{v \in V \mid v_1 = v_2 = \dots = v_m\}$  and its orthogonal complement  $W_2 = \{v \in V \mid \sum v_i = 0\} = W_1^\perp$ . The full symmetric group  $S(m)$  acts on  $V$  by permutation of its factors  $V_i$ . For any subgroup  $H$  of  $S(m)$   $D_i$  and  $S_i$  are  $H$ -spaces and  $f$  an  $H$ -map inducing another homeomorphism of pairs

$$(2) \quad (D_1/H * S_2/H, S_1/H * S_2/H) \cong (D/H, S/H) .$$

As  $H$  acts trivially on  $W_1$ ,  $D_1/H = \bar{D}_1$  and  $S_1/H = \bar{S}_1$  are again the disc and sphere. Moreover, for subgroups  $H_1 \subset H_2$  of  $S(m)$  it is easily checked that the quotient map  $p: D/H_1 \rightarrow D/H_2$  corresponds via

(2) to the join map

$$id * p_2: \bar{D}_1 * (S_2/H_1) \longrightarrow \bar{D}_1 * (S_2/H_2) .$$

Recall from [7] the definition of the spaces  $X_{m,l}^n$  which appear in the geometry of the symmetric product  $SP^m S^n$ . Let  $h_\tau: (D^n)^m \rightarrow (D^n)^m$  be the permutation homeomorphism defined by  $\tau \in S(m)$ , and set

$$A_{m,l}^n = (D^n)^{m-l} \times (S^{n-1})^l \quad \text{for } 0 \leq l \leq m .$$

Then

$$\tilde{X}_{m,l}^n = \bigcup_{\tau \in S(m)} h_\tau(A_{m,l}^n)$$

is an  $S(m)$ -subspace of  $(D^n)^m$  and so  $X_{m,l}^n = \tilde{X}_{m,l}^n/S(m)$  is well defined. To identify the pairs  $(D/S(m), S/S(m))$  and  $(X_{m,0}^n, X_{m,1}^n)$  we make the following change of norms: let  $V' = V$  as sets but set

$$\|v\|' = \max_i \|v_i\|_i$$

where  $\|v_i\|_i = \langle v_i \cdot v_i \rangle_i^{1/2}$ . Then  $x \rightarrow (\|x\|/\|x\|') \cdot x$  defines a norm preserving (non-linear)  $S(m)$ -homeomorphism  $V \rightarrow V'$  establishing the desired result  $(D/S(m), S/S(m)) \cong (X_{m,0}^n, X_{m,1}^n)$ . Thus

$$(3) \quad (\bar{D}_1 * (S_2/S(m)), \bar{S}_1 * (S_2/S(m))) \quad \text{and} \quad (X_{m,0}^n, X_{m,1}^n)$$

are homeomorphic pairs. Moreover the canonical map  $D^n \times X_{m-1,0}^n \rightarrow X_{m,0}^n$  is the quotient map  $(D^n)^m/S(m-1) \rightarrow (D^n)^m/S(m)$  induced by the inclusion homomorphism  $S(m-1) \rightarrow S(m)$  sending  $S(m-1)$  onto the subgroup  $\{e\} \times S(m-1)$  of  $S(m)$  which acts on  $(D^n)^m = D^n \times (D^n)^{m-1}$  by the identity on the first factor and by the usual symmetric action on the second factor. Combining the remark of the preceding paragraph with the above  $S(m)$ -homeomorphism  $V \rightarrow V'$  we see that the canonical map  $D^n \times X_{m-1,0}^n \rightarrow X_{m,0}^n$  corresponds to the join map

$$\bar{D}_1 * (S_2/S(m-1)) \longrightarrow \bar{D}_1 * (S_2/S(m)) .$$

Our first remark establishes a conjecture stated in [7].

**PROPOSITION 1.1.**  $X_{m,m-1}^n/X_{m-1,m-2}^n$  has the homotopy type of a space of the form  $S^{n-1} * K$  for  $K$  a suitable finite CW complex. Hence  $X_{m,m-1}^n/X_{m-1,m-2}^n$  has the homotopy type of an  $n^{\text{th}}$  suspension.

