Pacific Journal of Mathematics

GENERALIZED SYLOW TOWER GROUPS. II

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Vol. 45, No. 2 October 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

GENERALIZED SYLOW TOWER GROUPS, II

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A well-known result of P. Hall shows that finite solvable groups may be characterized by a permutability requirement on Sylow subgroups. The notion of a generalized Sylow tower group (*GSTG*) arises when this permutability condition on Sylow subgroups is replaced by a suitable normalizer condition. In an earlier papar, one of the authors showed that the nilpotent length of a *GSTG* cannot exceed the number of distinct primes which divide the order of the group. The present investigation utilizes the 'type' of a *GSTG* to obtain improved bounds for the nilpotent length of a *GSTG*. It is also shown that a *GSTG* with nilpotent length n possesses a Hall subgroup of nilpotent length n which is a Sylow tower group.

Let G be a finite group with order $p_1^{a_1} \cdots p_r^{a_r}$, where $p_1, \cdots, p_r$ are distinct primes and $a_1, \cdots, a_r$ are positive integers. For each integer i, $1 \leqslant i \leqslant r$, let G_i denote a Sylow p_i-subgroup of G. The collection of subgroups $\mathscr{S} = \{G_1, \cdots, G_r\}$ is then called a complete set of Sylow subgroups of G. If the elements of $\mathscr{S}$ are pairwise permutable as subgroups (that is, if $G_iG_j = G_jG_i$ holds for all i and j) then $\mathscr{S}$ will be called a Sylow basis for G. The notion of a generalized Sylow tower group arises when the permutability condition for a Sylow basis is replaced by a normalizer condition. Thus, we say that a finite group G is a generalized Sylow tower group (*GSTG*) if and only if some complete set of Sylow subgroups $\mathscr{S}$ of G satisfies the normalizer condition (N): if G_i and G_j are distinct elements of $\mathscr{S}$, at least one of these subgroups normalizes the other. It should be noted that not every complete set of Sylow subgroups of a *GSTG* need satisfy condition (N).

A well-known result of P. Hall states that a finite group is solvable if and only if the group possesses a Sylow basis. If a complete set of Sylow subgroups $\mathscr{S}$ of a group G satisfies condition (N) then any two elements of $\mathscr{S}$ are permutable as subgroups and $\mathscr{S}$ is a Sylow basis for G. Consequently every generalized Sylow tower group must be solvable.

A finite group G is called a Sylow tower group (*STG*) if every nontrivial epimorphic image of G possesses a nontrivial normal Sylow subgroup. Equivalently, the group G is a *STG* if the prime divisors $p_1, \cdots, p_r$ of the order of G can be labelled in such a way that a Sylow p_i-subgroup of G normalizes a Sylow p_j-subgroup of G whenever $i > j$. It is clear from this definition that a Sylow tower group is necessarily a generalized Sylow tower group. An example of a $GSTG$ which is not a Sylow tower group was given in ([1]; p. 638).

In order to handle generalized Sylow tower groups it will be necessary to introduce the 'type' of a $GSTG$. Suppose G is some given $GSTG$ and let $\mathscr{T}$ be a Sylow basis for G. Since any two Sylow bases for G are conjugate ([3]; p. 665), $\mathscr{T}$ satisfies the normalizer condition (N). Let R be a relation on the set of all primes with the property that either pRq or qRp (or both) holds for any primes p and q. If the Sylow p_i-subgroup of G in $\mathscr{T}$ normalizes the Sylow p_j-subgroup of G in $\mathscr{T}$ whenever p_iRp_j holds, then G will be called a $GSTG$ of type R. It follows directly from the conjugacy of Sylow bases that the type of a $GSTG$ is independent of the choice of a Sylow basis. It should be noted that a group can be a $GSTG$ of more than one type.

It was shown in [1] that the class of all generalized Sylow tower groups of a given type R is a formation. In addition, any subgroup of a $GSTG$ of type R was shown to be a $GSTG$ of type R. We list the main results about $GSTG$'s in [1] for easy reference.

THEOREM 1.7 [1]. *If G is a $GSTG$ then the nilpotent length of G does not exceed the number of distinct prime divisors of the order of G.*

THEOREM 1.8 [1]. *If G is a $GSTG$ and the nilpotent length of G is equal to the number of distinct prime divisors of the order of G then G is a Sylow tower group.*

All groups mentioned are assumed to be finite. The following notations will be used. For a group G

$c(G)$ denotes the set of distinct prime divisors of the order of G

$\pi(G)$ denotes the number of distinct prime divisors of the order of G

$\ell(G)$ denotes the nilpotent (Fitting) length of G.

