GENERALIZED SYLOW TOWER GROUPS, II

J. B. DERR AND N. P. MUKHERJEE

A well-known result of P. Hall shows that finite solvable groups may be characterized by a permutability requirement on Sylow subgroups. The notion of a generalized Sylow tower group (GSTG) arises when this permutability condition on Sylow subgroups is replaced by a suitable normalizer condition. In an earlier paper, one of the authors showed that the nilpotent length of a GSTG cannot exceed the number of distinct primes which divide the order of the group. The present investigation utilizes the 'type' of a GSTG to obtain improved bounds for the nilpotent length of a GSTG. It is also shown that a GSTG with nilpotent length \( n \) possesses a Hall subgroup of nilpotent length \( n \) which is a Sylow tower group.

Let \( G \) be a finite group with order \( p_1^{a_1} \cdots p_r^{a_r} \), where \( p_1, \ldots, p_r \) are distinct primes and \( a_1, \ldots, a_r \) are positive integers. For each integer \( i, 1 \leq i \leq r \), let \( G_i \) denote a Sylow \( p_i \)-subgroup of \( G \). The collection of subgroups \( \mathcal{S} = \{G_i, \ldots, G_r\} \) is then called a complete set of Sylow subgroups of \( G \). If the elements of \( \mathcal{S} \) are pairwise permutable as subgroups (that is, if \( G_iG_j = G_jG_i \) holds for all \( i \) and \( j \) then \( \mathcal{S} \) will be called a Sylow basis for \( G \). The notion of a generalized Sylow tower group arises when the permutability condition for a Sylow basis is replaced by a normalizer condition. Thus, we say that a finite group \( G \) is a generalized Sylow tower group (GSTG) if and only if some complete set of Sylow subgroups \( \mathcal{S} \) of \( G \) satisfies the normalizer condition \((N)\): if \( G_i \) and \( G_j \) are distinct elements of \( \mathcal{S} \), at least one of these subgroups normalizes the other. It should be noted that not every complete set of Sylow subgroups of a GSTG need satisfy condition \((N)\).

A well-known result of P. Hall states that a finite group is solvable if and only if the group possesses a Sylow basis. If a complete set of Sylow subgroups \( \mathcal{S} \) of a group \( G \) satisfies condition \((N)\) then any two elements of \( \mathcal{S} \) are permutable as subgroups and \( \mathcal{S} \) is a Sylow basis for \( G \). Consequently every generalized Sylow tower group must be solvable.

A finite group \( G \) is called a Sylow tower group (STG) if every nontrivial epimorphic image of \( G \) possesses a nontrivial normal Sylow subgroup. Equivalently, the group \( G \) is a STG if the prime divisors \( p_1, \ldots, p_r \) of the order of \( G \) can be labelled in such a way that a Sylow \( p_i \)-subgroup of \( G \) normalizes a Sylow \( p_j \)-subgroup of \( G \) when-
ever \( i > j \). It is clear from this definition that a Sylow tower group is necessarily a generalized Sylow tower group. An example of a \( GSTG \) which is not a Sylow tower group was given in ([1]; p. 638).

In order to handle generalized Sylow tower groups it will be necessary to introduce the ‘type’ of a \( GSTG \). Suppose \( G \) is some given \( GSTG \) and let \( \mathcal{T} \) be a Sylow basis for \( G \). Since any two Sylow bases for \( G \) are conjugate ([3]; p. 665), \( \mathcal{T} \) satisfies the normalizer condition \((N)\). Let \( R \) be a relation on the set of all primes with the property that either \( pRq \) or \( qRp \) (or both) holds for any primes \( p \) and \( q \). If the Sylow \( p_i \)-subgroup of \( G \) in \( \mathcal{T} \) normalizes the Sylow \( p_i \)-subgroup of \( G \) in \( \mathcal{T} \) whenever \( p_iRq \) holds, then \( G \) will be called a \( GSTG \) of type \( R \). It follows directly from the conjugacy of Sylow bases that the type of a \( GSTG \) is independent of the choice of a Sylow basis. It should be noted that a group can be a \( GSTG \) of more than one type.

