

Pacific Journal of Mathematics

MAXIMAL SUBFIELDS OF TENSOR PRODUCTS

BURTON I. FEIN AND MURRAY M. SCHACHER

MAXIMAL SUBFIELDS OF TENSOR PRODUCTS

BURTON FEIN AND MURRAY SCHACHER

Let D_1 and D_2 be finite-dimensional division rings with center K such that $D_1 \otimes_K D_2$ is a division ring. If L_1 and L_2 are maximal subfields of D_1 and D_2 , respectively, then clearly $L_1 \otimes_K L_2$ is a maximal subfield of $D_1 \otimes_K D_2$. In this note the converse question is considered: does there exist a maximal subfield L of $D_1 \otimes_K D_2$ which is not isomorphic to $L_1 \otimes_K L_2$ for maximal subfields L_1 and L_2 of D_1 and D_2 ? Examples are given to show that such noncomposite L may fail to exist even when K is a local field. For K an algebraic number field, however, it is shown that infinitely many noncomposite L always exist.

We say that a division algebra with center a field K is a K -division ring if it is finite-dimensional over K . Throughout this note D_1 and D_2 will denote K -division rings such that $D_1 \otimes_K D_2$ is a K -division ring. We say that a maximal subfield L of $D_1 \otimes_K D_2$ is a *composite* if $L \cong L_1 \otimes_K L_2$ where L_1 and L_2 are maximal subfields of D_1 and D_2 , respectively.

A sufficient condition for $D_1 \otimes_K D_2$ to be a division ring is for $([D_1:K], [D_2:K]) = 1$ [2, Theorem 10, p. 52]. This condition is necessary if K is either an algebraic number field or a local field since for these K the exponent of a K -division ring equals its index [2, Theorem 25, p. 144, and Theorem 32, p. 149]. This condition is not, however, necessary for K arbitrary, as is shown in [1]. We begin by determining, for the case when $([D_1:K], [D_2:K]) = 1$ necessary and sufficient conditions for a maximal subfield of $D_1 \otimes_K D_2$ to be a composite.

THEOREM 1. *Let D_1 and D_2 be K -division rings such that $([D_1:K], [D_2:K]) = 1$, and let L be a maximal subfield of $D_1 \otimes_K D_2$. Then L is a composite if and only if L has subfields L_1 and L_2 with $[L_1:K]^2 = [D_1:K]$ and $[L_2:K]^2 = [D_2:K]$.*

Proof. Let $n_i = [D_i:K]^{1/2}$, $i = 1, 2$. If L_i is a maximal subfield of D_i then $[L_i:K] = n_i$, $i = 1, 2$. It follows that if $L = L_1 \otimes_K L_2$ is a composite with L_i a maximal subfield of D_i , then $[L_i:K] = n_i$, $i = 1, 2$. This establishes one direction of the Theorem.

Suppose now that L has subfields L_1 and L_2 with $[L_i:K] = n_i$, $i = 1, 2$. Since L is a maximal subfield of $D_1 \otimes_K D_2$ we have $[L:K] = n_1 n_2$. As $(n_1, n_2) = 1$, it follows that $L \cong L_1 \otimes_K L_2$. Thus to conclude L is a composite we need only show that L_i splits D_i ,

$i = 1, 2$ [2, Theorem 27, p. 61]. We have $(D_1 \otimes_K D_2) \otimes_K L \cong [(D_1 \otimes_K L_1) \otimes_{L_1} L] \otimes_L [(D_2 \otimes_K L_1) \otimes_{L_1} L]$. Since L splits $D_1 \otimes_K D_2$, $(D_1 \otimes_K L_1) \otimes_{L_1} L = A_1$ is in the class of the opposite algebra of $A_2 = (D_2 \otimes_K L_1) \otimes_{L_1} L$ in the Brauer group of L . In particular, these algebras have the same exponent. Since $(n_1, n_2) = 1$ and the exponent of A_i divides n_i , it follows that A_1 and A_2 are complete matrix algebras. Thus L splits $D_1 \otimes_K L_1$. Since n_1 is prime to $[L: L_1] = n_2$, L_1 splits D_1 . Similarly, L_2 splits D_2 , proving the proposition.

COROLLARY 2. *Let D_1 and D_2 be K -division rings such that $([D_1: K], [D_2: K]) = 1$ and let L be a maximal subfield of $D_1 \otimes_K D_2$. If L is Galois over K with solvable Galois group, then L is a composite. In particular, if K is a local field and L is Galois over K , then L is a composite.*

Proof. Take G_i to be a Hall subgroup of order $[D_i: K]^{1/2}$ of the Galois group of L over K . Let L_1 and L_2 be the fixed fields of G_2 and G_1 , respectively. Then $L \cong L_1 \otimes_K L_2$, and L is composite by Theorem 1. The final assertion of the corollary follows from the result that Galois groups over local fields are solvable [6, Proposition 3.6.6, p. 101].

