GENERALIZED SYLOW TOWER GROUPS. II

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A well-known result of P. Hall shows that finite solvable groups may be characterized by a permutability requirement on Sylow subgroups. The notion of a generalized Sylow tower group \((GSTG)\) arises when this permutability condition on Sylow subgroups is replaced by a suitable normalizer condition. In an earlier paper, one of the authors showed that the nilpotent length of a \(GSTG\) cannot exceed the number of distinct primes which divide the order of the group. The present investigation utilizes the ‘type’ of a \(GSTG\) to obtain improved bounds for the nilpotent length of a \(GSTG\). It is also shown that a \(GSTG\) with nilpotent length \(n\) possesses a Hall subgroup of nilpotent length \(n\) which is a Sylow tower group.

Let \(G\) be a finite group with order \(p_1^{a_1} \cdots p_r^{a_r}\), where \(p_1, \ldots, p_r\) are distinct primes and \(a_1, \ldots, a_r\) are positive integers. For each integer \(i, 1 \leq i \leq r\), let \(G_i\) denote a Sylow \(p_i\)-subgroup of \(G\). The collection of subgroups \(\mathcal{S} = \{G_1, \ldots, G_r\}\) is then called a complete set of Sylow subgroups of \(G\). If the elements of \(\mathcal{S}\) are pairwise permutable as subgroups (that is, if \(G_iG_j = G_jG_i\) holds for all \(i\) and \(j\)) then \(\mathcal{S}\) will be called a Sylow basis for \(G\). The notion of a generalized Sylow tower group arises when the permutability condition for a Sylow basis is replaced by a normalizer condition. Thus, we say that a finite group \(G\) is a generalized Sylow tower group \((GSTG)\) if and only if some complete set of Sylow subgroups \(\mathcal{S}\) of \(G\) satisfies the normalizer condition \((N)\): if \(G_i\) and \(G_j\) are distinct elements of \(\mathcal{S}\), at least one of these subgroups normalizes the other. It should be noted that not every complete set of Sylow subgroups of a \(GSTG\) need satisfy condition \((N)\).

A well-known result of P. Hall states that a finite group is solvable if and only if the group possesses a Sylow basis. If a complete set of Sylow subgroups \(\mathcal{S}\) of a group \(G\) satisfies condition \((N)\) then any two elements of \(\mathcal{S}\) are permutable as subgroups and \(\mathcal{S}\) is a Sylow basis for \(G\). Consequently every generalized Sylow tower group must be solvable.

A finite group \(G\) is called a Sylow tower group \((STG)\) if every nontrivial epimorphic image of \(G\) possesses a nontrivial normal Sylow subgroup. Equivalently, the group \(G\) is a \(STG\) if the prime divisors \(p_i, \ldots, p_r\) of the order of \(G\) can be labelled in such a way that a Sylow \(p_i\)-subgroup of \(G\) normalizes a Sylow \(p_j\)-subgroup of \(G\) when-
ever \( i > j \). It is clear from this definition that a Sylow tower group is necessarily a generalized Sylow tower group. An example of a GSTG which is not a Sylow tower group was given in ([1]; p. 638).

In order to handle generalized Sylow tower groups it will be necessary to introduce the ‘type’ of a GSTG. Suppose \( G \) is some given GSTG and let \( \mathcal{T} \) be a Sylow basis for \( G \). Since any two Sylow bases for \( G \) are conjugate ([3]; p. 665), \( \mathcal{T} \) satisfies the normalizer condition \((N)\). Let \( R \) be a relation on the set of all primes with the property that either \( pRq \) or \( qRp \) (or both) holds for any primes \( p \) and \( q \). If the Sylow \( p_i \)-subgroup of \( G \) in \( \mathcal{T} \) normalizes the Sylow \( p_j \)-subgroup of \( G \) in \( \mathcal{T} \) whenever \( p_iRp_j \) holds, then \( G \) will be called a GSTG of type \( R \). It follows directly from the conjugacy of Sylow bases that the type of a GSTG is independent of the choice of a Sylow basis. It should be noted that a group can be a GSTG of more than one type.

It was shown in [1] that the class of all generalized Sylow tower groups of a given type \( R \) is a formation. In addition, any subgroup of a GSTG of type \( R \) was shown to be a GSTG of type \( R \). We list the main results about GSTG’s in [1] for easy reference.

