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RESTRICTING ISOTOPIES OF SPHERES

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J. A. CHILDRESS

In this note we consider the problem of determining whether isotopic homeomorphisms of S^n that agree on a subset X of S^n are isotopic by an isotopy that is fixed on X . In particular, in the PL category, an affirmative answer is obtained for X a locally unknotted closed cell or an unknotted sphere.

If X and Y are polyhedra and h_0 and h_1 are homeomorphisms of X onto Y , then an *isotopy* between h_0 and h_1 is a homeomorphism $H: X \times I \rightarrow Y \times I$ ($I = [0, 1]$) such that $H(x, t) = (h_t(x), t)$ for all $(x, t) \in X \times I$. Two embeddings f, g of X in Y are said to be *ambient isotopic* if there is an isotopy $H: Y \times I \rightarrow Y \times I$ such that $h_0 = \text{id.}$, and $h_1 f = g$. The isotopy H is *fixed on* $A \subset Y$ if $H(x, t) = (x, t)$ for all $(x, t) \in A \times I$. Let S^n denote the standard n -sphere, E^n Euclidean n -space, Δ^k a k -simplex in some combinatorial triangulation of S^n or E^n , and let " PL " denote "piecewise linear." If $k < n$ we regard S^n as the $(n - k)$ -fold suspension of S^k , so there is a natural inclusion $S^k \subset S^n$. A PL embedding $i: S^k \rightarrow S^n$ is *unknotted* if $(S^n, i(S^k)) \underset{\approx}{\approx} (S^n, S^k)$, which is always the case if $k \leq n - 3$. Clearly an unknotted sphere S^k in S^n is PL locally flat; i.e., for each point $x \in S^k$, there is a neighborhood U of x in S^n such that

$$(U, U \cap S^k) \underset{\approx}{\approx} (E^n, E^k).$$

The main results of this paper are the following:

THEOREM 1. *Let $X = \Delta^k$ or $X = S^k$, and let $i: X \rightarrow S^n$ be a PL -embedding, unknotted if $X = S^k$, locally unknotted if $X = \Delta^k$. If f and g are PL -homeomorphisms of S^n that are ambient isotopic, and if $f|_i(X) = g|_i(X)$, then f and g are PL ambient isotopic fixing $i(X)$.*

THEOREM 2. *Let $\Sigma^k \subset S^n$ be unknotted, $n \geq 5$, $k \neq 3$, and f and g be homeomorphisms S^n that are isotopic and agree on Σ . Then f and g are ambient isotopic fixing Σ .*

If $k \leq n - 3$, then Theorem 1 is a special case of [2]. Note that in Theorem 2, we do not require f and g to be PL .

The key step in the proof of these theorems is

LEMMA 3. *Let X be a k -simplex in S^n or the standard k -sphere $S^k \subset S^n$. If f is an orientation preserving PL -homeomorphism of S^n*

that is the identity on X , then f is PL -isotopic to the identity keeping X fixed.

Proof. The proof is by induction on n , with the case $n = 0$ trivial. Assume the lemma is true for X a simplex or a sphere in S^{n-1} .

Case 1. $X = \Delta^k$.

Let D be a second derived neighborhood of $X \bmod \partial X$; if $k = 0$, D is a regular neighborhood of X ; if $k = n$, $D = X$. Observe that $f(D)$ is also such a regular neighborhood. Thus there is an isotopy H of S^n , keeping X fixed, such that $H_0 = \text{id.}$, and $H_1 f(D) = D$ [1].

Now $H_1 f | \partial D$ is an orientation preserving PL homeomorphism; since $H_1 f | (\partial D \cap X = S^{k-1}) = \text{id.}$, and $\partial D \overset{\approx}{\cong} S^{n-1}$, by induction, $H_1 f | \partial D$ is isotopic (in ∂D) to the id. fixing $\partial D \cap X$. Thus there is a PL isotopy G'_t of ∂D such that $G'_0 = \text{id.}$, $G'_1 H_1 f | \partial D = \text{id.}$, and $G'_t H_1 f | \partial D \cap X = \text{id.}$ for $0 \leq t \leq 1$. Suspend this isotopy to obtain an isotopy, G , of S^n that keeps X fixed; to do this, pick suspension points $x \in X, y \in S^n \setminus D$, and note that we may assume that X is then a subcone of D . Thus G has similar properties to G' ; i.e., $G_t H_1 f | \partial D \cup X = \text{id.}$ for $0 \leq t \leq 1$, and $G_0 = \text{id.}$ The PL -homeomorphism $G_1 H_1 f$ of S^n is the id. on $\partial D \cup X$, so the Alexander technique yields an isotopy F of S^n such that $F_0 = G_1 H_1 f, F_1 = \text{id.}$, and F keeps $\partial D \cup X$ fixed. The isotopy

