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# GENERALIZED SYLOW TOWER GROUPS, II

J. B. DERR AND N. P. MUKHERJEE

**A well-known result of P. Hall shows that finite solvable groups may be characterized by a permutability requirement on Sylow subgroups. The notion of a generalized Sylow tower group (*GSTG*) arises when this permutability condition on Sylow subgroups is replaced by a suitable normalizer condition. In an earlier papar, one of the authors showed that the nilpotent length of a *GSTG* cannot exceed the number of distinct primes which divide the order of the group. The present investigation utilizes the 'type' of a *GSTG* to obtain improved bounds for the nilpotent length of a *GSTG*. It is also shown that a *GSTG* with nilpotent length $n$ possesses a Hall subgroup of nilpotent length $n$ which is a Sylow tower group.**

Let $G$ be a finite group with order $p_1^{a_1} \cdots p_r^{a_r}$, where $p_1, \cdots, p_r$ are distinct primes and $a_1, \cdots, a_r$ are positive integers. For each integer $i$, $1 \leqslant i \leqslant r$, let $G_i$ denote a Sylow $p_i$-subgroup of $G$. The collection of subgroups $\mathscr{S} = \{G_1, \cdots, G_r\}$ is then called a complete set of Sylow subgroups of $G$. If the elements of $\mathscr{S}$ are pairwise permutable as subgroups (that is, if $G_iG_j = G_jG_i$ holds for all $i$ and $j$) then $\mathscr{S}$ will be called a Sylow basis for $G$. The notion of a generalized Sylow tower group arises when the permutability condition for a Sylow basis is replaced by a normalizer condition. Thus, we say that a finite group $G$ is a generalized Sylow tower group ($GSTG$) if and only if some complete set of Sylow subgroups $\mathscr{S}$ of $G$ satisfies the normalizer condition ($N$): if $G_i$ and $G_j$ are distinct elements of $\mathscr{S}$, at least one of these subgroups normalizes the other. It should be noted that not every complete set of Sylow subgroups of a $GSTG$ need satisfy condition ($N$).

A well-known result of P. Hall states that a finite group is solvable if and only if the group possesses a Sylow basis. If a complete set of Sylow subgroups $\mathscr{S}$ of a group $G$ satisfies condition ($N$) then any two elements of $\mathscr{S}$ are permutable as subgroups and $\mathscr{S}$ is a Sylow basis for $G$. Consequently every generalized Sylow tower group must be solvable.

A finite group $G$ is called a Sylow tower group ($STG$) if every nontrivial epimorphic image of $G$ possesses a nontrivial normal Sylow subgroup. Equivalently, the group $G$ is a $STG$ if the prime divisors $p_1, \cdots, p_r$ of the order of $G$ can be labelled in such a way that a Sylow $p_i$-subgroup of $G$ normalizes a Sylow $p_j$-subgroup of $G$ whenever $i > j$. It is clear from this definition that a Sylow tower group is necessarily a generalized Sylow tower group. An example of a *GSTG* which is not a Sylow tower group was given in ([1]; p. 638).

In order to handle generalized Sylow tower groups it will be necessary to introduce the 'type' of a *GSTG*. Suppose $G$ is some given *GSTG* and let $\mathscr{T}$ be a Sylow basis for $G$. Since any two Sylow bases for $G$ are conjugate ([3]; p. 665), $\mathscr{T}$ satisfies the normalizer condition $(N)$. Let $R$ be a relation on the set of all primes with the property that either $pRq$ or $qRp$ (or both) holds for any primes $p$ and $q$. If the Sylow $p_i$-subgroup of $G$ in $\mathscr{T}$ normalizes the Sylow $p_j$-subgroup of $G$ in $\mathscr{T}$ whenever $p_iRp_j$ holds, then $G$ will be called a *GSTG* of type $R$. It follows directly from the conjugacy of Sylow bases that the type of a *GSTG* is independent of the choice of a Sylow basis. It should be noted that a group can be a *GSTG* of more than one type.

It was shown in [1] that the class of all generalized Sylow tower groups of a given type $R$ is a formation. In addition, any subgroup of a *GSTG* of type $R$ was shown to be a *GSTG* of type $R$. We list the main results about *GSTG*'s in [1] for easy reference.