If H is a subgroup of G then $N_G(H)$ means the normalizer of H in G and $C_G(H)$ means the centralizer of H in G.

If p_i is a prime, G_i will denote a Sylow p_i-subgroup of G.

The following lemma will be used in several of our arguments.

LEMMA 1. *If G is a nontrivial $GSTG$ then at least one of the following holds*:

(1) *G contains a nontrivial normal Sylow subgroup P with $C_G(P) \subseteqq P$*

(2) *G contains nontrivial normal subgroups having relatively prime orders.*

Proof. Let G be a $GSTG$ and suppose that $\mathscr{S} = \{G_1, \cdots, G_n\}$ is a Sylow basis for G. Since a $GSTG$ is necessarily solvable, G possesses a nontrivial minimal normal subgroup M with order a power of some prime q. Let Q denote the maximum normal q-subgroup of G and suppose that G_1 is the Sylow q-subgroup of G belonging to $\mathscr{S}$. Since $\mathscr{S}$ satisfies the normalizer condition (N), either

$$G_1 \subseteqq N_G(G_k) \quad \text{or} \quad G_k \subseteqq N_G(G_1)$$

must hold for each integer k, $1 \leqslant k \leqslant n$. We distinguish two cases.

First suppose that $G_1 \subseteqq N_G(G_k)$ holds for some k, $1 < k \leqslant n$. Since Q is normal in G and has order prime to the order of G_k, G_k centralizes Q. Hence $C = C_G(Q)$ is not a q-group. Set $Q_0 = Q \cap C_G(Q)$ and consider the factor group C/Q_0. Since C/Q_0 is a nontrivial solvable group, C/Q_0 contains a nontrivial minimal normal subgroup L/Q_0 with order a power of some prime p. Let T/Q_0 be the maximum normal p-subgroup of C/Q_0. Since Q is the maximum normal q-subgroup of G it follows that $p \neq q$. If N is a Sylow p-subgroup of T then T is the direct product of N and Q_0. The normality of T in G then implies the normality of N in G. Therefore G has nontrivial normal subgroups with relatively prime orders.

Now suppose that $G_1 \nsubseteq N_G(G_k)$ holds for all integers k, $1 < k \leqslant n$. Since $\mathscr{S}$ satisfies the normalizer condition (N), $G_k \subseteqq G_G(G_1)$ must then hold for all integers k. Thus G_1 is a normal Sylow q-subgroup of G. Then $G_1 \cap C_G(G_1)$ is a normal Sylow q-subgroup of $C_G(G_1)$ and $C_G(G_1)$ has a normal q-complement W. If W is nontrivial then W is a normal subgroup of G having q'-order and (2) holds. If W is trivial then $G_1 \cap C_G(G_1) = C_G(G_1)$. In this case $C_G(G_1) \subseteqq G_1$ and (1) holds.

THEOREM S. *Let G be a $GSTG$. If $H_1, \cdots, H_n$ are pairwise permutable Hall subgroups of G with $G = H_1 \cdots H_n$, then the nilpotent length of G does not exceed the sum of the nilpotent lengths of $H_1, \cdots, H_n$.*

Proof. (By induction on the order of G.) Since the product $H_2 \cdots H_n$ is a Hall subgroup of G permutable with H_1 we may assume that $n = 2$. First suppose that G possesses nontrivial normal subgroups A and B with $A \cap B = 1$. Then G is isomorphic to a subgroup of the direct of G/A and G/B. Hence

$$\ell(G) = \max\{\ell(G/A), \ell(G/B)\}$$

and it suffices to show that

$$\ell(G/A) \leqslant \ell(H_1) + \ell(H_2) \quad \text{and} \quad \ell(G/B) \leqslant \ell(H_1) + \ell(H_2)\,.$$

Since G/A is the product of the permutable Hall subgroups H_1A/A and H_2A/A of G/A, the induction hypothesis gives

$$\ell(G/A) \leqslant \ell(H_1A/A) + \ell(H_2A/A)\,.$$

Since H_1A/A and H_2A/A are epimorphic images of H_1 and H_2 (respectively), we know that $\ell(H_iA/A) \leqslant \ell(H_i)$ for $i = 1, 2$. Therefore $\ell(G/A) \leqslant \ell(H_1) + \ell(H_2)$. The same argument applied to G/B will show $\ell(G/B) \leqslant \ell(H_1) + \ell(H_2)$. This verifies the theorem in this case.