*Proof.* Proposition 2.6 of [7] asserts the existence of a homotopy equivalence

$$X_{m,m-1}^n/X_{m-1,m-2}^n \sim EX_{m,1}^{n-1} \bigcup_{E \mathcal{V}} C(E(X_{1,1}^{n-1} * X_{m-1,1}^{n-1}))$$

where  $\psi$  is given by the canonical maps

$$X_{1,1}^{n-1} \times X_{m-1,0}^{n-1} \longrightarrow X_{m,1}^{n-1}, \quad X_{1,0}^{n-1} \times X_{m-1,1}^{n-1} \longrightarrow X_{m,1}^{n-1}.$$

$\psi$  is just the restriction of the canonical map

$$\psi': X_{1,0}^{n-1} \times X_{m-1,0}^{n-1} \longrightarrow X_{m,0}^{n-1}$$

which, as we have already noted, can be identified with the join map  $id * \bar{\psi}': \bar{D}_1 * (S_2/S(m-1)) \longrightarrow \bar{D}_1 * (S_2/S(m))$ . Under this identification the subspaces  $X_{1,1}^{n-1} \times X_{m-1,0}^{n-1} \cup X_{1,0}^{n-1} \times X_{m-1,1}^{n-1}$  and  $X_{m,1}^{n-1}$  correspond to  $\bar{S}_1 * (S_2/S(m-1))$  and  $\bar{S}_1 * (S_2/S(m))$ , and so the map  $\psi$  corresponds to the join map  $id * \bar{\psi}: \bar{S}_1 * (S_2/S(m-1)) \rightarrow \bar{S}_1 * (S_2/S(m))$  which is the restriction of  $id * \bar{\psi}'$ . Hence there is a homotopy equivalence

$$\begin{aligned} X_{m,m-1}^n / X_{m-1,m-2}^n &\sim E((E^{n-1}(S_2/S(m))) \bigcup_{E^{n-1}\bar{\psi}} C(E^{n-1}X_{m-1,1}^{n-1})) \\ &\sim E^n((S_2/S(m)) \bigcup_{\bar{\psi}} CX_{m-1,1}^{n-1}) \end{aligned}$$

and the result is proved.

For  $p$ -fold cyclic products ( $p$  any prime) there is an analogous result to 1.1 whose proof differs only slightly from the preceding. For this let now  $h: (D^n)^p \rightarrow (D^n)^p$  be the (cyclic) permutation homeomorphism defined by  $\tau \in Z_p \subset S(p)$  and set  $A_{p,l}^n = (D^n)^{p-l} \times (S^{n-1})^l$ . Then as before

$$\widetilde{X}_{p,l}^n = \bigcup_{\tau \in Z_p} h_\tau(A_{p,l}^n)$$

is a  $Z_p$ -space and  $X_{p,l}^n = \widetilde{X}_{p,l}^n / Z_p$  is well defined. Let  $W_i^{n-1} \subset (S^{n-1})^l$  be the subspace  $\{x \in (S^{n-1})^l \mid x_i = \text{basepoint for some } i\}$ ,  $\widetilde{Z}_{p,l}^n = (D^n)^{p-l} \times W_i^{n-1}$  and  $Z_{p,l}^n$  the image of  $\widetilde{Z}_{p,l}^n$  under the canonical projection

$$(D^n)^{p-l} \times (S^{n-1})^l \longrightarrow X_{p,l}^n.$$

Then  $Z_{p,p-1}^n \subset X_{p,p-1}^n$  and formula (3.3) of [8] asserts the existence of a homeomorphism

$$(4) \quad X_{p,p-1}^n / Z_{p,p-1}^n \cong EX_{p,1}^{n-1} \cup e^{np-p+1}.$$

The top cell  $e^{np-p+1}$  arises from the product  $D^n \times (D^{n-1})^{p-1}$  and the attaching map of (4)  $S^{n-1} \times (D^{n-1})^{p-1} \cup D^n \times \partial [(D^{n-1})^{p-1}] \rightarrow EX_{p,1}^{n-1}$  sends the contractible subspace

$$A = S^{n-1} \times \partial [(D^{n-1})^{p-1}] \cup \text{point} \times (D^{n-1})^{p-1}$$

to the basepoint of  $EX_{p,1}^{n-1}$  and so factors as  $(E\gamma) \circ p$ , where  $p$  is the collapsing homotopy equivalence

$$S^{n-1} * \partial [(D^{n-1})^{p-1}] \longrightarrow S^{n-1} * \partial [(D^{n-1})^{p-1}]/A$$

and  $\psi$  the canonical projections

$$S^{n-2} \times (D^{n-1})^{p-1} \longrightarrow X_{p,1}^{n-1}, \quad D^{n-1} \times \partial [(D^{n-1})^{p-1}] \longrightarrow X_{p,1}^{n-1}$$

(see the proof of Prop. 2.6 of [7] with  $Z_p$  replacing  $S(m)$ ). By the above  $X_{p,1}^{n-1}$  is homeomorphic to the join  $S_1 * (S_2/Z_p) - V$  is now  $R^{np} -$  and the map  $\psi$  can be identified via this homeomorphism with the join map  $id * p: S_1 * S_2 \rightarrow S_1 * (S_2/Z_p)$ . Thus we obtain the analogous result to 1.1.