**Theorem 1.7** [1]. If \( G \) is a GSTG then the nilpotent length of \( G \) does not exceed the number of distinct prime divisors of the order of \( G \).

**Theorem 1.8** [1]. If \( G \) is a GSTG and the nilpotent length of \( G \) is equal to the number of distinct prime divisors of the order of \( G \) then \( G \) is a Sylow tower group.

All groups mentioned are assumed to be finite. The following notations will be used. For a group \( G \)
- \( c(G) \) denotes the set of distinct prime divisors of the order of \( G \)
- \( \pi(G) \) denotes the number of distinct prime divisors of the order of \( G \)
- \( \ell(G) \) denotes the nilpotent (Fitting) length of \( G \).
- If \( H \) is a subgroup of \( G \) then \( N_o(H) \) means the normalizer of \( H \) in \( G \) and \( C_o(H) \) means the centralizer of \( H \) in \( G \).
- If \( p_i \) is a prime, \( G_i \) will denote a Sylow \( p_i \)-subgroup of \( G \).

The following lemma will be used in several of our arguments.

**Lemma 1.** If \( G \) is a nontrivial GSTG then at least one of the following holds:
1. \( G \) contains a nontrivial normal Sylow subgroup \( P \) with \( C_o(P) \trianglelefteq P \)
(2) \( G \) contains nontrivial normal subgroups having relatively prime orders.

Proof. Let \( G \) be a GSTG and suppose that \( \mathcal{S} = \{G_1, \ldots, G_n\} \) is a Sylow basis for \( G \). Since a GSTG is necessarily solvable, \( G \) possesses a nontrivial minimal normal subgroup \( M \) with order a power of some prime \( q \). Let \( Q \) denote the maximum normal \( q \)-subgroup of \( G \) and suppose that \( G_k \) is the Sylow \( q \)-subgroup of \( G \) belonging to \( \mathcal{S} \). Since \( \mathcal{S} \) satisfies the normalizer condition (\( N \)), either

\[
G_1 \subseteq N_G(G_k) \quad \text{or} \quad G_k \subseteq N_G(G_i)
\]

must hold for each integer \( k, 1 \leq k \leq n \). We distinguish two cases.

First suppose that \( G_1 \subseteq N_G(G_k) \) holds for some \( k, 1 < k < n \). Since \( Q \) is normal in \( G \) and has order prime to the order of \( G_k \), \( G_k \) centralizes \( Q \). Hence \( C = C_G(Q) \) is not a \( q \)-group. Set \( Q_0 = Q \cap C_G(Q) \) and consider the factor group \( C/Q_0 \). Since \( C/Q_0 \) is a nontrivial solvable group, \( C/Q_0 \) contains a nontrivial minimal normal subgroup \( L/Q_0 \) with order a power of some prime \( p \). Let \( T/Q_0 \) be the maximum normal \( p \)-subgroup of \( C/Q_0 \). Since \( Q \) is the maximum normal \( q \)-subgroup of \( G \) it follows that \( p \neq q \). If \( N \) is a Sylow \( p \)-subgroup of \( T \) then \( T \) is the direct product of \( N \) and \( Q_0 \). The normality of \( T \) in \( G \) then implies the normality of \( N \) in \( G \). Therefore \( G \) has nontrivial normal subgroups with relatively prime orders.

Now suppose that \( G_1 \nsubseteq N_G(G_k) \) holds for all integers \( k, 1 < k < n \). Since \( \mathcal{S} \) satisfies the normalizer condition (\( N \)), \( G_k \subseteq G_o(G_i) \) must then hold for all integers \( k \). Thus \( G_1 \) is a normal Sylow \( q \)-subgroup of \( G \). Then \( G_1 \cap C_G(G_i) \) is a normal Sylow \( q \)-subgroup of \( C_G(G_i) \) and \( C_G(G_i) \) has a normal \( q \)-complement \( W \). If \( W \) is nontrivial then \( W \) is a normal subgroup of \( G \) having \( q^r \)-order and (2) holds. If \( W \) is trivial then \( G_1 \cap C_G(G_i) = C_G(G_i) \). In this case \( C_G(G_i) \subseteq G_i \) and (1) holds.

Theorem 5. Let \( G \) be a GSTG. If \( H_1, \ldots, H_n \) are pairwise permutable Hall subgroups of \( G \) with \( G = H_1 \cdots H_n \), then the nilpotent length of \( G \) does not exceed the sum of the nilpotent lengths of \( H_1, \ldots, H_n \).