THEOREM 1.7 [1]. *If $G$ is a GSTG then the nilpotent length of $G$ does not exceed the number of distinct prime divisors of the order of $G$.*

THEOREM 1.8 [1]. *If $G$ is a GSTG and the nilpotent length of $G$ is equal to the number of distinct prime divisors of the order of $G$ then $G$ is a Sylow tower group.*

All groups mentioned are assumed to be finite. The following notations will be used. For a group $G$

$c(G)$ denotes the set of distinct prime divisors of the order of $G$

$\pi(G)$ denotes the number of distinct prime divisors of the order of $G$

$\ell(G)$ denotes the nilpotent (Fitting) length of $G$.

If $H$ is a subgroup of $G$ then $N_G(H)$ means the normalizer of $H$ in $G$ and $C_G(H)$ means the centralizer of $H$ in $G$.

If $p_i$ is a prime, $G_i$ will denote a Sylow $p_i$-subgroup of $G$.

The following lemma will be used in several of our arguments.

LEMMA 1. *If $G$ is a nontrivial GSTG then at least one of the following holds:*

(1) *$G$ contains a nontrivial normal Sylow subgroup $P$ with $C_G(P) \subseteqq P$*

(2) *$G$ contains nontrivial normal subgroups having relatively prime orders.*

*Proof.* Let $G$ be a $GSTG$ and suppose that $\mathscr{S} = \{G_1, \cdots, G_n\}$ is a Sylow basis for $G$. Since a $GSTG$ is necessarily solvable, $G$ possesses a nontrivial minimal normal subgroup $M$ with order a power of some prime $q$. Let $Q$ denote the maximum normal $q$-subgroup of $G$ and suppose that $G_1$ is the Sylow $q$-subgroup of $G$ belonging to $\mathscr{S}$. Since $\mathscr{S}$ satisfies the normalizer condition $(N)$, either

$$G_1 \subseteqq N_G(G_k) \quad \text{or} \quad G_k \subseteqq N_G(G_1)$$

must hold for each integer $k$, $1 \leqslant k \leqslant n$. We distinguish two cases.

First suppose that $G_1 \subseteqq N_G(G_k)$ holds for some $k$, $1 < k \leqslant n$. Since $Q$ is normal in $G$ and has order prime to the order of $G_k$, $G_k$ centralizes $Q$. Hence $C = C_G(Q)$ is not a $q$-group. Set $Q_0 = Q \cap C_G(Q)$ and consider the factor group $C/Q_0$. Since $C/Q_0$ is a nontrivial solvable group, $C/Q_0$ contains a nontrivial minimal normal subgroup $L/Q_0$ with order a power of some prime $p$. Let $T/Q_0$ be the maximum normal $p$-subgroup of $C/Q_0$. Since $Q$ is the maximum normal $q$-subgroup of $G$ it follows that $p \neq q$. If $N$ is a Sylow $p$-subgroup of $T$ then $T$ is the direct product of $N$ and $Q_0$. The normality of $T$ in $G$ then implies the normality of $N$ in $G$. Therefore $G$ has nontrivial normal subgroups with relatively prime orders.

Now suppose that $G_1 \not\subseteqq N_G(G_k)$ holds for all integers $k$, $1 < k \leqslant n$. Since $\mathscr{S}$ satisfies the normalizer condition $(N)$, $G_k \subseteqq G_G(G_1)$ must then hold for all integers $k$. Thus $G_1$ is a normal Sylow $q$-subgroup of $G$. Then $G_1 \cap C_G(G_1)$ is a normal Sylow $q$-subgroup of $C_G(G_1)$ and $C_G(G_1)$ has a normal $q$-complement $W$. If $W$ is nontrivial then $W$ is a normal subgroup of $G$ having $q'$-order and (2) holds. If $W$ is trivial then $G_1 \cap C_G(G_1) = C_G(G_1)$. In this case $C_G(G_1) \subseteqq G_1$ and (1) holds.