**PROPOSITION 1.2.** *For the  $p$ -fold cyclic product of spheres the space  $X_{p,p-1}^n/Z_{p,p-1}^n$  has the homotopy type of a space of the form  $S^{n-1} * K$  for  $K$  a suitable finite CW complex. Hence  $X_{p,p-1}^n/Z_{p,p-1}^n$  has the homotopy type of an  $n^{\text{th}}$  suspension.*

Application of 1.2 was made in [8].

Consider again the symmetric product situation. Lemma 2.5 (iii) of [7] provides a homeomorphism

$$X_{m,l}^n/X_{m-1,l-1}^n \cong (X_{m,l+1}^n/X_{m-1,l}^n) \cup C(X_{m-l,1}^n * X_{l,1}^{n-1}).$$

For  $l = m$  and  $l = m - 1$  the spaces  $X_{m,l}^n/X_{m-1,l-1}^n$  have now been shown to have the homotopy type of a space of the form  $S^{n-1} * K$ . It seems reasonable to expect the same to be true for the remaining values of  $l$ ,  $2 \leq l \leq m - 2$ .

**2. Geometry of  $SP^m EX$ .** Our second remark extends the Steenrod adjunction formula [3]

$$SP^2 S^n \cong E(SP^2 S^{n-1}) \cup e^{2n}$$

to higher symmetric products  $SP^m EX$  of suspension spaces. Let

$$\begin{aligned} I^n &= \{x \in R^n \mid 0 \leq x_i \leq 1\} \\ A_n &= \{x \in I^n \mid x_n = 1\} \\ T_n &= \{x \in I^n \mid x_1 \geq x_2 \geq \dots \geq x_n\} \\ p &= \{(1, 1, \dots, 1)\} \subset I^n. \end{aligned}$$

For  $i = 1, 2, \dots, n - 1$  define  $f_i: I^n \rightarrow I^n$  by  $f_i(t_1, \dots, t_n) = (t'_1, \dots, t'_n)$  where  $t'_j = t_j$  if  $j \neq i$  and  $t'_i = t_{i+1} + t_i(1 - t_{i+1})$ . One shows easily that the composite  $g_n = f_1 \circ f_2 \circ \dots \circ f_{n-1}$  defines a relative homeomorphism  $(I^n, A_n) \cong (T_n, p)$ . The map  $g_n$  is useful in studying the quotients  $A_{i+1}/A_i$  arising from a filtration

$$(5) \quad SP^m EX = A_m \supset A_{m-1} \supset \cdots \supset A_1 = E(SP^m X)$$

which we define as follows. For  $x' = [x, t] \in EX$  call  $t$  the *height* of  $x'$ . As each element  $[x'_1, \dots, x'_m] \in SP^m EX$  has a representative with heights  $t_1 \geq t_2 \geq \dots \geq t_m$  we can set  $A_i$  to be the subset of  $SP^m EX$  of all elements having representatives with at most  $i$  distinct heights. The  $A_i$  define a filtration (5) of  $SP^m EX$ .

For any partition  $\pi = [i_1: i_2: \dots: i_q]$  of  $m$  let  $A_{q\pi} \subset A_q$  be the set of all points having representatives with heights  $t_1 \geq t_2 \geq \dots \geq t_q, i_1$  of the  $m$  coordinates at height  $t_1, i_2$  of them at height  $t_2$ , etc. Set  $Y_1 = C(SP^{i_1} X)$ ,  $Y_j = (SP^{i_j} X) \times I$  for  $2 \leq j < q$ ,  $Y_q = \tilde{C}(SP^{i_q} X)$  and  $Y = Y_1 \times \dots \times Y_q$  where

$$C(Z) = Z \times I/Z \times \{1\} \quad \text{and} \quad \tilde{C}(Z) = Z \times I/Z \times \{0\}.$$

Set

$$\partial C(Z) = \{[z, t] \in C(Z) \mid t = 0\}, \quad \partial \tilde{C}(Z) = \{[z, t] \in \tilde{C}(Z) \mid t = 1\}$$

and

$$\partial(SP^k X \times I) = SP^k X \times \{0, 1\}.$$

This defines  $\partial Y_i$  for  $i = 1, \dots, q$ . Finally set

$$\partial Y = \bigcup_{i=1}^q Y_1 \times \dots \times \partial Y_i \times \dots \times Y_q.$$

Clearly  $\partial$  is just a kind of boundary operator for cones and related spaces.