Proof. (By induction on the order of \( G \).) Since the product \( H_1 \cdots H_n \) is a Hall subgroup of \( G \) permutable with \( H_i \) we may assume that \( n = 2 \). First suppose that \( G \) possesses nontrivial normal subgroups \( A \) and \( B \) with \( A \cap B = 1 \). Then \( G \) is isomorphic to a subgroup of the direct of \( G/A \) and \( G/B \). Hence

\[
\varrho(G) = \max \{\varrho(G/A), \varrho(G/B)\}
\]
and it suffices to show that
\[ \langle G/A \rangle \leq \langle H_1 \rangle + \langle H_2 \rangle \quad \text{and} \quad \langle G/B \rangle \leq \langle H_1 \rangle + \langle H_2 \rangle. \]

Since \( G/A \) is the product of the permutable Hall subgroups \( H_1A/A \) and \( H_2A/A \) of \( G/A \), the induction hypothesis gives
\[ \langle G/A \rangle \leq \langle H_1A/A \rangle + \langle H_2A/A \rangle. \]

Since \( H_1A/A \) and \( H_2A/A \) are epimorphic images of \( H_1 \) and \( H_2 \) (respectively), we know that \( \langle H_iA/A \rangle \leq \langle H_i \rangle \) for \( i = 1, 2 \). Therefore \( \langle G/A \rangle \leq \langle H_1 \rangle + \langle H_2 \rangle \). The same argument applied to \( G/B \) will show \( \langle G/B \rangle \leq \langle H_1 \rangle + \langle H_2 \rangle \). This verifies the theorem in this case.

Now suppose that \( G \) possesses a unique minimal normal subgroup. Then Lemma 1 shows that \( G \) contains a nontrivial normal Sylow subgroup \( P \) with \( C_G(P) \subseteq P \). Since the theorem is trivially true if \( G = P \) we may assume this is not the case. Set
\[ \overline{G} = G/P, \overline{H}_1 = H_1P/P, \overline{H}_2 = H_2P/P \]
and consider the nontrivial GSTG \( \overline{G} \). The induction hypothesis applied to \( \overline{G} = \overline{H}_1 \overline{H}_2 \) gives \( \langle \overline{G} \rangle \leq \langle \overline{H}_1 \rangle + \langle \overline{H}_2 \rangle \). We first observe that \( \langle G \rangle = \langle \overline{G} \rangle + 1 \). Since \( P \) is a nilpotent normal subgroup of \( G \), \( P \) must lie in the Fitting subgroup \( F \) of \( G \). If \( P \neq F \) then \( F \) contains a nonidentity element of order prime to the order of \( P \) which belongs to the centralizer in \( G \) of \( P \). This contradicts \( C_G(P) \subseteq P \). Therefore \( P = F \) and \( \langle G \rangle = \langle \overline{G} \rangle + 1 \).

Since \( H_1 \) and \( H_2 \) are Hall subgroups of \( G \) satisfying \( G = H_1H_2 \), the Sylow subgroup \( P \) must lie in \( H_1 \) or \( H_2 \). We may suppose that \( H_1 \) contains \( P \). If \( P = H_1 \) then \( \overline{G} = \overline{H}_2 \) and so \( \langle \overline{G} \rangle = \langle \overline{H}_2 \rangle \leq \langle H_2 \rangle \). Then \( \langle G \rangle = \langle \overline{G} \rangle + 1 \leq \langle H_2 \rangle + \langle H_1 \rangle \), which is what we wanted to show. If \( P \neq H_1 \) then the argument used in the preceding paragraph can be repeated to show \( \langle H_1 \rangle = \langle \overline{H}_1 \rangle + 1 \). It follows from this that
\[ \langle G \rangle = \langle \overline{G} \rangle + 1 \leq \langle \overline{H}_1 \rangle + 1 + \langle \overline{H}_2 \rangle \leq \langle H_1 \rangle + \langle H_2 \rangle. \]

This completes the argument.