THEOREM S. *Let $G$ be a $GSTG$. If $H_1, \cdots, H_n$ are pairwise permutable Hall subgroups of $G$ with $G = H_1 \cdots H_n$, then the nilpotent length of $G$ does not exceed the sum of the nilpotent lengths of $H_1, \cdots, H_n$.*

*Proof.* (By induction on the order of $G$.) Since the product $H_2 \cdots H_n$ is a Hall subgroup of $G$ permutable with $H_1$ we may assume that $n = 2$. First suppose that $G$ possesses nontrivial normal subgroups $A$ and $B$ with $A \cap B = 1$. Then $G$ is isomorphic to a subgroup of the direct of $G/A$ and $G/B$. Hence

$$\ell(G) = \max\{\ell(G/A), \ell(G/B)\}$$

and it suffices to show that

$$\ell(G/A) \leqslant \ell(H_1) + \ell(H_2) \quad \text{and} \quad \ell(G/B) \leqslant \ell(H_1) + \ell(H_2) .$$

Since $G/A$ is the product of the permutable Hall subgroups $H_1A/A$ and $H_2A/A$ of $G/A$, the induction hypothesis gives

$$\ell(G/A) \leqslant \ell(H_1A/A) + \ell(H_2A/A) .$$

Since $H_1A/A$ and $H_2A/A$ are epimorphic images of $H_1$ and $H_2$ (respectively), we know that $\ell(H_iA/A) \leqslant \ell(H_i)$ for $i = 1, 2$. Therefore $\ell(G/A) \leqslant \ell(H_1) + \ell(H_2)$. The same argument applied to $G/B$ will show $\ell(G/B) \leqslant \ell(H_1) + \ell(H_2)$. This verifies the theorem in this case.

Now suppose that $G$ possesses a unique minimal normal subgroup. Then Lemma 1 shows that $G$ contains a nontrivial normal Sylow subgroup $P$ with $C_G(P) \subseteqq P$. Since the theorem is trivially true if $G = P$ we may assume this is not the case. Set

$$\bar{G} = G/P, \bar{H}_1 = H_1P/P, \bar{H}_2 = H_2P/P$$

and consider the nontrivial $GSTG$ $\bar{G}$. The induction hypothesis applied to $\bar{G} = \bar{H}_1\bar{H}_2$ gives $\ell(\bar{G}) \leqslant \ell(\bar{H}_1) + \ell(\bar{H}_2)$. We first observe that $\ell(G) = \ell(\bar{G}) + 1$. Since $P$ is a nilpotent normal subgroup of $G$, $P$ must lie in the Fitting subgroup $F$ of $G$. If $P \neq F$ then $F$ contains a nonidentity element of order prime to the order of $P$ which belongs to the centralizer in $G$ of $P$. This contradicts $C_G(P) \subseteqq P$. Therefore $P = F$ and $\ell(G) = \ell(\bar{G}) + 1$.

Since $H_1$ and $H_2$ are Hall subgroups of $G$ satisfying $G = H_1H_2$, the Sylow subgroup $P$ must lie in $H_1$ or $H_2$. We may suppose that $H_1$ contains $P$. If $P = H_1$ then $\bar{G} = \bar{H}_2$ and so $\ell(\bar{G}) = \ell(\bar{H}_2) \leqslant \ell(H_2)$. Then $\ell(G) = \ell(\bar{G}) + 1 \leqslant \ell(H_2) + \ell(H_1)$, which is what we wanted to show. If $P \neq H_1$ then the argument used in the preceding paragraph can be repeated to show $\ell(H_1) = \ell(\bar{H}_1) + 1$. It follows from this that

$$\ell(G) = \ell(\bar{G}) + 1 \leqslant \ell(\bar{H}_1) + 1 + \ell(\bar{H}_2) \leqslant \ell(H_1) + \ell(H_2) .$$

This completes the argument.

It seems interesting to ask if this theorem has a converse, in the following sense. Does a $GSTG$ $G$ necessarily possess pairwise permutable proper Hall subgroups $H_1, \cdots, H_n$ satisfying $G = H_1 \cdots H_n$ so that $\ell(G) = \ell(H_1) + \cdots + \ell(H_n)$ holds? The answer is obviously no, since any nilpotent group is a $GSTG$. If we insist that the group $G$ not be nilpotent, the answer to the question is still no. This can easily be verified using the example of an $N$-group which is not a Sylow tower group (see [1]; p. 638). We now mention some consequences of the theorem.

