Pacific Journal of Mathematics

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Vol. 45, No. 2 October 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

RESTRICTING ISOTOPIES OF SPHERES

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In this note we consider the problem of determining whether isotopic homeomorphisms of S^n that agree on a subset X of S^n are isotopic by an isotopy that is fixed on X. In particular, in the PL category, an affirmative answer is obtained for X a locally unknotted closed cell or an unknotted sphere.

If X and Y are polyhedra and h_0 and h_1 are homeomorphisms of X onto Y, then an *isotopy* between h_0 and h_1 is a homeomorphism $H: X \times I \rightarrow Y \times I (I = [0, 1])$ such that $H(x, t) = (h_t(x), t)$ for all $(x, t) \in X \times I$. Two embeddings f, g of X in Y are said to be *ambient isotopic* if there is an isotopy $H: Y \times I \rightarrow Y \times I$ such that $h_0 = \text{id.}$, and $h_1 f = g$. The isotopy H is *fixed on* $A \subset Y$ if $H(x, t) = (x, t)$ for all $(x, t) \in A \times I$. Let S^n denote the standard n-sphere, E^n Euclidean n-space, Δ^k a k-simplex in some combinatorial triangulation of S^n or E^n, and let "PL" denote "piecewise linear." If $k < n$ we regard S^n as the $(n-k)$-fold suspension of S^k, so there is a natural inclusion $S^k \subset S^n$. A PL embedding $i: S^k \rightarrow S^n$ is *unknotted* if $(S^n, i(S^k)) \underset{\approx}{PL} (S^n, S^k)$, which is always the case if $k \leqq n - 3$. Clearly an unknotted sphere Σ^k in S^n is PL locally flat; i.e., for each point $x \in \Sigma^k$, there is a neighborhood U of x in S^n such that

$$(U, U \cap \Sigma^k) \underset{\approx}{PL} (E^n, E^k) .$$

The main results of this paper are the following:

THEOREM 1. *Let $X = \Delta^k$ or $X = S^k$, and let $i: X \rightarrow S^n$ be a PL-embedding, unknotted if $X = S^k$, locally unknotted if $X = \Delta^k$. If f and g are PL-homeomorphisms of S^n that are ambient isotopic, and if $f \mid i(X) = g \mid i(X)$, then f and g are PL ambient isotopic fixing $i(X)$.*

THEOREM 2. *Let $\Sigma^k \subset S^n$ be unknotted, $n \geqq 5$, $k \neq 3$, and f and g be homeomorphisms S^n that are isotopic and agree on Σ. Then f and g are ambient isotopic fixing Σ.*

If $k \leqq n - 3$, then Theorem 1 is a special case of [2]. Note that in Theorem 2, we do not require f and g to be PL.

The key step in the proof of these theorems is

LEMMA 3. *Let X be a k-simplex in S^n or the standard k-sphere $S^k \subset S^n$. If f is an orientation preserving PL-homeomorphism of S^n*

that is the identity on X, then f is PL-isotopic to the identity keeping X fixed.

Proof. The proof is by induction on n, with the case $n = 0$ trivial. Assume the lemma is true for X a simplex or a sphere in S^{n-1}.

Case 1. $X = \Delta^k$.

Let D be a second derived neighborhood of X mod ∂X; if $k = 0$, D is a regular neighborhood of X; if $k = n$, $D = X$. Observe that $f(D)$ is also such a regular neighborhood. Thus there is an isotopy H of S^n, keeping X fixed, such that $H_0 = \text{id.}$, and $H_1 f(D) = D$ [1].

Now $H_1 f \mid \partial D$ is an orientation preserving PL homeomorphism; since $H_1 f \mid (\partial D \cap X = S^{k-1}) = \text{id.}$, and $\partial D \underset{\approx}{PL} S^{n-1}$, by induction, $H_1 f \mid \partial D$ is isotopic (in ∂D) to the id. fixing $\partial D \cap X$. Thus there is a PL isotopy G'_t of ∂D such that $G'_0 = \text{id.}$, $G'_1 H_1 f \mid \partial D = id.$, and $G'_t H_1 f \mid \partial D \cap X = \text{id.}$ for $0 \leqq t \leqq 1$. Suspend this isotopy to obtain an isotopy, G, of S^n that keeps X fixed; to do this, pick suspension points $x \in X$, $y \in S^n \backslash D$, and note that we may assume that X is then a subcone of D. Thus G has similar properties to G'; i.e., $G_t H_1 f \mid \partial D \cup X = \text{id.}$ for $0 \leqq t \leqq 1$, and $G_0 = \text{id.}$ The PL-homeomorphism $G_1 H_1 f$ of S^n is the *id.* on $\partial D \cup X$, so the Alexander technique yields an isotopy F of S^n such that $F_0 = G_1 H_1 f$, $F_1 = id.$, and F keeps $\partial D \cup X$ fixed. The isotopy

$$\begin{array}{ll} H_{4t}(f(x)) & 0 \leqq t \leqq \frac{1}{4},\ x \in S^n\ , \\ G_{4t-1}(H_1 f(x)) & \frac{1}{4} \leqq t \leqq \frac{1}{2},\ x \in S^n \\ F_{2t-1}(x) & \frac{1}{2} \leqq t \leqq 1,\ x \in S^n \end{array}$$

is the required result.