It seems interesting to ask if this theorem has a converse, in the following sense. Does a GSTG \( G \) necessarily possess pairwise permutable proper Hall subgroups \( H_1, \ldots, H_n \) satisfying \( G = H_1 \cdots H_n \) so that \( \langle G \rangle = \langle H_1 \rangle + \cdots + \langle H_n \rangle \) holds? The answer is obviously no, since any nilpotent group is a GSTG. If we insist that the group \( G \) not be nilpotent, the answer to the question is still no. This can easily be verified using the example of an \( N \)-group which is not a Sylow tower group (see [1]; p. 638). We now mention some consequences of the theorem.
Let $G$ be a GSTG and suppose that $S = \{G_1, \ldots, G_n\}$ is a Sylow basis for $G$. Since $G_1, \ldots, G_n$ are pairwise permutable Sylow subgroups of $G$ satisfying $G = G_1 \cdots G_n$, Theorem 5 gives

$$\langle G \rangle \leq \langle G_1 \rangle + \cdots + \langle G_n \rangle = \pi(G).$$

Consequently Theorem 1.7 [1] follows from Theorem 5.

Now we show how the type of a GSTG can be used to improve the bound on the nilpotent length of a GSTG given by Theorem 1.7 [1]. It will be helpful to first introduce some terminology. Let $R$ be a relation on the set of all primes and let $\sigma$ denote some given set of primes. Then $\sigma$ will be called a complete $R$-symmetric set provided both $pRq$ and $qRp$ hold for all distinct primes $p$ and $q$ belonging to $\sigma$. If $\sigma$ contains a single prime then $\sigma$ is (trivially) a complete $R$-symmetric set. It is clear from this that any set of primes can be written as a union of complete $R$-symmetric subsets. The set $\sigma$ will be called an $R$-cyclic set if $\sigma$ contains distinct primes $p, q, w$ such that $pRq$, $qRw$, and $wRp$ hold.

**Corollary 1.** Let $G$ be a GSTG of type $R$. If $\sigma_1, \ldots, \sigma_d$ are complete $R$-symmetric subsets of $c(G)$ such that the union of the $\sigma_i$ is $c(G)$ then $\langle G \rangle \leq d$.

**Proof.** Let $S = \{G_1, \ldots, G_n\}$ be a Sylow basis for $G$. For each $i, 1 \leq i \leq d$, define the subgroup $H_i$ of $G$ to be the product of all Sylow $p_i$-subgroups $G_k$ for which $p_k \in \sigma_i$. Since $pRq$ and $qRp$ hold for all distinct primes $p$ and $q$ from $\sigma_i$, each $H_i$ is seen to be a nilpotent Hall $\sigma_i$-subgroup of $G$. Since the union of the $\sigma_i$'s is $c(G)$, clearly $G = H_1 \cdots H_d$. The theorem then shows that

$$\langle G \rangle \leq \langle H_1 \rangle + \cdots + \langle H_d \rangle = d.$$

**Corollary 2.** Let $G$ be a GSTG of type $R$. If $\sigma_1, \ldots, \sigma_d$ are disjoint $R$-cyclic subsets of $c(G)$ then $\langle G \rangle \leq \pi(G) - d$.

**Proof.** Let $S = \{G_1, \ldots, G_n\}$ be a Sylow basis for $G$. For each integer $i, 1 \leq i \leq d$, define the subgroup $H_i$ of $G$ to be the product of all Sylow $p_i$-subgroups $G_k$ for which $p_k \in \sigma_i$. It is clear from the definition that the $H_i$ are pairwise permutable Hall $\sigma_i$-subgroups of $G$. Since the product $H = H_1 \cdots H_d$ has a Hall complement in $G$, it suffices to show that $\langle H \rangle \leq \pi(H) - d$. Since the $\sigma_i$'s are disjoint sets, this will follow if $\langle H_i \rangle \leq \pi(H_i) - 1$ holds for each $i, 1 \leq i \leq d$. Let $p$, $q$, and $w$ be distinct primes in $\sigma_i$ satisfying $pRq$, $qRp$ and $wRp$. Consider a Hall $\{p, q, w\}$-subgroup $T$ of $H_i$. If $T$ has a normal Sylow $p$-subgroup then $pRq$ shows that $T$ has a nilpotent Hall
\(\{p, q\}\)-subgroup. Then Corollary 1 shows \(\zeta(T) \leq \pi(T) - 1\). It now follows from Theorem S that \(\zeta(H_i) \leq \pi(H_i) - 1\). Consequently we may assume \(T\) has no nontrivial normal Sylow subgroup. By Lemma 1, \(T\) then has nontrivial normal subgroups \(A\) and \(B\) with \(A \cap B = 1\). Since \(T/A\) and \(T/B\) are GSTG's of type \(R\), induction shows that \(\zeta(T/A) \leq 2\) and \(\zeta(T/B) \leq 2\). Using the fact that \(T\) is isomorphic to a subgroup of the direct product of \(T/A\) and \(T/B\) we obtain

\[\zeta(T) \leq 2 = \pi(T) - 1.\]