Let $G$ be a $GSTG$ and suppose that $\mathscr{S} = \{G_1, \cdots, G_n\}$ is a Sylow basis for $G$. Since $G_1, \cdots, G_n$ are pairwise permutable Sylow subgroups of $G$ satisfying $G = G_1 \cdots G_n$, Theorem S gives

$$\ell(G) \leqslant \ell(G_1) + \cdots + \ell(G_n) = \pi(G) .$$

Consequently Theorem 1.7 [1] follows from Theorem S.

Now we show how the type of a $GSTG$ can be used to improve the bound on the nilpotent length of a $GSTG$ given by Theorem 1.7 [1]. It will be helpful to first introduce some terminology. Let $R$ be a relation on the set of all primes and let $\sigma$ denote some given set of primes. Then $\sigma$ will be called a complete $R$-symmetric set provided both $pRq$ and $qRp$ hold for all primes $p$ and $q$ belonging to $\sigma$. If $\sigma$ contains a single prime then $\sigma$ is (trivially) a complete $R$-symmetric set. It is clear from this that any set of primes can be written as a union of complete $R$-symmetric subsets. The set $\sigma$ will be called an $R$-cyclic set if $\sigma$ contains distinct primes $p$, $q$, and $w$ such that $pRq$, $qRw$, and $wRp$ hold.

COROLLARY 1. *Let $G$ be a $GSTG$ of type $R$. If $\sigma_1, \cdots, \sigma_d$ are complete $R$-symmetric subsets of $c(G)$ such that the union of the $\sigma_i$ is $c(G)$ then $\ell(G) \leqslant d$.*

*Proof.* Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be a Sylow basis for $G$. For each $i$, $1 \leqslant i \leqslant d$, define the subgroup $H_i$ of $G$ to be the product of all Sylow $p_k$-subgroups $G_k$ for which $p_k \in \sigma_i$. Since $pRq$ and $qRp$ hold for all distinct primes $p$ and $q$ from $\sigma_i$, each $H_i$ is seen to be a nilpotent Hall $\sigma_i$-subgroup of $G$. Since the union of the $\sigma_i$'s is $c(G)$, clearly $G = H_1 \cdots H_d$. The theorem then shows that

$$\ell(G) \leqslant \ell(H_1) + \cdots + \ell(H_d) = d .$$

COROLLARY 2. *Let $G$ be a $GSTG$ of type $R$. If $\sigma_1, \cdots, \sigma_d$ are disjoint $R$-cyclic subsets of $c(G)$ then $\ell(G) \leqslant \pi(G) - d$.*

*Proof.* Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be a Sylow basis for $G$. For each integer $i$, $1 \leqslant i \leqslant d$, define the subgroup $H_i$ of $G$ to be the product of all Sylow $p_k$-subgroups $G_k$ for which $p_k \in \sigma_i$. It is clear from the definition that the $H_i$ are pairwise permutable Hall $\sigma_i$-subgroups of $G$. Since the product $H = H_1 \cdots H_d$ has a Hall complement in $G$, it suffices to show that $\ell(H) \leqslant \pi(H) - d$. Since the $\sigma_i$'s are disjoint sets, this will follow if $\ell(H_i) \leqslant \pi(H_i) - 1$ holds for each $i$, $1 \leqslant i \leqslant d$. Let $p$, $q$, and $w$ be distinct primes in $\sigma_i$ satisfying $pRq$, $qRp$ and $wRp$. Consider a Hall $\{p, q, w\}$-subgroup $T$ of $H_i$. If $T$ has a normal Sylow $p$-subgroup then $pRq$ shows that $T$ has a nilpotent Hall