It was shown in [1] that the class of all generalized Sylow tower groups of a given type \( R \) is a formation. In addition, any subgroup of a \( GSTG \) of type \( R \) was shown to be a \( GSTG \) of type \( R \). We list the main results about \( GSTG \)'s in [1] for easy reference.

**Theorem 1.7** [1]. If \( G \) is a \( GSTG \) then the nilpotent length of \( G \) does not exceed the number of distinct prime divisors of the order of \( G \).

**Theorem 1.8** [1]. If \( G \) is a \( GSTG \) and the nilpotent length of \( G \) is equal to the number of distinct prime divisors of the order of \( G \) then \( G \) is a Sylow tower group.

All groups mentioned are assumed to be finite. The following notations will be used. For a group \( G \)
- \( \pi(G) \) denotes the set of distinct prime divisors of the order of \( G \)
- \( \varpi(G) \) denotes the number of distinct prime divisors of the order of \( G \)
- \( \mathcal{F}(G) \) denotes the nilpotent (Fitting) length of \( G \).

If \( H \) is a subgroup of \( G \) then \( N_o(H) \) means the normalizer of \( H \) in \( G \) and \( C_o(H) \) means the centralizer of \( H \) in \( G \).

If \( p_i \) is a prime, \( G_i \) will denote a Sylow \( p_i \)-subgroup of \( G \).

The following lemma will be used in several of our arguments.

**Lemma 1.** If \( G \) is a nontrivial \( GSTG \) then at least one of the following holds:

1. \( G \) contains a nontrivial normal Sylow subgroup \( P \) with \( C_o(P) \subseteq P \)
(2) $G$ contains nontrivial normal subgroups having relatively prime orders.

Proof. Let $G$ be a GSTG and suppose that $\mathcal{S} = \{G_1, \ldots, G_n\}$ is a Sylow basis for $G$. Since a GSTG is necessarily solvable, $G$ possesses a nontrivial minimal normal subgroup $M$ with order a power of some prime $q$. Let $Q$ denote the maximum normal $q$-subgroup of $G$ and suppose that $G_i$ is the Sylow $q$-subgroup of $G$ belonging to $\mathcal{S}$. Since $\mathcal{S}$ satisfies the normalizer condition $(N)$, either

$$G_i \subseteq N_o(G_i) \quad \text{or} \quad G_k \subseteq N_o(G_k)$$

must hold for each integer $k$, $1 \leq k \leq n$. We distinguish two cases.

First suppose that $G_i \subseteq N_o(G_k)$ holds for some $k$, $1 < k < n$. Since $Q$ is normal in $G$ and has order prime to the order of $G_k$, $G_k$ centralizes $Q$. Hence $C = C_o(Q)$ is not a $q$-group. Set $Q_o = Q \cap C_o(Q)$ and consider the factor group $C/Q_o$. Since $C/Q_o$ is a nontrivial solvable group, $C/Q_o$ contains a nontrivial minimal normal subgroup $L/Q_o$ with order a power of some prime $p$. Let $T/Q_o$ be the maximum normal $p$-subgroup of $C/Q_o$. Since $Q$ is the maximum normal $q$-subgroup of $G$ it follows that $p \neq q$. If $N$ is a Sylow $p$-subgroup of $T$ then $T$ is the direct product of $N$ and $Q_o$. The normality of $T$ in $G$ then implies the normality of $N$ in $G$. Therefore $G$ has nontrivial normal subgroups with relatively prime orders.

Now suppose that $G_i \subseteq N_o(G_k)$ holds for all integers $k$, $1 < k < n$. Since $\mathcal{S}$ satisfies the normalizer condition $(N)$, $G_k \subseteq G_o(G_i)$ must then hold for all integers $k$. Thus $G_i$ is a normal Sylow $q$-subgroup of $G$. Then $G_i \cap C_o(G_i)$ is a normal Sylow $q$-subgroup of $C_o(G_i)$ and $C_o(G_i)$ has a normal $q$-complement $W$. If $W$ is nontrivial then $W$ is a normal subgroup of $G$ having $q'$-order and (2) holds. If $W$ is trivial then $G_i \cap C_o(G_i) = C_o(G_i)$. In this case $C_o(G_i) \subseteq G_i$ and (1) holds.