Theorem S applied to \(H_i\) now gives \(\zeta(H_i) \leq \pi(H_i) - 1\). Therefore we have shown that \(\zeta(H_i) \leq \pi(H_i) - 1\) holds for arbitrary \(i\) and the assertion follows.

The next consequence of the theorem is Theorem 1.8 [1].

**Corollary 3.** Let \(G\) be GSTG with \(\zeta(G) = \pi(G)\). Then \(G\) is a Sylow tower group of exactly one type \(R\), in the sense that the relation \(R\) is uniquely determined for pairs of primes \(p, q\) in \(c(G)\).

**Proof.** Let \(\mathcal{S} = \{G_1, \ldots, G_n\}\) be Sylow basis for \(G\). Define the relation \(R\) on the set of all primes as follows: \(R\) is reflexive and for distinct primes \(p\) and \(q\), \(pRq\) holds if and only if either \(p\) or \(q\) does not divide the order of \(G\) or both \(p\) and \(q\) do divide the order of \(G\) and the Sylow \(p\)-subgroup of \(G\) belonging to \(\mathcal{S}\) normalizes the Sylow \(q\)-subgroup of \(G\) belonging to \(\mathcal{S}\). Clearly \(G\) is of type \(R\). Since \(\zeta(G) = \pi(G)\), Corollary 1 shows that both \(pRq\) and \(qRp\) hold for no distinct primes \(p, q \in c(G)\). In addition, Corollary 2 shows that \(pRq, qRw,\) and \(wRp\) hold for no distinct primes \(p, q, w \in c(G)\). Since either \(pRq\) or \(qRp\) holds for any primes \(p, q \in c(G)\), the restriction of \(R\) to \(c(G)\) must be a linear order. Therefore \(G\) is a Sylow tower group of type \(R\). Suppose that \(G\) is also a STG of type \(S\) and the restriction of \(S\) to \(c(G)\) differs from the restriction of \(R\) to \(c(G)\). Then \(G\) would necessarily have a nilpotent Hall \(\{p, q\}\)-subgroup for some distinct \(p, q \in c(G)\). The conjugacy of Hall \(\{p, q\}\)-subgroups in \(G\) then implies that \(\zeta(G) \leq \pi(G) - 1\), a contradiction. Therefore \(G\) is a STG of exactly one type, in the sense mentioned.

We next give an example to show that the nilpotent length of a GSTG cannot be found from the type alone. Let \(A\) be the holomorph of a cyclic group of order 7 and let \(B\) denote the Hall \(\{7, 3\}\)-subgroup of \(A\). Define the group \(G_1\) as the direct product of \(A\) and a symmetric group of degree 3 and define \(G_2\) as the wreath product of \(B\) by a cyclic group of order 2. Both \(G_1\) and \(G_2\) are Sylow tower groups of type \(7 < 3 < 2\) and no distinct Sylow subgroups of \(G_1\) or \(G_2\) centralize one another. Hence, for a given relation \(R\) on the set
of all primes, \(G_1\) is a GSTG of type \(R\) if and only if \(G_2\) is a GSTG of type \(R\). Yet the nilpotent length of \(G_1\) is 2 and the nilpotent length of \(G_2\) is 3.

**Theorem T.** Let \(G\) be a GSTG with nilpotent length \(k\). Then \(G\) contains a Hall subgroup which is a Sylow tower group and has nilpotent length \(k\).

**Proof.** By Theorem 1.8 [1] it is sufficient to show that \(G\) contains a Hall subgroup \(L\) with \(\langle L \rangle = \pi(L) = k\). We proceed by induction on the order of \(G\).

