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SOME RESULTS ON LACUNARY WALSH SERIES

JOHN EDWARD CORY

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JOHN COURY

It is known that if a lacunary trigonometric series converges to 0 on a set of positive measure, then the series vanishes identically. In the present paper, the following analogue for the Walsh system is proved: a lacunary Walsh series which converges to 0 almost everywhere is identically zero.

In particular, let $S(x) = \sum_1^\infty c_k w_{n_k}(x)$ be a lacunary Walsh series, with $n_{k+1}/n_k \geq q > 1$ for each k . We prove that if $S(x)$ converges to 0 on a set of positive measure, or on a set of the second category having the property of Baire, then the series is a finite sum. If $S(x)$ converges to 0 on a set of sufficiently large measure (the measure depending only on the degree of lacunarity q), then $S(x)$ is identically zero. Hence we prove that the only lacunary Walsh series converging to 0 almost everywhere is the identically zero series. Finally, sufficient conditions are given for a set to be a set of uniqueness for lacunary Walsh series.

1. Preliminaries. If $x \in [0, 1[$ has the dyadic development $\sum_{k=1}^\infty x_k 2^{-k}$, where x_k is 0 or 1, then the $(k-1)$ st Rademacher function r_{k-1} evaluated at x has the value $(-1)^{x_k}$. (For dyadic rationals in $[0, 1[$, which have two such expansions, we agree to take the finite development.) If we write a positive integer n as $2^{n_1} + 2^{n_2} + \dots + 2^{n_v}$, where $n_1 > n_2 > \dots > n_v \geq 0$, then the n th Walsh function is given by $w_n = r_{n_1} \cdot r_{n_2} \cdot \dots \cdot r_{n_v}$ (following Paley's modification). Define $w_0(x) \equiv 1$; then the functions $\{w_n\}_{n=0}^\infty$ form a complete orthonormal set on $[0, 1[$. For $n \geq 1$, w_n is plainly discontinuous.

Fine [2] has shown that the Walsh functions may be identified with the full character group $\{w_n^*\}$ of 2^ω , where 2^ω denotes the countable product of the two-element group $\{0, 1\}$. For $\mathbf{x} = (x_1, x_2, \dots)$ in 2^ω , define $w_n^*(\mathbf{x}) = r_{n_1}^*(\mathbf{x}) \cdot \dots \cdot r_{n_v}^*(\mathbf{x})$, where $n = 2^{n_1} + \dots + 2^{n_v}$ and $r_{k-1}^*(\mathbf{x}) = (-1)^{x_k}$. To simplify notation, we shall suppose henceforth that $r_{k-1}^*(\mathbf{x}) = (-1)^{x_{k-1}}$, where $k \geq 2$.

Let $\phi(\mathbf{x}) = \sum_{k=1}^\infty x_k 2^{-k}$, where $\mathbf{x} = (x_1, x_2, \dots) \in 2^\omega$. Then $\phi(\mathbf{x})$ is a continuous measure-preserving map of 2^ω onto $[0, 1[$ but is not injective: a dyadic rational in $[0, 1[$ is the image of two points in 2^ω . If \mathcal{S} denotes the set of sequences in 2^ω that are eventually 1, then

$\phi(\mathbf{x})$ is one-to-one on the complement of \mathcal{F} and $\phi(\mathbf{x})$ maps \mathcal{F} onto the dyadic rationals of $[0, 1[$. This 'exceptional' set \mathcal{F} corresponds to the infinite expansion of dyadic rationals; \mathcal{F} is plainly countable and hence has Haar measure 0.

With a Walsh series $\sum c_n w_n(x)$ on $[0, 1[$, we may associate the corresponding 'Walsh' series $\sum c_n w_n^*(\mathbf{x})$ on 2^ω , where \mathbf{x} is such that $\phi(\mathbf{x}) = x$. Clearly, if we neglect the points of \mathcal{F} , the former series converges to a value c when and only when the latter series also converges to c .