Theorem S. Let $G$ be a GSTG. If $H_1, \ldots, H_n$ are pairwise permutable Hall subgroups of $G$ with $G = H_1 \cdots H_n$, then the nilpotent length of $G$ does not exceed the sum of the nilpotent lengths of $H_1, \ldots, H_n$.

Proof. (By induction on the order of $G$.) Since the product $H_1 \cdots H_n$ is a Hall subgroup of $G$ permutable with $H_i$ we may assume that $n = 2$. First suppose that $G$ possesses nontrivial normal subgroups $A$ and $B$ with $A \cap B = 1$. Then $G$ is isomorphic to a subgroup of the direct of $G/A$ and $G/B$. Hence

$$\ell(G) = \max \{\ell(G/A), \ell(G/B)\}$$
and it suffices to show that

\[ \nu(G/A) \leq \nu(H_1) + \nu(H_2) \quad \text{and} \quad \nu(G/B) \leq \nu(H_1) + \nu(H_2). \]

Since \( G/A \) is the product of the permutable Hall subgroups \( H_1A/A \) and \( H_2A/A \) of \( G/A \), the induction hypothesis gives

\[ \nu(G/A) \leq \nu(H_1A/A) + \nu(H_2A/A). \]

Since \( H_1A/A \) and \( H_2A/A \) are epimorphic images of \( H_1 \) and \( H_2 \) (respectively), we know that \( \nu(H_1A/A) \leq \nu(H_i) \) for \( i = 1, 2 \). Therefore \( \nu(G/A) \leq \nu(H_1) + \nu(H_2) \). The same argument applied to \( G/B \) will show \( \nu(G/B) \leq \nu(H_1) + \nu(H_2) \). This verifies the theorem in this case.

Now suppose that \( G \) possesses a unique minimal normal subgroup. Then Lemma 1 shows that \( G \) contains a nontrivial normal Sylow subgroup \( P \) with \( C_G(P) \subseteq P \). Since the theorem is trivially true if \( G = P \) we may assume this is not the case. Set

\[ \bar{G} = G/P, \bar{H}_1 = H_1P/P, \bar{H}_2 = H_2P/P \]

and consider the nontrivial GSTG \( \bar{G} \). The induction hypothesis applied to \( \bar{G} = \bar{H}_1\bar{H}_2 \) gives \( \nu(\bar{G}) \leq \nu(\bar{H}_1) + \nu(\bar{H}_2) \). We first observe that \( \nu(G) = \nu(\bar{G}) + 1 \). Since \( P \) is a nilpotent normal subgroup of \( G \), \( P \) must lie in the Fitting subgroup \( F \) of \( G \). If \( P \neq F \) then \( F \) contains a nonidentity element of order prime to the order of \( P \) which belongs to the centralizer in \( G \) of \( P \). This contradicts \( C_G(P) \subseteq P \). Therefore \( P = F \) and \( \nu(G) = \nu(\bar{G}) + 1 \).

Since \( H_1 \) and \( H_2 \) are Hall subgroups of \( G \) satisfying \( G = H_1H_2 \), the Sylow subgroup \( P \) must lie in \( H_1 \) or \( H_2 \). We may suppose that \( H_1 \) contains \( P \). If \( P = H_1 \) then \( \bar{G} = \bar{H}_2 \) and so \( \nu(\bar{G}) = \nu(\bar{H}_2) \leq \nu(H_2) \). Then \( \nu(G) = \nu(\bar{G}) + 1 \leq \nu(H_1) + \nu(H_2) \), which is what we wanted to show. If \( P \neq H_1 \) then the argument used in the preceding paragraph can be repeated to show \( \nu(H_1) = \nu(\bar{H}_1) + 1 \). It follows from this that

\[ \nu(G) = \nu(\bar{G}) + 1 \leq \nu(\bar{H}_1) + 1 + \nu(\bar{H}_2) \leq \nu(H_1) + \nu(H_2). \]

This completes the argument.