$\{p, q\}$-subgroup. Then Corollary 1 shows $\ell(T) \leqslant \pi(T) - 1$. It now follows from Theorem S that $\ell(H_i) \leqslant \pi(H_i) - 1$. Consequently we may assume $T$ has no nontrivial normal Sylow subgroup. By Lemma 1, $T$ then has nontrivial normal subgroups $A$ and $B$ with $A \cap B = 1$. Since $T/A$ and $T/B$ are $GSTG$'s of type $R$, induction shows that $\ell(T/A) \leqslant 2$ and $\ell(T/B) \leqslant 2$. Using the fact that $T$ is isomorphic to a subgroup of the direct product of $T/A$ and $T/B$ we obtain

$$\ell(T) \leqslant 2 = \pi(T) - 1 .$$

Theorem S applied to $H_i$ now gives $\ell(H_i) \leqslant \pi(H_i) - 1$. Therefore we have shown that $\ell(H_i) \leqslant \pi(H_i) - 1$ holds for arbitrary $i$ and the assertion follows.

The next consequence of the theorem is Theorem 1.8 [1].

COROLLARY 3. *Let $G$ be GSTG with $\ell(G) = \pi(G)$. Then $G$ is a Sylow tower group of exactly one type $R$, in the sense that the relation $R$ is uniquely determined for pairs of primes $p$, $q$ in $c(G)$.*

*Proof.* Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be Sylow basis for $G$. Define the relation $R$ on the set of all primes as follows: $R$ is reflexive and for distinct primes $p$ and $q$, $pRq$ holds if and only if either $p$ or $q$ does not divide the order of $G$ or both $p$ and $q$ do divide the order of $G$ and the Sylow $p$-subgroup of $G$ belonging to $\mathscr{S}$ normalizes the Sylow $q$-subgroup of $G$ belonging to $\mathscr{S}$. Clearly $G$ is of type $R$. Since $\ell(G) = \pi(G)$, Corollary 1 shows that both $pRq$ and $qRp$ hold for no distinct primes $p, q \in c(G)$. In addition, Corollary 2 shows that $pRq$, $qRw$, and $wRp$ hold for no distinct primes $p$, $q$, and $w$ from $c(G)$. Since either $pRq$ or $qRp$ holds for any primes $p$, $q \in c(G)$, the restriction of $R$ to $c(G)$ must be a linear order. Therefore $G$ is a Sylow tower group of type $R$. Suppose that $G$ is also a $STG$ of type $S$ and the restriction of $S$ to $c(G)$ differs from the restriction of $R$ to $c(G)$. Then $G$ would necessarily have a nilpotent Hall $\{p, q\}$-subgroup for some distinct $p, q \in c(G)$. The conjugacy of Hall $\{p, q\}$-subgroups in $G$ then implies that $\ell(G) \leqslant \pi(G) - 1$, a contradiction. Therefore $G$ is a $STG$ of exactly one type, in the sense mentioned.

We next give an example to show that the nilpotent length of a $GSTG$ cannot be found from the type alone. Let $A$ be the holomorph of a cyclic group of order 7 and let $B$ donote the Hall $\{7, 3\}$-subgroup of $A$. Define the group $G_1$ as the direct product of $A$ and a symmetric group of degree 3 and define $G_2$ as the wreath product of $B$ by a cyclic group of order 2. Both $G_1$ and $G_2$ are Sylow tower groups of type $7 < 3 < 2$ and no distinct Sylow subgroups of $G_1$ or $G_2$ centralize one another. Hence, for a given relation $R$ on the set of all primes, $G_1$ is a $GSTG$ of type $R$ if and only if $G_2$ is a $GSTG$ of type $R$. Yet the nilpotent length of $G_1$ is 2 and the nilpotent length of $G_2$ is 3.

THEOREM T. *Let $G$ be a $GSTG$ with nilpotent length $k$. Then $G$ contains a Hall subgroup which is a Sylow tower group and has nilpotent length $k$.*

*Proof.* By Theorem 1.8 [1] it is sufficient to show that $G$ contains a Hall subgroup $L$ with $\ell(L) = \pi(L) = k$. We proceed by induction on the order of $G$.