Thus, since \mathcal{F} is of the first category and has Haar measure zero in 2^ω , it suffices to prove the theorems stated in the introduction on the group 2^ω , using lacunary series of the form $S^*(\mathbf{x}) = \sum c_k w_{n_k}^*(\mathbf{x})$. The advantage is that the functions w_n^* are continuous with respect to the usual (product) topology of 2^ω ; however, all of the results of §§ 2 and 3 are valid for lacunary Walsh series defined on the interval $[0, 1[$. Henceforth we write $S(\mathbf{x})$ for $S^*(\mathbf{x})$ and w_n for w_n^* .

2. The main results. A Walsh series $S(\mathbf{x}) = \sum_{k=1}^{\infty} c_k w_{n_k}(\mathbf{x})$ is called q -lacunary if $n_{k+1}/n_k \geq q > 1$ for all k , where q is the supremum of all such numbers.

LEMMA. Let $S(\mathbf{x})$ be a q -lacunary Walsh series, with $q \geq 2$. Suppose that $S(\mathbf{x})$ converges to 0 (or is constant) on a set E of Haar measure exceeding $1/2$. Then $S(\mathbf{x})$ vanishes identically.

Proof. Since $q \geq 2$, each n_k contains a power of 2 greater than each power of 2 in n_{k-1} . Thus the series $S(\mathbf{x})$ is simply a series $\sum c_k X_k$, where the X_k are independent random variables of the Bernoulli type. This latter series may in turn be viewed as a Rademacher series $R(\mathbf{x}) = \sum_{k=1}^{\infty} c_k r_k(\mathbf{x})$. Thus, as far as measure is concerned, the series $R(\mathbf{x})$ and $S(\mathbf{x})$ have the same properties, and so it suffices to prove the lemma for Rademacher series.

For an arbitrary positive integer N , let \mathbf{u}_N denote the point in 2^ω with entry 1 in the N th coordinate and 0's elsewhere. Since adding \mathbf{u}_N to a point $\mathbf{x} \in 2^\omega$ affects only the N th coordinate of \mathbf{x} , it follows that $r_k(\mathbf{x} + \mathbf{u}_N) = r_k(\mathbf{x})$ for every $k \neq N$, and $r_N(\mathbf{x} + \mathbf{u}_N) = -r_N(\mathbf{x})$.

Let μ denote (normalized) Haar measure on 2^ω . Since $\mu(E) > 1/2$, the set $E \cap (E + \mathbf{u}_N)$ has positive measure for every N , where $E + \mathbf{u}_N = \{\mathbf{x} + \mathbf{u}_N : \mathbf{x} \in E\}$. Thus there exists $z \in E$ such that $z + \mathbf{u}_N \in E$. If R_n denotes the n th partial sum of R , we have

$$\begin{aligned} \text{and} \quad R_n(z + \mathbf{u}_N) - R_n(z) &= 0 && \text{for } n < N, \\ R_n(z + \mathbf{u}_N) - R_n(z) &= -2c_N r_N(z) && \text{for } n \geq N. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we obtain $-2c_N r_N(z) = 0$, whence

$c_N = 0$. Since N was arbitrary, we conclude that $R(\mathbf{x})$ vanishes identically.

It is well-known (see, for example, [1, vol. II, p. 233]) that for a given $q > 1$, any lacunary sequence of integers can be written as the finite union of pairwise disjoint lacunary subsequences each of whose degree of lacunarity is at least q . The least number of q -lacunary subseries, with $q \geq 2$, into which a given lacunary series can be partitioned will be called the *index of 2-lacunarity* of the series. We now generalize the result of the lemma to arbitrary lacunary Walsh series.

THEOREM 1. *Let $S(\mathbf{x})$ be a lacunary Walsh series, and let M denote its index of 2-lacunarity. Suppose that $S(\mathbf{x})$ converges to 0 (or is constant) on a set E with $\mu(E) > 1 - 1/2^M$. Then $S(\mathbf{x})$ is identically zero.*

Proof. By induction on M . For $M = 1$, the result follows from the lemma. Thus assume the theorem is true for an index $M - 1$; write $S(\mathbf{x}) = \sum_{i=1}^M S_i(\mathbf{x})$, where $S_i(\mathbf{x}) = \sum_{k=1}^{\infty} c_{i,k} w_{n_{i,k}}(\mathbf{x})$ and for each fixed i the sequence $\{n_{i,k}\}$ is q -lacunary with $q \geq 2$.