Now suppose that G possesses a unique minimal normal subgroup. Then Lemma 1 shows that G contains a nontrivial normal Sylow subgroup P with $C_G(P) \subseteqq P$. Since the theorem is trivially true if $G = P$ we may assume this is not the case. Set

$$\bar{G} = G/P,\ \bar{H}_1 = H_1P/P,\ \bar{H}_2 = H_2P/P$$

and consider the nontrivial $GSTG$ $\bar{G}$. The induction hypothesis applied to $\bar{G} = \bar{H}_1\bar{H}_2$ gives $\ell(\bar{G}) \leqslant \ell(\bar{H}_1) + \ell(\bar{H}_2)$. We first observe that $\ell(G) = \ell(\bar{G}) + 1$. Since P is a nilpotent normal subgroup of G, P must lie in the Fitting subgroup F of G. If $P \neq F$ then F contains a nonidentity element of order prime to the order of P which belongs to the centralizer in G of P. This contradicts $C_G(P) \subseteqq P$. Therefore $P = F$ and $\ell(G) = \ell(\bar{G}) + 1$.

Since H_1 and H_2 are Hall subgroups of G satisfying $G = H_1H_2$, the Sylow subgroup P must lie in H_1 or H_2. We may suppose that H_1 contains P. If $P = H_1$ then $\bar{G} = \bar{H}_2$ and so $\ell(\bar{G}) = \ell(\bar{H}_2) \leqslant \ell(H_2)$. Then $\ell(G) = \ell(\bar{G}) + 1 \leqslant \ell(H_2) + \ell(H_1)$, which is what we wanted to show. If $P \neq H_1$ then the argument used in the preceding paragraph can be repeated to show $\ell(H_1) = \ell(\bar{H}_1) + 1$. It follows from this that

$$\ell(G) = \ell(\bar{G}) + 1 \leqslant \ell(\bar{H}_1) + 1 + \ell(\bar{H}_2) \leqslant \ell(H_1) + \ell(H_2)\,.$$

This completes the argument.

It seems interesting to ask if this theorem has a converse, in the following sense. Does a $GSTG$ G necessarily possess pairwise permutable proper Hall subgroups $H_1, \cdots, H_n$ satisfying $G = H_1 \cdots H_n$ so that $\ell(G) = \ell(H_1) + \cdots + \ell(H_n)$ holds? The answer is obviously no, since any nilpotent group is a $GSTG$. If we insist that the group G not be nilpotent, the answer to the question is still no. This can easily be verified using the example of an N-group which is not a Sylow tower group (see [1]; p. 638). We now mention some consequences of the theorem.

Let G be a $GSTG$ and suppose that $\mathscr{S} = \{G_1, \cdots, G_n\}$ is a Sylow basis for G. Since $G_1, \cdots, G_n$ are pairwise permutable Sylow subgroups of G satisfying $G = G_1 \cdots G_n$, Theorem S gives

$$\ell(G) \leqslant \ell(G_1) + \cdots + \ell(G_n) = \pi(G) .$$

Consequently Theorem 1.7 [1] follows from Theorem S.

Now we show how the type of a $GSTG$ can be used to improve the bound on the nilpotent length of a $GSTG$ given by Theorem 1.7 [1]. It will be helpful to first introduce some terminology. Let R be a relation on the set of all primes and let σ denote some given set of primes. Then σ will be called a complete R-symmetric set provided both pRq and qRp hold for all primes p and q belonging to σ. If σ contains a single prime then σ is (trivially) a complete R-symmetric set. It is clear from this that any set of primes can be written as a union of complete R-symmetric subsets. The set σ will be called an R-cyclic set if σ contains distinct primes p, q, and w such that pRq, qRw, and wRp hold.