It seems interesting to ask if this theorem has a converse, in the following sense. Does a GSTG \( G \) necessarily possess pairwise permutable proper Hall subgroups \( H_1, \ldots, H_n \) satisfying \( G = H_1 \cdots H_n \) so that \( \nu(G) = \nu(H_1) + \cdots + \nu(H_n) \) holds? The answer is obviously no, since any nilpotent group is a GSTG. If we insist that the group \( G \) not be nilpotent, the answer to the question is still no. This can easily be verified using the example of an N-group which is not a Sylow tower group (see [1]; p. 638). We now mention some consequences of the theorem.
Let $G$ be a GSTG and suppose that $\mathcal{S} = \{G_1, \ldots, G_n\}$ is a Sylow basis for $G$. Since $G_1, \ldots, G_n$ are pairwise permutable Sylow subgroups of $G$ satisfying $G = G_1 \cdots G_n$, Theorem S gives

$$\lef(G) \leq \lef(G_1) + \cdots + \lef(G_n) = \pi(G).$$

Consequently, Theorem 1.7 [1] follows from Theorem S.

Now we show how the type of a GSTG can be used to improve the bound on the nilpotent length of a GSTG given by Theorem 1.7 [1]. It will be helpful to first introduce some terminology. Let $R$ be a relation on the set of all primes and let $\sigma$ denote some given set of primes. Then $\sigma$ will be called a complete $R$-symmetric set provided both $pRq$ and $qRp$ hold for all primes $p$ and $q$ belonging to $\sigma$. If $\sigma$ contains a single prime then $\sigma$ is (trivially) a complete $R$-symmetric set. It is clear from this that any set of primes can be written as a union of complete $R$-symmetric subsets. The set $\sigma$ will be called an $R$-cyclic set if $\sigma$ contains distinct primes $p, q, w$ such that $pRq, qRw$, and $wRp$ hold.

**Corollary 1.** Let $G$ be a GSTG of type $R$. If $\sigma_1, \ldots, \sigma_d$ are complete $R$-symmetric subsets of $c(G)$ such that the union of the $\sigma_i$ is $c(G)$ then $\lef(G) \leq d$.

**Proof.** Let $\mathcal{S} = \{G_1, \ldots, G_n\}$ be a Sylow basis for $G$. For each $i$, $1 \leq i \leq d$, define the subgroup $H_i$ of $G$ to be the product of all Sylow $p_i$-subgroups $G_k$ for which $p_k \in \sigma_i$. Since $pRq$ and $qRp$ hold for all distinct primes $p$ and $q$ from $\sigma_i$, each $H_i$ is seen to be a nilpotent Hall $\sigma_i$-subgroup of $G$. Since the union of the $\sigma_i$'s is $c(G)$, clearly $G = H_1 \cdots H_d$. The theorem then shows that

$$\lef(G) \leq \lef(H_1) + \cdots + \lef(H_d) = d.$$

**Corollary 2.** Let $G$ be a GSTG of type $R$. If $\sigma_1, \ldots, \sigma_d$ are disjoint $R$-cyclic subsets of $c(G)$ then $\lef(G) \leq \pi(G) - d$.

**Proof.** Let $\mathcal{S} = \{G_1, \ldots, G_n\}$ be a Sylow basis for $G$. For each integer $i$, $1 \leq i \leq d$, define the subgroup $H_i$ of $G$ to be the product of all Sylow $p_i$-subgroups $G_k$ for which $p_k \in \sigma_i$. It is clear from the definition that the $H_i$ are pairwise permutable Hall $\sigma_i$-subgroups of $G$. Since the product $H = H_1 \cdots H_d$ has a Hall complement in $G$, it suffices to show that $\lef(H) \leq \pi(H) - d$. Since the $\sigma_i$'s are disjoint sets, this will follow if $\lef(H_i) \leq \pi(H_i) - 1$ holds for each $i$, $1 \leq i \leq d$. Let $p, q,$ and $w$ be distinct primes in $\sigma_i$ satisfying $pRq, qRp$ and $wRp$. Consider a Hall $(p, q, w)$-subgroup $T$ of $H_i$. If $T$ has a normal Sylow $p$-subgroup then $pRq$ shows that $T$ has a nilpotent Hall
{p, q}-subgroup. Then Corollary 1 shows \( \ell(T) \leq \pi(T) - 1 \). It now follows from Theorem S that \( \ell(H_i) \leq \pi(H_i) - 1 \). Consequently, we may assume \( T \) has no nontrivial normal Sylow subgroup. By Lemma 1, \( T \) then has nontrivial normal subgroups \( A \) and \( B \) with \( A \cap B = 1 \). Since \( T/A \) and \( T/B \) are GSTG's of type \( R \), induction shows that \( \ell(T/A) \leq 2 \) and \( \ell(T/B) \leq 2 \). Using the fact that \( T \) is isomorphic to a subgroup of the direct product of \( T/A \) and \( T/B \) we obtain

\[
\ell(T) \leq 2 = \pi(T) - 1.
\]