For a positive integer N , write $N = 2^{N_1} + \dots + 2^{N_v}$, with $N_1 > N_2 > \dots > N_v \geq 0$. Then for $\mathbf{x} \in 2^\omega$, we shall say that the coordinates of \mathbf{x} which “correspond to N ” are the N_i th entries, $i = 1, 2, \dots, v$.

We may suppose that $n_{M,1} < n_{1,1}$. Since $n_{1,1}$ has a power of 2 that $n_{M,1}$ does not, choose \mathbf{z}_1 for $n_{1,1}$ so that in those coordinates of \mathbf{z}_1 corresponding to $n_{M,1}$ there are an odd number of 1’s, and in the coordinates corresponding to every $n_{i,k}$, $k \geq 1$, there are an even number of 1’s. For arbitrary $\mathbf{x} \in 2^\omega$, we have

$$\begin{aligned} S(\mathbf{x} + \mathbf{z}_1) - S(\mathbf{x}) &= \sum_{k=1}^{\infty} c_{1,k} [w_{n_{1,k}}(\mathbf{z}_1) - 1] w_{n_{1,k}}(\mathbf{x}) \\ &\quad + \sum_{i=2}^M \sum_{k=1}^{\infty} c_{i,k} [w_{n_{i,k}}(\mathbf{x} + \mathbf{z}_1) - w_{n_{i,k}}(\mathbf{x})] \\ &= \sum_{i=2}^M \sum_{k=1}^{\infty} c_{i,k} [w_{n_{i,k}}(\mathbf{z}_1) - 1] w_{n_{i,k}}(\mathbf{x}), \end{aligned}$$

since $w_{n_{1,k}}(\mathbf{z}_1) = 1$ for every $k \geq 1$. The left-side of the equation is 0 on the set $E \cap (E + \mathbf{z}_1)$; since $\mu(E \cap (E + \mathbf{z}_1)) > 1 - 1/2^{M-1}$, it follows from the induction hypothesis that $c_{i,k} [w_{n_{i,k}}(\mathbf{z}_1) - 1] = 0$ for each $i \geq 2$ and all k . In particular, taking $i = M$ and $k = 1$, we have $-2c_{M,1} = 0$, since $w_{n_{M,1}}(\mathbf{z}_1) = -1$. Thus $c_{M,1} = 0$. Similarly, we prove that $c_{1,1} = c_{2,1} = \dots = c_{M-1,1} = 0$.

Now suppose that we have shown $c_{i,1} = c_{i,2} = \dots = c_{i,p-1} = 0$ for $i = 1, 2, \dots, M$, and consider $c_{M,p}$. Let $n_{i,p}$ ($i \neq M$) be such that

$n_{M,p} < n_{i,p}$. Choose z_p for $n_{i,p}$ so that in the coordinates of z_p corresponding to $n_{M,p}$ there are an odd number of 1's, and an even number of 1's in the coordinates which correspond to each $n_{i,j} (j \geq p)$. Then, as above, we conclude that $c_{M,p} = 0$. Thus $c_{i,k} = 0$ for each $i = 1, 2, \dots, M$ and every $k \geq 1$, and hence the given series $S(\mathbf{x})$ vanishes identically.

COROLLARY. *A lacunary Walsh series which converges to 0 (or is constant) almost everywhere vanishes identically.*

This latter result is the Walsh analogue of the following important theorem for trigonometric series: a lacunary trigonometric series converging to 0 on a set of positive measure is the identically zero series ([8, vol. I, p. 206]). The statement in the corollary is the best possible in that convergence a.e. cannot be replaced by convergence on a set of measure less than 1. This follows at once from the fact that in $[0, 1]$ (or 2^ω), the Walsh-Fourier series of the characteristic function of a dyadic interval (basic open set) has only finitely many nonzero coefficients ([7, p. 288]).

The trigonometric result does, however, have the following counterpart in the Walsh system.