Suppose $G$ contains a proper subgroup $W$ with $\ell(W) = k$. The induction hypothesis then shows that $W$ contains a Hall subgroup $T$ with $\ell(T) = \pi(T) = k$. Choose a Hall subgroup $L$ of $G$ with $T \subseteqq L$ and $c(T) = c(L)$. Then $\ell(G) = k = \ell(T) \leqslant \ell(L) \leqslant \ell(G)$ shows that $\ell(L) = \pi(L) = k$. This proves the theorem in the case where $G$ contains a proper subgroup with nilpotent length $k$. Now suppose that every proper subgroup of $G$ has nilpotent length strictly less than $k$.

Since $G$ is a $GSTG$, either $G$ possesses nontrivial normal subgroups $A$ and $B$ with $A \cap B = 1$ or $G$ contains a nontrivial normal Sylow subgroup $P$ with $C_G(P) \subseteqq P$. We consider these possibilities separately. First suppose that $A$ and $B$ are distinct minimal normal subgroups of $G$. If the Frattini subgroup $\phi$ of $G$ is trivial then $G$ contains a maximal subgroup $M_1$ not containing $A$ and a maximal subgroup $M_2$ not containing $B$. Then $M_1$ complements $A$ in $G$ and $M_2$ complements $B$ in $G$. Since we have assumed that all proper subgroups of $G$ have nilpotent length less than $k$, the isomorphism of $G/A$ and $M_1$ gives $\ell(G/A) < k$. Similarly one sees that $\ell(G/B) < k$. Since $G$ is isomorphic to a subgroup of the direct product of $G/A$ and $G/B$, it follows that $\ell(G) = \max\{\ell(G/A), \ell(G/B)\}$ is less than $k$, a contradiction. Therefore $G$ has nontrivial Frattini subgroup. Since $\ell(G/\phi) = \ell(G) = k$, the induction hypothesis shows that $G$ contains a Hall subgroup $L$ satisfying $\ell(L\phi/\phi) = \pi(L\phi/\phi) = k$. Now

$$k = \ell(L\phi/\phi) \leqslant \ell(L) \leqslant \ell(G) = k$$

shows $\ell(L) = k$. Hence $L = G$. Since the Frattini subgroup of $G$ contains no Sylow subgroup of $G$, $\pi(G) = \pi(L) = \pi(L\phi/\phi) = k$. Therefore $\ell(G) = \pi(G) = k$, which completes the argument in this case.

Now suppose $G$ contains a nontrivial normal Sylow subgroup $P$ with $C_G(P) \subseteqq P$. It follows that $P$ must be the Fitting subgroup of $G$. Therefore $\ell(G) = \ell(G/P) + 1$ or $G = P$. In the latter case the theorem is trivially true. If $\ell(G) = \ell(G/P) + 1$, the induction hypothesis shows that $G/P$ contains a nontrivial Hall subgroup $L/P$ satisfying $\ell(L/P) = \pi(L/P) = k - 1 = \ell(G/P)$. Clearly $L$ is then a Hall subgroup of $G$ with $\pi(L) = k$. Since $C_G(P) \subseteqq P$, $P$ must be the Fitting subgroup of $L$. Hence $\ell(L) = \ell(L/P) + 1$. Therefore

$$\ell(L) = k = \pi(L) .$$

This completes the proof of the theorem.

Let $G$ be a given $GSTG$ and suppose $\mathscr{S}$ is a Sylow basis for $G$. Define the relation $R$ on the set of all primes as follows: for any primes $p$ and $q$ (possibly equal), $pRq$ holds if and only if $p \notin c(G)$ or $q \notin c(G)$ or both $p$ and $q$ belong to $c(G)$ and the Sylow $p$-subgroup of $G$ in $\mathscr{S}$ normalizes the Sylow $q$-subgroup of $G$ in $\mathscr{S}$. Clearly $G$ is a $GSTG$ of type $R$. If $H$ is a Hall subgroup of $G$ which is a Sylow tower group and $H$ satisfies $\ell(H) = \pi(H) = \ell(G)$, then the restriction of $R$ to $c(H)$ is a transitive relation (see the proof of Corollary 3). This leads to the following bound for the nilpotent length of $G$ in terms of the relation $R$ defined above. The nilpotent length of the $GSTG$ $G$ cannot exceed $n$, where $n$ is the largest integer such that the restriction of $R$ to some subset of $c(G)$ having $n$ elements is a transitive relation.

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