Theorem S applied to \( H_i \) now gives \( \ell(H_i) \leq \pi(H_i) - 1 \). Therefore we have shown that \( \ell(H_i) \leq \pi(H_i) - 1 \) holds for arbitrary \( i \) and the assertion follows.

The next consequence of the theorem is Theorem 1.8 [1].

**Corollary 3.** Let \( G \) be GSTG with \( \ell(G) = \pi(G) \). Then \( G \) is a Sylow tower group of exactly one type \( R \), in the sense that the relation \( R \) is uniquely determined for pairs of primes \( p, q \) in \( c(G) \).

**Proof.** Let \( \mathcal{S} = \{G_1, \ldots, G_n\} \) be Sylow basis for \( G \). Define the relation \( R \) on the set of all primes as follows: \( R \) is reflexive and for distinct primes \( p \) and \( q \), \( pRq \) holds if and only if either \( p \) or \( q \) does not divide the order of \( G \) or both \( p \) and \( q \) do divide the order of \( G \) and the Sylow \( p \)-subgroup of \( G \) belonging to \( \mathcal{S} \) normalizes the Sylow \( q \)-subgroup of \( G \) belonging to \( \mathcal{S} \). Clearly \( G \) is of type \( R \). Since \( \ell(G) = \pi(G) \), Corollary 1 shows that both \( pRq \) and \( qRp \) hold for no distinct primes \( p, q \in c(G) \). In addition, Corollary 2 shows that \( pRq, qRw, \) and \( wRp \) hold for no distinct primes \( p, q, \) and \( w \) from \( c(G) \). Since either \( pRq \) or \( qRp \) holds for any primes \( p, q \in c(G) \), the restriction of \( R \) to \( c(G) \) must be a linear order. Therefore \( G \) is a Sylow tower group of type \( R \). Suppose that \( G \) is also a STG of type \( S \) and the restriction of \( S \) to \( c(G) \) differs from the restriction of \( R \) to \( c(G) \). Then \( G \) would necessarily have a nilpotent Hall \( \{p, q\} \)-subgroup for some distinct \( p, q \in c(G) \). The conjugacy of Hall \( \{p, q\} \)-subgroups in \( G \) then implies that \( \ell(G) \leq \pi(G) - 1 \), a contradiction. Therefore \( G \) is a STG of exactly one type, in the sense mentioned.

We next give an example to show that the nilpotent length of a GSTG cannot be found from the type alone. Let \( A \) be the holomorph of a cyclic group of order 7 and let \( B \) denote the Hall \( \{7, 3\} \)-subgroup of \( A \). Define the group \( G_1 \) as the direct product of \( A \) and a symmetric group of degree 3 and define \( G_2 \) as the wreath product of \( B \) by a cyclic group of order 2. Both \( G_1 \) and \( G_2 \) are Sylow tower groups of type \( 7 < 3 < 2 \) and no distinct Sylow subgroups of \( G_1 \) or \( G_2 \) centralize one another. Hence, for a given relation \( R \) on the set
of all primes, $G_1$ is a GSTG of type $R$ if and only if $G_2$ is a GSTG of type $R$. Yet the nilpotent length of $G_1$ is 2 and the nilpotent length of $G_2$ is 3.

**Theorem T.** Let $G$ be a GSTG with nilpotent length $k$. Then $G$ contains a Hall subgroup which is a Sylow tower group and has nilpotent length $k$.

*Proof.* By Theorem 1.8 [1] it is sufficient to show that $G$ contains a Hall subgroup $L$ with $\pi(L) = k$. We proceed by induction on the order of $G$.