THEOREM 2. *A lacunary Walsh series which converges to 0 (or is constant) on a set of positive measure has only finitely many nonzero terms.*

Proof. The function $\pi: \mathbf{x} \rightarrow \mu(E \cap (E + \mathbf{x}))$ is continuous ([3, (20.17)]). Hence if \mathbf{x} is 'close' to 0 (that is, if \mathbf{x} has sufficiently many 0's initially), then $\mu(E \cap (E + \mathbf{x})) > 0$. Let N be the smallest integer for which $E \cap (E + \mathbf{x})$ has positive measure whenever \mathbf{x} has its first N coordinates equal to 0.

Let M be the index of 2-lacunarity of the series $S(\mathbf{x})$; we prove the theorem by induction on M . If $M = 1$, let R be the largest power of 2 appearing in the base 2 development of n_1, n_2, \dots, n_N . Define \mathbf{s}_{N+1} as follows: let \mathbf{s}_{N+1} have 0's in every coordinate up to and including the R th coordinate; an odd number of 1's in the coordinates which correspond to n_{N+1} ; an even number of 1's in the coordinates corresponding to each $n_j, j > N + 1$; and 0's in the coordinates not otherwise determined. Then we have

$$S(\mathbf{x} + \mathbf{s}_{N+1}) - S(\mathbf{x}) = -2c_{N+1}w_{n_{N+1}}(\mathbf{x})$$

for every $\mathbf{x} \in 2^\omega$. Since each of n_1, \dots, n_N has a power of 2 greater than each power of 2 in its predecessor, it follows that $R \geq N$, hence \mathbf{s}_{N+1} has at least N zeros initially. Thus $E \cap (E + \mathbf{s}_{N+1})$ is nonempty. As in the proof of the lemma, it can then be shown that $c_{N+1} = 0$.

Proceeding by induction, we conclude that $c_n = 0$ for every $n > N$.

We retain the notation in the proof of Theorem 1 and suppose the theorem is true for an index $M - 1$; that is, if the given series is the sum of $M - 1$ q -lacunary series ($q \geq 2$) and converges to 0 on a set of positive measure, then $c_{i,j} = 0$ for each i and all $j > N$, where N is as in the first paragraph. Consider now the series $S(\mathbf{x}) = S_1(\mathbf{x}) + \dots + S_M(\mathbf{x})$. Let R be the largest power of 2 appearing in the base 2 expansion of $n_{M,1}, n_{M,2}, \dots, n_{M,N}$. Choose $n_{i,J}$ ($i \neq M$) so that $n_{i,J} > n_{M,N+1}$ and $n_{i,J}$ contains a power of 2 greater than each power of 2 in $n_{M,N+1}$ (this is always possible unless each subsequence $\{n_{i,k}\}$ is finite, in which case we are done). Define \mathbf{s}_J for $n_{i,J}$ as follows: let \mathbf{s}_J have 0's in every coordinate up to and including the R th; let \mathbf{s}_J have an odd number of 1's in the coordinates corresponding to $n_{M,N+1}$; let \mathbf{s}_J have an even number of 1's in the coordinates which correspond to each $n_{i,j}$, for every $j \geq J$. Assign 0 to those coordinates not already determined. Then for each $\mathbf{x} \in 2^\omega$,

$$S(\mathbf{x} + \mathbf{s}_J) - S(\mathbf{x}) = \sum_{k=J}^{J-1} c_{i,k} [w_{n_{i,k}}(\mathbf{s}_J) - 1] w_{n_{i,k}}(\mathbf{x}) + \sum_{\substack{r=1 \\ r \neq i}}^M [S_r(\mathbf{x} + \mathbf{s}_J) - S_r(\mathbf{x})],$$

since $w_{n_{i,k}}(\mathbf{s}_J) = 1$ for $k \geq J$. Because \mathbf{s}_J has at least N zeros initially, $E \cap (E + \mathbf{s}_J)$ has positive measure and so the left-side of this equation vanishes for \mathbf{x} in a set of positive measure. Also, since the first sum on the right-side is finite, it is necessarily constant on sets of positive measure. Thus,

$$\sum_{\substack{r=1 \\ r \neq i}}^M [S_r(\mathbf{x} + \mathbf{s}_J) - S_r(\mathbf{x})] = \sum_{\substack{r=1 \\ r \neq i}}^M \sum_{\substack{m=1 \\ m \neq i}}^\infty c_{r,m} [w_{n_{r,m}}(\mathbf{s}_J) - 1] w_{n_{r,m}}(\mathbf{x})$$