Case 2. $X = S^0$

Let $X = \{a, b\}$, and let N be a second derived neighborhood of a mod b in S^n. Let M be a second derived neighborhood of b mod $(N \cup f(N))$ in S^n. Then N and $f(N)$ are regular neighborhoods of a in $Q = \text{cl}\ (S^n - M)$ that meet ∂Q regularly. Thus there exists an ambient isotopy H of Q, keeping $\partial Q \cup a$ fixed, such that $H_1 f(N) = N$. Extend H to S^n by the identity on M.

$H_1 f \mid \partial N$: $\partial N \rightarrow \partial N$ is an orientation preserving PL homeomorphism $(\partial N \overset{PL}{\approx} S^{n-1})$, so we may use the Alexander technique to obtain an ambient isotopy G of S^n such that $G_1 H_1 f = \text{id.}$ and $G_t \mid X = \text{id.}$ As before, this yields the desired result.

Case 3. $X = S^k$, $k \geqq 1$.

Clearly we may assume $k < n$, and that if $S^n = \Sigma^{n-1}S^1$, then $S^k = \Sigma^{k-1}S^1$. Let $a, b \in S^1 \subset S^n$, and let $S_*^{n-1} = \Sigma^{n-1}\{a, b\}$. (In each of these suspensions, we are using the same suspension points in the same order.) Let B_+^n, B_-^n be the closed complementary domains of S_*^{n-1}. Let $B_+^k = S^k \cap B_+^n$; $B_-^k = S^k \cap B_-^n$.

Observe that B_+^n is a regular neighborhood of B_+^k mod B_-^k, as is $f(B_+^n)$. Thus by Theorem 3 of [1], there exists an isotopy H of S^n, fixed on $B_+^k \cup B_-^k = S^k$, such that

$$H_0 = \text{id}, \quad \text{and} \quad H_1f(B_+^n) = B_+^n .$$

Note that $H_1f(S_*^{n-1}) = S_*^{n-1}$, and that

$$H_1f \mid S^k \cap S_*^{n-1}\colon S^k \cap S_*^{n-1}(=S^{k-1}) \longrightarrow S_*^{n-1} \text{ is the id} .$$

Thus $H_1f \mid S_*^{n-1}$ is isotopic to the identity keeping $S^k \cap S_*^{n-1}$ fixed. Proceed as before to complete the proof.

COROLLARY 4. *If f is an orientation preserving PL homeomorphism of E^n such that $f \mid \Delta^k =$ id., then f is PL-isotopic to the* id. *fixing Δ^k.*

COROLLARY 5. *Let $g\colon \Delta^k \longrightarrow E^n(S^n)$ be a PL-embedding, locally unknotted if $k = n - 2$. If f is an orientation preserving PL-homeomorphism of $E^n(S^n)$ and if $f \mid g(\Delta) =$ id., then f is PL-isotopic to the identity fixing $g(\Delta)$.*

COROLLARY 6. *Let $g\colon S^k \longrightarrow S^n$ be an unknotted PL-embedding. If f is a PL-homeomorphism of S^n that is orientation preserving and the identity on $g(S^k)$, then f is PL-isotopic to the identity fixing $g(S^k)$.*

Proof of Theorem 1. Observe that gf^{-1} is an orientation preserving PL-homeomorphism of S^n that is the identity on $i(X)$. Thus there is a PL ambient isotopy h_t of S^n such that

$$\begin{aligned} h_0 &= \text{id} . \\ h_1 &= gf^{-1}, \text{ and} \\ h_t \mid i(X) &= \text{id} . \end{aligned}$$

$h_t\colon S^n \longrightarrow S^n$ is the desired isotopy.

Proof of Theorem 2. As in the proof of Theorem 1, it suffices to consider the case when f is orientation preserving and g is the identity. By [3], f is isotopic to a PL-homeomorphism f' fixing Σ^k. Apply Lemma 3 to f'.

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Received November 11, 1971. This is a portion of the author's dissertation, written at the University of Georgia under the direction of J. C. Cantrell. The author wishes to thank Professors Cantrell and Nelson L. Max for advice concerning these results.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 45, No. 2 October, 1973