**PROPOSITION 2.1.** *The map  $Y \rightarrow SP^m EX$  given by*

$$\begin{aligned} &([x_1, t_1], (x_2, t_2), \dots, (x_{q-1}, t_{q-1}), [x_q, t_q]) \\ &\longrightarrow [[x_1, t'_1], [x_2, t'_2], \dots, [x_q, t'_q]] \end{aligned}$$

where  $t'_j$  is the  $j^{\text{th}}$  coordinate of  $g_q(t_1, \dots, t_q)$ , induces a relative homeomorphism  $(Y, \partial Y) \cong (A_{q\pi}, A_{q-1})$  for each  $2 \leq q < m$  and each partition  $\pi = [i_1: \dots: i_q]$  of  $m$ .

The proof is straightforward.

To obtain an expression for  $A_q/A_{q-1}$  first observe that

$$A_q/A_{q-1} = \bigvee_{\pi} (A_{q\pi}/A_{q-1}),$$

the wedge taken over all partitions of  $m$ . Thus by 2.1 it suffices to consider for each  $\pi$  the corresponding quotient  $Y/\partial Y$ .

**PROPOSITION 2.2.** For  $\pi = [i_1: \dots: i_q]$  and  $Y = Y_1 \times \dots \times Y_q$  as above the space  $Y/\partial Y$  has the same homotopy type as the space

$$E^q(SP^{i_1}X \wedge SP^{i_q}X) \vee E^q\left(SP^{i_1}X \wedge \prod_{j=2}^{q-1} SP^{i_j}X \wedge SP^{i_q}X\right).$$

*Proof.* Let  $B$  be a space with  $\partial B = \phi$ . Then the obvious quotient map  $Q = CA \times B \times I^{q-2} \times CA' \rightarrow P = EA \times B \times S^{q-2} \times EA'$  induces a relative homeomorphism  $(Q, \partial Q) = (P, P')$  where

$$P' = EA \times B \times S^{q-2} \vee EA \times B \times EA' \vee B \times S^{q-2} \times EA'.$$

So  $P/P' = B \times (EA \wedge S^{q-2} \wedge EA')/B \times \text{point}$  and the latter quotient has the homotopy type of the wedge  $E^q(A \wedge A') \vee E^q(A \wedge B \wedge A')$  [5]. Therefore 2.2 is obtained by setting  $A = SP^{i_1}X$ ,  $B = \prod_{j=2}^{q-1} SP^{i_j}X$  and  $A' = SP^{i_q}X$ .

As an illustration of the preceding analysis let us return to the Steenrod formula for the symmetric square of a sphere. The space  $Y$  is just  $CX \times \tilde{C}X$  and the map  $CX \times \tilde{C}X \rightarrow SP^2EX$  is

$$([x_1, t_1], [x_2, t_2]) \longmapsto [[x_1, t_2 + t_1(1 - t_2)], [x_2, t_2]].$$

The subspace  $X \times \tilde{C}X \cup CX \times X$  (given by  $t_1 = 0$  or  $t_2 = 1$ ) is mapped to  $ESP^2X$ . It is well known that there are homeomorphisms

$$X \times \tilde{C}X \cup CX \times X \cong X * X$$

and

$$CX \times \tilde{C}X \cong C(X \times \tilde{C}X \cup CX \times X).$$

Hence we obtain the adjunction formula  $SP^2EX \cong ESP^2X \cup C(X * X)$  extending the Steenrod result from spheres to suspensions.

**REMARK.** For  $X = S^{n-1}$  2.2 can be used to recompute Nakaoka's results [4] on the integral cohomology of  $SP^mS^n$  for low values of  $m$ .

**3. Group theoretic construction of symmetric maps.** Let  $H \subset G \subset S(m)$  be subgroups of the symmetric group  $S(m)$  and let  $S(G/H)$  be the symmetric group on the set of right cosets  $G/H$ . Define a homomorphism  $\alpha: G \rightarrow S(G/H)$  by  $\alpha(g)(Hg_1) = Hg_1g^{-1}$ . Kernel of  $\alpha$  is just the normal subgroup  $B = \bigcap_{g \in G} gHg^{-1}$  and so there is an injection  $G/B \rightarrow S(G/H)$ . Let  $A$  denote the image of  $\alpha$  and  $|G/H|$  the cardinality of  $G/H$ .