Suppose $G$ contains a proper subgroup $W$ with $\pi(W) = k$. The induction hypothesis then shows that $W$ contains a Hall subgroup $T$ with $\pi(T) = k$. Choose a Hall subgroup $L$ of $G$ with $T \leq L$ and $c(T) = c(L)$. Then $\pi(G) = k = \pi(T) \leq \pi(L) \leq \pi(G)$ shows that $\pi(L) = \pi(G) = k$. This proves the theorem in the case where $G$ contains a proper subgroup with nilpotent length $k$. Now suppose that every proper subgroup of $G$ has nilpotent length strictly less than $k$.

Since $G$ is a GSTG, either $G$ possesses nontrivial normal subgroups $A$ and $B$ with $A \cap B = 1$ or $G$ contains a nontrivial normal Sylow subgroup $P$ with $C_G(P) \leq P$. We consider these possibilities separately. First suppose that $A$ and $B$ are distinct minimal normal subgroups of $G$. If the Frattini subgroup $\phi$ of $G$ is trivial then $G$ contains a maximal subgroup $M_1$ not containing $A$ and a maximal subgroup $M_2$ not containing $B$. Then $M_1$ complements $A$ in $G$ and $M_2$ complements $B$ in $G$. Since we have assumed that all proper subgroups of $G$ have nilpotent length less than $k$, the isomorphism of $G/A$ and $M_1$ gives $\pi(G/A) < k$. Similarly one sees that $\pi(G/B) < k$. Since $G$ is isomorphic to a subgroup of the direct product of $G/A$ and $G/B$, it follows that $\pi(G) = \max\{\pi(G/A), \pi(G/B)\}$ is less than $k$, a contradiction. Therefore $G$ has nontrivial Frattini subgroup. Since $\pi(G/\phi) = \pi(G) = k$, the induction hypothesis shows that $G$ contains a Hall subgroup $L$ satisfying $\pi(L/\phi) = \pi(L/\phi) = k$. Now

$$k = \pi(L/\phi) \leq \pi(L) \leq \pi(G) = k$$

shows $\pi(L) = k$. Hence $L = G$. Since the Frattini subgroup of $G$ contains no Sylow subgroup of $G$, $\pi(G) = \pi(L) = \pi(L/\phi) = k$. Therefore $\pi(G) = \pi(G) = k$, which completes the argument in this case.

Now suppose $G$ contains a nontrivial normal Sylow subgroup $P$ with $C_G(P) \leq P$. It follows that $P$ must be the Fitting subgroup of $G$. Therefore $\pi(G) = \pi(G/P) + 1$ or $G = P$. In the latter case the theorem is trivially true. If $\pi(G) = \pi(G/P) + 1$, the induction
hypothesis shows that $G/P$ contains a nontrivial Hall subgroup $L/P$ satisfying $\lhd(L/P) = \pi(L/P) = k - 1 = \lhd(G/P)$. Clearly $L$ is then a Hall subgroup of $G$ with $\pi(L) = k$. Since $C_L(P) \lhd P$, $P$ must be the Fitting subgroup of $L$. Hence $\lhd(L) = \lhd(L/P) + 1$. Therefore

$$\lhd(L) = k = \pi(L).$$

This completes the proof of the theorem.

Let $G$ be a given GSTG and suppose $\mathcal{S}$ is a Sylow basis for $G$. Define the relation $R$ on the set of all primes as follows: for any primes $p$ and $q$ (possibly equal), $pRq$ holds if and only if $p \in c(G)$ or $q \in c(G)$ or both $p$ and $q$ belong to $c(G)$ and the Sylow $p$-subgroup of $G$ in $\mathcal{S}$ normalizes the Sylow $q$-subgroup of $G$ in $\mathcal{S}$. Clearly $G$ is a GSTG of type $\tau$. If $H$ is a Hall subgroup of $G$ which is a Sylow tower group and $H$ satisfies $\lhd(H) = \pi(H) = \lhd(G)$, then the restriction of $R$ to $c(H)$ is a transitive relation (see the proof of Corollary 3). This leads to the following bound for the nilpotent length of $G$ in terms of the relation $R$ defined above. The nilpotent length of the GSTG $G$ cannot exceed $n$, where $n$ is the largest integer such that the restriction of $R$ to some subset of $c(G)$ having $n$ elements is a transitive relation.

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