Suppose \(G\) contains a proper subgroup \(W\) with \(\langle W \rangle = k\). The induction hypothesis then shows that \(W\) contains a Hall subgroup \(T\) with \(\langle T \rangle = \pi(T) = k\). Choose a Hall subgroup \(L\) of \(G\) with \(T \subseteq L\) and \(c(T) = c(L)\). Then \(\langle G \rangle = k = \langle T \rangle \leq \langle L \rangle \leq \langle G \rangle\) shows that \(\langle L \rangle = \pi(L) = k\). This proves the theorem in the case where \(G\) contains a proper subgroup with nilpotent length \(k\). Now suppose that every proper subgroup of \(G\) has nilpotent length strictly less than \(k\).

Since \(G\) is a GSTG, either \(G\) possesses nontrivial normal subgroups \(A\) and \(B\) with \(A \cap B = 1\) or \(G\) contains a nontrivial normal Sylow subgroup \(P\) with \(C_G(P) \subseteq P\). We consider these possibilities separately. First suppose that \(A\) and \(B\) are distinct minimal normal subgroups of \(G\). If the Frattini subgroup \(\phi\) of \(G\) is trivial then \(G\) contains a maximal subgroup \(M_1\) not containing \(A\) and a maximal subgroup \(M_2\) not containing \(B\). Then \(M_1\) complements \(A\) in \(G\) and \(M_2\) complements \(B\) in \(G\). Since we have assumed that all proper subgroups of \(G\) have nilpotent length less than \(k\), the isomorphism of \(G/A\) and \(M_1\) gives \(\langle G/A \rangle < k\). Similarly one sees that \(\langle G/B \rangle < k\). Since \(G\) is isomorphic to a subgroup of the direct product of \(G/A\) and \(G/B\), it follows that \(\langle G \rangle = \max \{\langle G/A \rangle, \langle G/B \rangle\}\) is less than \(k\), a contradiction. Therefore \(G\) has nontrivial Frattini subgroup. Since \(\langle G/\phi \rangle = \langle G \rangle = k\), the induction hypothesis shows that \(G\) contains a Hall subgroup \(L\) satisfying \(\langle L/\phi \rangle = \pi(L/\phi) = k\). Now

\[k = \langle L/\phi \rangle \leq \langle L \rangle \leq \langle G \rangle = k\]

shows \(\langle L \rangle = k\). Hence \(L = G\). Since the Frattini subgroup of \(G\) contains no Sylow subgroup of \(G\), \(\pi(G) = \pi(L) = \pi(L/\phi) = k\). Therefore \(\langle G \rangle = \pi(G) = k\), which completes the argument in this case.

Now suppose \(G\) contains a nontrivial normal Sylow subgroup \(P\) with \(C_G(P) \subseteq P\). It follows that \(P\) must be the Fitting subgroup of \(G\). Therefore \(\langle G \rangle = \langle G/P \rangle + 1\) or \(G = P\). In the latter case the theorem is trivially true. If \(\langle G \rangle = \langle G/P \rangle + 1\), the induction
hypothesis shows that $G/P$ contains a nontrivial Hall subgroup $L/P$ satisfying $\epsilon(L/P) = \pi(L/P) = k - 1 = \epsilon(G/P)$. Clearly $L$ is then a Hall subgroup of $G$ with $\pi(L) = k$. Since $C_G(P) \triangleleft P$, $P$ must be the Fitting subgroup of $L$. Hence $\epsilon(L) = \epsilon(L/P) + 1$. Therefore

$$\epsilon(L) = k = \pi(L).$$

This completes the proof of the theorem.

Let $G$ be a given GSTG and suppose $\mathscr{S}$ is a Sylow basis for $G$. Define the relation $R$ on the set of all primes as follows: for any primes $p$ and $q$ (possibly equal), $pRq$ holds if and only if $p \in c(G)$ or $q \in c(G)$ or both $p$ and $q$ belong to $c(G)$ and the Sylow $p$-subgroup of $G$ in $\mathscr{S}$ normalizes the Sylow $q$-subgroup of $G$ in $\mathscr{S}$. Clearly $G$ is a GSTG of type $i_2$. If $H$ is a Hall subgroup of $G$ which is a Sylow tower group and $H$ satisfies $\epsilon(H) = \pi(H) = \epsilon(G)$, then the restriction of $R$ to $c(H)$ is a transitive relation (see the proof of Corollary 3). This leads to the following bound for the nilpotent length of $G$ in terms of the relation $R$ defined above. The nilpotent length of the GSTG $G$ cannot exceed $n$, where $n$ is the largest integer such that the restriction of $R$ to some subset of $c(G)$ having $n$ elements is a transitive relation.

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