**PROPOSITION 3.1.** If  $v: X^{|G/H|} \rightarrow X$  and  $w: X^m \rightarrow X$  are  $A$  and  $H$ -maps respectively, then  $F: X^m \rightarrow X$  given by

$$F(x) = v(w(g_1 \cdot x), w(g_2 \cdot x), \dots, w(g_l \cdot x))$$

for  $g_1, \dots, g_l$  a complete set of coset representatives in  $G/H$ , is a  $G$ -map.

*Proof.* As  $w$  is an  $H$ -map we have for any  $g \in G$  and any

$$1 \leq i \leq l = |G/H|$$

the existence of an  $h \in H$  and a unique  $1 \leq j \leq l$  such that

$$w(g_i \cdot (g \cdot x)) = w(h \cdot (g_j \cdot x)) = w(g_j \cdot x),$$

where  $h$  arises from the coset equality  $Hg_i g = Hg_j$ . Hence there exists an element  $\sigma \in S(G/H)$  in  $A = \text{image}(\alpha)$  satisfying

$$\begin{aligned} F(g \cdot x) &= v(w(g_1 \cdot (g \cdot x)), \dots, w(g_l \cdot (g \cdot x))) \\ &= v(w(g_{\sigma(1)} \cdot x), \dots, w(g_{\sigma(l)} \cdot x)) \\ &= v(\sigma \cdot (w(g_1 \cdot x), \dots, w(g_l \cdot x))) \\ &= v(w(g_1 \cdot x), \dots, w(g_l \cdot x)) = F(x). \end{aligned}$$

The result follows.

To compute the James number of  $F$  when  $X = S^n$  note that the degree of the composite  $S^n \xrightarrow{\Delta} (S^n)^m \xrightarrow{F} S^n$  ( $\Delta$  the diagonal map) equals the product  $\text{deg}(v \circ \Delta) \cdot \text{deg}(w \circ \Delta)$ , since  $F \circ \Delta = v \circ \Delta \circ w \circ \Delta$  as maps. Therefore the James number of  $F$  is easily computed from those of  $v$  and  $w$  via the Künneth formula.

*Applications.* Let  $n = 2t + 1$  in the following four applications.

1. Let  $H = \{id, (123), (132)\} \cong Z_3$  so  $H \triangleleft S(3) = G$ . Choose  $v: (S^n)^{|G/H|} \rightarrow S^n$  to be an  $S(2)$ -map with  $J_v = 2^{\phi(2t)}$  [2] and  $w: (S^n)^3 \rightarrow S^n$  to be an  $H$ -map with  $J_w = 3^t$  [8]. Then  $J_F = 2^{\phi(2t)+1} \cdot 3^t$ . However obstruction theory can improve this result as follows. From [7] we know that there exists a map  $SP^m S^n \rightarrow S^n$  of James number  $N$  if and only if the composite  $X_{n,m-1}^n \xrightarrow{\phi} S^n \xrightarrow{f_N} S^n$ , is nullhomotopic where  $\text{deg } f_N = N$ . Here  $\phi$  arises from the geometry of  $SP^m S^n$  given in [7, § 2]. As  $X_{2,1}^n \subset X_{3,2}^n$  the obstructions to extending an  $S(2)$ -map  $g_1: (S^n)^2 \rightarrow S^n$  to an  $S(3)$ -map  $g: (S^n)^3 \rightarrow S^n$  lie in the groups  $H^i(X_{3,2}^n, X_{2,1}^n; \pi_i S^n)$ , which by Nakaoka [4] (see also [1], Lemma (4.3)) are 3-primary. Hence there exists an  $S(3)$ -map  $G: (S^n)^3 \rightarrow S^n$  with  $J_G = 2^{\phi(2t)} \cdot 3^r$  for some  $r$ . As the set of all possible James numbers of  $S(m)$ -maps forms an ideal [1], there must also exist an  $S(3)$ -map  $G': (S^n)^3 \rightarrow S^n$  with  $J_{G'} = 2^{\phi(2t)} \cdot 3^t$  and so we recover the main result



of [7].