COROLLARY 1. *Let G be a $GSTG$ of type R. If $\sigma_1, \cdots, \sigma_d$ are complete R-symmetric subsets of $c(G)$ such that the union of the σ_i is $c(G)$ then $\ell(G) \leqslant d$.*

Proof. Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be a Sylow basis for G. For each i, $1 \leqslant i \leqslant d$, define the subgroup H_i of G to be the product of all Sylow p_k-subgroups G_k for which $p_k \in \sigma_i$. Since pRq and qRp hold for all distinct primes p and q from σ_i, each H_i is seen to be a nilpotent Hall σ_i-subgroup of G. Since the union of the σ_i's is $c(G)$, clearly $G = H_1 \cdots H_d$. The theorem then shows that

$$\ell(G) \leqslant \ell(H_1) + \cdots + \ell(H_d) = d .$$

COROLLARY 2. *Let G be a $GSTG$ of type R. If $\sigma_1, \cdots, \sigma_d$ are disjoint R-cyclic subsets of $c(G)$ then $\ell(G) \leqslant \pi(G) - d$.*

Proof. Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be a Sylow basis for G. For each integer i, $1 \leqslant i \leqslant d$, define the subgroup H_i of G to be the product of all Sylow p_k-subgroups G_k for which $p_k \in \sigma_i$. It is clear from the definition that the H_i are pairwise permutable Hall σ_i-subgroups of G. Since the product $H = H_1 \cdots H_d$ has a Hall complement in G, it suffices to show that $\ell(H) \leqslant \pi(H) - d$. Since the σ_i's are disjoint sets, this will follow if $\ell(H_i) \leqslant \pi(H_i) - 1$ holds for each i, $1 \leqslant i \leqslant d$. Let p, q, and w be distinct primes in σ_i satisfying pRq, qRp and wRp. Consider a Hall $\{p, q, w\}$-subgroup T of H_i. If T has a normal Sylow p-subgroup then pRq shows that T has a nilpotent Hall

$\{p, q\}$-subgroup. Then Corollary 1 shows $\ell(T) \leqslant \pi(T) - 1$. It now follows from Theorem S that $\ell(H_i) \leqslant \pi(H_i) - 1$. Consequently we may assume T has no nontrivial normal Sylow subgroup. By Lemma 1, T then has nontrivial normal subgroups A and B with $A \cap B = 1$. Since T/A and T/B are $GSTG$'s of type R, induction shows that $\ell(T/A) \leqslant 2$ and $\ell(T/B) \leqslant 2$. Using the fact that T is isomorphic to a subgroup of the direct product of T/A and T/B we obtain

$$\ell(T) \leqslant 2 = \pi(T) - 1 .$$

Theorem S applied to H_i now gives $\ell(H_i) \leqslant \pi(H_i) - 1$. Therefore we have shown that $\ell(H_i) \leqslant \pi(H_i) - 1$ holds for arbitrary i and the assertion follows.

The next consequence of the theorem is Theorem 1.8 [1].

COROLLARY 3. *Let G be GSTG with $\ell(G) = \pi(G)$. Then G is a Sylow tower group of exactly one type R, in the sense that the relation R is uniquely determined for pairs of primes p, q in $c(G)$.*

Proof. Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be Sylow basis for G. Define the relation R on the set of all primes as follows: R is reflexive and for distinct primes p and q, pRq holds if and only if either p or q does not divide the order of G or both p and q do divide the order of G and the Sylow p-subgroup of G belonging to $\mathscr{S}$ normalizes the Sylow q-subgroup of G belonging to $\mathscr{S}$. Clearly G is of type R. Since $\ell(G) = \pi(G)$, Corollary 1 shows that both pRq and qRp hold for no distinct primes $p, q \in c(G)$. In addition, Corollary 2 shows that pRq, qRw, and wRp hold for no distinct primes p, q, and w from $c(G)$. Since either pRq or qRp holds for any primes p, $q \in c(G)$, the restriction of R to $c(G)$ must be a linear order. Therefore G is a Sylow tower group of type R. Suppose that G is also a STG of type S and the restriction of S to $c(G)$ differs from the restriction of R to $c(G)$. Then G would necessarily have a nilpotent Hall $\{p, q\}$-subgroup for some distinct $p, q \in c(G)$. The conjugacy of Hall $\{p, q\}$-subgroups in G then implies that $\ell(G) \leqslant \pi(G) - 1$, a contradiction. Therefore G is a STG of exactly one type, in the sense mentioned.

We next give an example to show that the nilpotent length of a $GSTG$ cannot be found from the type alone. Let A be the holomorph of a cyclic group of order 7 and let B donote the Hall $\{7, 3\}$-subgroup of A. Define the group G_1 as the direct product of A and a symmetric group of degree 3 and define G_2 as the wreath product of B by a cyclic group of order 2. Both G_1 and G_2 are Sylow tower groups of type $7 < 3 < 2$ and no distinct Sylow subgroups of G_1 or G_2 centralize one another. Hence, for a given relation R on the set of all primes, G_1 is a *GSTG* of type R if and only if G_2 is a *GSTG* of type R. Yet the nilpotent length of G_1 is 2 and the nilpotent length of G_2 is 3.