$$\begin{aligned} H_{4t}(f(x)) & \quad 0 \leq t \leq \frac{1}{4}, x \in S^n, \\ G_{4t-1}(H_1 f(x)) & \quad \frac{1}{4} \leq t \leq \frac{1}{2}, x \in S^n \\ F_{2t-1}(x) & \quad \frac{1}{2} \leq t \leq 1, x \in S^n \end{aligned}$$

is the required result.

Case 2. $X = S^0$

Let $X = \{a, b\}$, and let N be a second derived neighborhood of $a \bmod b$ in S^n . Let M be a second derived neighborhood of $b \bmod (N \cup f(N))$ in S^n . Then N and $f(N)$ are regular neighborhoods of a in $Q = \text{cl}(S^n - M)$ that meet ∂Q regularly. Thus there exists an ambient isotopy H of Q , keeping $\partial Q \cup a$ fixed, such that $H_1 f(N) = N$. Extend H to S^n by the identity on M .

$H_1 f | \partial N: \partial N \rightarrow \partial N$ is an orientation preserving PL homeomorphism ($\partial N \overset{\approx}{\cong} S^{n-1}$), so we may use the Alexander technique to obtain an ambient isotopy G of S^n such that $G_1 H_1 f = \text{id.}$ and $G_t | X = \text{id.}$ As before, this yields the desired result.

Case 3. $X = S^k, k \geq 1$.

Clearly we may assume $k < n$, and that if $S^n = \Sigma^{n-1}S^1$, then $S^k = \Sigma^{k-1}S^1$. Let $a, b \in S^1 \subset S^n$, and let $S_*^{n-1} = \Sigma^{n-1}\{a, b\}$. (In each of these suspensions, we are using the same suspension points in the same order.) Let B_+^n, B_-^n be the closed complementary domains of S_*^{n-1} . Let $B_+^k = S^k \cap B_+^n; B_-^k = S^k \cap B_-^n$.

Observe that B_+^n is a regular neighborhood of $B_+^k \text{ mod } B_-^k$, as is $f(B_+^n)$. Thus by Theorem 3 of [1], there exists an isotopy H of S^n , fixed on $B_+^k \cup B_-^k = S^k$, such that

$$H_0 = \text{id}, \text{ and } H_1f(B_+^n) = B_+^n .$$

Note that $H_1f(S_*^{n-1}) = S_*^{n-1}$, and that

$$H_1f|S^k \cap S_*^{n-1}: S^k \cap S_*^{n-1}(=S^{k-1}) \rightarrow S_*^{n-1} \text{ is the id.}$$

Thus $H_1f|S_*^{n-1}$ is isotopic to the identity keeping $S^k \cap S_*^{n-1}$ fixed. Proceed as before to complete the proof.

COROLLARY 4. *If f is an orientation preserving PL homeomorphism of E^n such that $f| \Delta^k = \text{id.}$, then f is PL-isotopic to the id. fixing Δ^k .*

COROLLARY 5. *Let $g: \Delta^k \rightarrow E^n(S^n)$ be a PL-embedding, locally unknotted if $k = n - 2$. If f is an orientation preserving PL-homeomorphism of $E^n(S^n)$ and if $f|g(\Delta) = \text{id.}$, then f is PL-isotopic to the identity fixing $g(\Delta)$.*

COROLLARY 6. *Let $g: S^k \rightarrow S^n$ be an unknotted PL-embedding. If f is a PL-homeomorphism of S^n that is orientation preserving and the identity on $g(S^k)$, then f is PL-isotopic to the identity fixing $g(S^k)$.*

Proof of Theorem 1. Observe that gf^{-1} is an orientation preserving PL-homeomorphism of S^n that is the identity on $i(X)$. Thus there is a PL ambient isotopy h_t of S^n such that

$$\begin{aligned} h_0 &= \text{id} . \\ h_1 &= gf^{-1}, \text{ and} \\ h_t | i(X) &= \text{id} . \end{aligned}$$

$h_t: S^n \rightarrow S^n$ is the desired isotopy.

Proof of Theorem 2. As in the proof of Theorem 1, it suffices to consider the case when f is orientation preserving and g is the identity. By [3], f is isotopic to a PL-homeomorphism f' fixing Σ^k . Apply Lemma 3 to f' .

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