is constant in a set of positive measure and so by the induction hypothesis, $c_{r,m} [w_{n_{r,m}}(\mathbf{s}_J) - 1] = 0$ for $r = 1, 2, \dots, M, r \neq i$, and all $m > N$. In particular, for $r = M$ and $m = N + 1$, we have $w_{n_{M,N+1}}(\mathbf{s}_J) = -1$ and therefore $c_{M,N+1} = 0$. Proceeding by induction, we show that $c_{M,j} = 0$ for every $j > N$. Thus, $S_M(\mathbf{x})$ is a finite sum.

Write $S_1(\mathbf{x}) + \dots + S_{M-1}(\mathbf{x}) = -S_M(\mathbf{x})$; since $S_M(\mathbf{x})$ is finite, it assumes constant values on sets of positive measure. The induction hypothesis then implies that each of $S_1(\mathbf{x}), \dots, S_{M-1}(\mathbf{x})$ is also a finite sum, and so therefore is $S(\mathbf{x})$.

The previous proof uses the hypothesis that E has positive measure only to ensure that the set $E \cap (E + \mathbf{x})$ is nonempty for sufficiently 'small' \mathbf{x} . However, even for E of measure zero, this will still be true if E is of the second category and has the property of Baire (see [5, p. 21]). (A set is said to have the property of Baire if it can

be expressed as the symmetric difference of an open set and a set of the first category.) If a set has the property of Baire, so does its complement ([5, p. 19]); it follows that any residual set (i.e., the complement of a first category set) has the property of Baire. With a slight modification to the proof of Theorem 2, we have the following result.

THEOREM 3. *Suppose that E is of the second category and has the property of Baire. Then a lacunary Walsh series which converges to 0 (or is constant) on E is necessarily a finite sum. In particular, if E is residual, then the series is identically constant.*

Proof. Let \mathbf{s}_j be defined as in the previous proof. Then $E \cap (E + \mathbf{s}_j)$ is of the second category ([5, p. 21]); since the class of all sets having the property of Baire is a σ -algebra, $E \cap (E + \mathbf{s}_j)$ also has the property of Baire. The first sum on the right-side of the equation in the previous proof is constant on dyadic intervals; since $E \cap (E + \mathbf{s}_j)$ must meet some one of these intervals in a set of the second category, the series $\sum_{r=1}^M [S_r(\mathbf{x} + \mathbf{s}_j) - S_r(\mathbf{x})]$ is constant on a second category set having the property of Baire. The proof of Theorem 2 may now be applied, with the obvious modification to the induction hypothesis, to show that the given series is a finite sum.

Finally, if E is residual, then E is dense. Because the series is finite and hence continuous, it can assume only one value.

3. Sets of uniqueness. A set C is called a *set of uniqueness* for lacunary Walsh series, or a *Walsh U_L -set*, if the only lacunary series converging to 0 on the complement of C is the identically zero series.

THEOREM 4. *Suppose that C satisfies one of the following:*

- (i) C is a null set;
- (ii) C is of the first category;
- (iii) the complement of C is a dense second category set having the property of Baire.

Then C is a Walsh U_L -set.

Proof. If C satisfies (i), the result follows from the corollary to Theorem 1. Suppose now that (ii) or (iii) holds; in view of Theorem 3, a lacunary Walsh series converging to 0 on the complement of C is a finite sum and so continuous. Because the complement of C is dense, the series must vanish identically.

None of the conditions in the theorem are necessary. That (i) and

(ii) are not necessary follows from the fact that a null set may be of either category and that there exist sets of the first category of any given measure. To show that (iii) is not necessary, we take C to be a residual set of measure zero; then C is a U_L -set in view of the corollary to Theorem 1.

Lastly, we note that a Walsh U_L -set must have empty interior and so, in particular, cannot be open: for otherwise the Walsh-Fourier series of the characteristic function of a dyadic subinterval, a finite series, would converge to 0 on the complement of the set but not vanish identically.

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