THEOREM T. *Let G be a GSTG with nilpotent length k. Then G contains a Hall subgroup which is a Sylow tower group and has nilpotent length k.*

Proof. By Theorem 1.8 [1] it is sufficient to show that G contains a Hall subgroup L with $\ell(L) = \pi(L) = k$. We proceed by induction on the order of G.

Suppose G contains a proper subgroup W with $\ell(W) = k$. The induction hypothesis then shows that W contains a Hall subgroup T with $\ell(T) = \pi(T) = k$. Choose a Hall subgroup L of G with $T \subseteqq L$ and $c(T) = c(L)$. Then $\ell(G) = k = \ell(T) \leqslant \ell(L) \leqslant \ell(G)$ shows that $\ell(L) = \pi(L) = k$. This proves the theorem in the case where G contains a proper subgroup with nilpotent length k. Now suppose that every proper subgroup of G has nilpotent length strictly less than k.

Since G is a *GSTG*, either G possesses nontrivial normal subgroups A and B with $A \cap B = 1$ or G contains a nontrivial normal Sylow subgroup P with $C_G(P) \subseteqq P$. We consider these possibilities separately. First suppose that A and B are distinct minimal normal subgroups of G. If the Frattini subgroup ϕ of G is trivial then G contains a maximal subgroup M_1 not containing A and a maximal subgroup M_2 not containing B. Then M_1 complements A in G and M_2 complements B in G. Since we have assumed that all proper subgroups of G have nilpotent length less than k, the isomorphism of G/A and M_1 gives $\ell(G/A) < k$. Similarly one sees that $\ell(G/B) < k$. Since G is isomorphic to a subgroup of the direct product of G/A and G/B, it follows that $\ell(G) = \max\{\ell(G/A), \ell(G/B)\}$ is less than k, a contradiction. Therefore G has nontrivial Frattini subgroup. Since $\ell(G/\phi) = \ell(G) = k$, the induction hypothesis shows that G contains a Hall subgroup L satisfying $\ell(L\phi/\phi) = \pi(L\phi/\phi) = k$. Now

$$k = \ell(L\phi/\phi) \leqslant \ell(L) \leqslant \ell(G) = k$$

shows $\ell(L) = k$. Hence $L = G$. Since the Frattini subgroup of G contains no Sylow subgroup of G, $\pi(G) = \pi(L) = \pi(L\phi/\phi) = k$. Therefore $\ell(G) = \pi(G) = k$, which completes the argument in this case.

Now suppose G contains a nontrivial normal Sylow subgroup P with $C_G(P) \subseteqq P$. It follows that P must be the Fitting subgroup of G. Therefore $\ell(G) = \ell(G/P) + 1$ or $G = P$. In the latter case the theorem is trivially true. If $\ell(G) = \ell(G/P) + 1$, the induction hypothesis shows that G/P contains a nontrivial Hall subgroup L/P satisfying $\ell(L/P) = \pi(L/P) = k - 1 = \ell(G/P)$. Clearly L is then a Hall subgroup of G with $\pi(L) = k$. Since $C_G(P) \subseteqq P$, P must be the Fitting subgroup of L. Hence $\ell(L) = \ell(L/P) + 1$. Therefore

$$\ell(L) = k = \pi(L) .$$

This completes the proof of the theorem.

Let G be a given $GSTG$ and suppose $\mathscr{S}$ is a Sylow basis for G. Define the relation R on the set of all primes as follows: for any primes p and q (possibly equal), pRq holds if and only if $p \notin c(G)$ or $q \notin c(G)$ or both p and q belong to $c(G)$ and the Sylow p-subgroup of G in $\mathscr{S}$ normalizes the Sylow q-subgroup of G in $\mathscr{S}$. Clearly G is a $GSTG$ of type R. If H is a Hall subgroup of G which is a Sylow tower group and H satisfies $\ell(H) = \pi(H) = \ell(G)$, then the restriction of R to $c(H)$ is a transitive relation (see the proof of Corollary 3). This leads to the following bound for the nilpotent length of G in terms of the relation R defined above. The nilpotent length of the $GSTG$ G cannot exceed n, where n is the largest integer such that the restriction of R to some subset of $c(G)$ having n elements is a transitive relation.

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Received February 29, 1972 and in revised form June 19, 1972.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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