2. Let  $H \subset S(4)$  be the subgroup generated by  $\{(12), (34), (13)(24)\}$ , so  $|H| = 8$ ,  $H \triangleleft G$  and

$$B = \bigcap_{g \in G} gHg^{-1} = \{id, (14)(23), (13)(24), (12)(34)\} \cong Z_2 \times Z_2.$$

Hence  $|B| = 4$  and  $A = S(3)$ . Apply 3.1 with  $v$  an  $S(3)$ -map with  $J_v = 2^{\phi(2t)} \cdot 3^t$  and  $w$  the  $H$ -map  $(S^n)^4 \xrightarrow{h \circ h^2} S^n$  where  $h: (S^n)^2 \rightarrow S^n$  is an  $S(2)$ -map with  $J_h = 2^{\phi(2t)}$ . Clearly  $J_w = 2^{2 \cdot \phi(2t)}$  and so we obtain an  $S(4)$ -map  $F: (S^n)^4 \rightarrow S^n$  with  $J_F = 2^{3\phi(2t)} \cdot 3^{t+1}$ . Now an exactly analogous argument to that of (1) shows that the obstructions to extending an  $S(3)$ -map  $(S^n)^3 \rightarrow S^n$  to an  $S(4)$ -map  $(S^n)^4 \rightarrow S^n$  lie in the groups  $H^i(X_{4,3}^n, X_{3,2}^n; \pi_i S^n)$ , which again by Nakaoka are 2-primary. Thus there is an  $S(4)$ -map of James number  $J = 2^r \cdot 2^{\phi(2t)} \cdot 3^t$  for some  $r$ . This as above implies the existence of an  $S(4)$ -map with James number  $2^{3\phi(2t)} \cdot 3^t$ . Note it is not difficult using  $K$ -theory to show that the James number of any  $S(4)$ -map  $(S^n)^4 \rightarrow S^n$  must be a multiple of  $2^{2t} \cdot 3^t$  (the first named author has improved this bound to  $2^{\phi(2t)} \cdot 2^t \cdot 3^t$  via ad hoc considerations).

3. For  $G = G^r$  the Sylow  $p$ -subgroup of  $S(p^r)$  given by the  $r$ -fold Wreath product of  $G^1 \cong Z_p$  with itself and  $H = \prod_{k=1}^r G^{r-1} \triangleleft G = G^r$  (see [8, § 2]) we have  $G/H \cong Z_p$ . Let  $w$  be the composite

$$(S^n)^{p^r} \xrightarrow{\pi_1} (S^n)^{p^{r-1}} \xrightarrow{w_1} S^n$$

where  $w_1$  is a  $G^{r-1}$ -map with James number  $J_{w_1}$  and  $\pi_1$  is projection onto the first  $p^{r-1}$  factors of  $(S^n)^{p^r}$ ; let  $v: (S^n)^p \rightarrow S^n$  be a  $Z_p$ -map with James number  $J_v$ . Then  $J_F = J_{w_1} \cdot J_v$  where  $F$  is given by 3.1. From a  $Z_p$ -map  $h$  with  $J_h = p^t$  [2], this result plus induction on  $r$  provides a  $G^r$ -map  $h'$  with  $J_{h'} = p^{rt}$ . This iteration of 3.1 applied to the  $G^1$ -map  $h$  gives precisely the composite  $G^r$ -map  $h \circ h^p \circ \dots \circ h^{p^{r-1}}: (S^n)^{p^r} \rightarrow S^n$ .

4. For  $G = Z_{m^n}$  and  $H = Z_n \triangleleft G$  we have  $A = Z_m$ . In this situation 3.1 provides a  $G$ -map  $F$  with  $J_F = J_w \cdot J_v$  where  $w = w_1 \circ \pi_1$  is the composite of a  $Z_n$ -map  $w_1$  and projection  $\pi_1: X^m \rightarrow X^n$ . Thus 3.1 provides the construction of the "best" cyclic map of order  $m$  from the "best" cyclic maps of prime-power orders occurring in the prime decomposition of  $m$ . The latter are studied in [6].

In conclusion we remark that if  $B$  is the trivial subgroup, 3.1 provides no useful information at all e.g.  $G = S(m)$  for  $m \geq 5$ . Also the appearance of obstruction theory in applications 1 and 2 above

indicate the limitations of 3.1. It would appear now from the results of [9] that the most natural approach to constructing  $S(m)$ -maps of minimal James number is via obstruction theory using [8] and Nakaoka's results relating the cohomology of  $SP^m S^n$  to that of iterated cyclic products of spheres.

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