Corollary 2 is false without the restriction that L have a solvable Galois group. By [5, Theorem 9.1, p. 472] there is a field K , a K -division ring D , and a maximal subfield L of D such that L is a Galois extension of K with group A_5 . By [2, Theorem 18, p. 77], $D \cong D_1 \otimes_K D_2$ where D_1 and D_2 are K -division rings with D_1 of index 20 and D_2 of index 3. However, L clearly has no subfield L_2 with $[L_2: K] = 3$, since A_5 has no subgroup of order 20.

Theorem 1 is false without the assumption that $([D_1: K], [D_2: K]) = 1$. In [1] an example is presented of two quaternion algebras D_1 and D_2 central over a field K such that $D_1 \otimes_K D_2$ is a cyclic division algebra. If L is a maximal subfield of $D_1 \otimes_K D_2$ with $L|K$ cyclic, then L contains a subfield of degree two over K but is not a composite as composites would have Galois group $Z_2 \times Z_2$.

While one might expect that there should always exist maximal subfields of $D_1 \otimes_K D_2$ which are not composites, this is not the case even when K is a local field. Our next result treats the case when K is local and $[D_1 \otimes_K D_2: K]^{1/2}$ is a product of two primes. The general case may be expected to be much more complicated.

THEOREM 3. *Let p and r be distinct primes, $p < r$, and let K be a local field with residue class field $GF(q)$ where $p \nmid q$, $r \nmid q$. Let D_1 and D_2 be K -division rings of indices p and r respectively. If either $p \nmid r - 1$ or $q \equiv 1 \pmod{pr}$, then every maximal subfield of*

$D_1 \otimes_K D_2$ is a composite. If $p \mid r - 1$ there are infinitely many primes q and Q_q -division rings D_1 and D_2 (where Q_q is the q -adic field) of indices p and r , respectively, having maximal subfields which are not composites.

Proof. Suppose $p \nmid r - 1$ or $q \equiv 1 \pmod{pr}$. Let L be a maximal subfield of $D_1 \otimes_K D_2$. Then $[L:K] = pr$. Since $p \nmid q$, $r \nmid q$, L is tamely ramified over K . L will have subfields of degrees p and r over K if L is either unramified or totally ramified over K . From Corollary 2 we also see that L will be a composite if L is Galois over K . Let e and f be, respectively, the ramification and residue class degrees of L over K . Thus $ef = pr$ and we may assume that $e > 1$ and $f > 1$. If $q \equiv 1 \pmod{e}$ then L is normal over K [3, Theorem 6, p. 680]. Thus L is a composite if $q \equiv 1 \pmod{pr}$, so we assume that $p \nmid r - 1$ and $e \nmid q - 1$. By [3, Theorem 2, p. 678], we may assume that $L = K(\zeta, \alpha)$, where ζ is a primitive $(q^f - 1)$ th root of unity, $\alpha^e = \zeta^i \pi$, i is an integer, and π is a prime element of K . Let $q^f - 1 = (q - 1)t$. If e divided t , then $q^f \equiv 1 \pmod{e}$. But $(f, e - 1) = 1$ since $p \nmid r - 1$ and $p < r$. Thus $q \equiv 1 \pmod{e}$, against our assumption. Thus $(e, t) = 1$ so there is an integer j with $jt \equiv i \pmod{e}$. Let β be any root of $x^e - \zeta^{jt} \pi$ in an algebraic closure of K . Then $K(\zeta, \beta)$ is isomorphic to L by [3, Theorem 3, p. 679]. But $\zeta^t \in K$ since K contains all $(q - 1)$ th roots of unity, so $[K(\beta):K] = e$. Thus L has a subfield isomorphic to $K(\beta)$ which is of degree e over K . Since L also contains an unramified extension of degree f over K , Theorem 1 shows L is a composite.

Now suppose $p \mid r - 1$. Let b be an integer, $b \not\equiv 1 \pmod{r}$, $b^p \equiv 1 \pmod{r}$. Take q a prime, $q \equiv b \pmod{r}$. There are infinitely many such q by Dirichlet's theorem. If $q^p - 1 = (q - 1)t$, then r divides t . Let D_1 and D_2 be Q_q -division rings of indices p and r respectively. Let ζ be a primitive $(q^p - 1)$ th root of unity and let $\alpha^r = \zeta q$. Since $[Q_q(\zeta, \alpha):Q_q] = pr$, $Q_q(\zeta, \alpha)$ is a maximal subfield of $D_1 \otimes_K D_2$ [2, Theorem 23, p. 144]. If $Q_q(\zeta, \alpha)$ were a composite, it would have a subfield E with $[E:Q_q] = r$. E would be totally and tamely ramified over Q_q , and so $E \cong Q_q(\beta)$ where $\beta^r = \zeta^{tj} q$ for some integer j . Thus $Q_q(\zeta, \alpha) \cong Q_q(\zeta, \beta)$ so $1 \equiv jt \pmod{d}$ where $d = (r, q^p - 1)$ by [3, Theorem 3, p. 678]. Since $d = r$, we have $jt \equiv 1 \pmod{r}$. But $r \mid t$, a contradiction.

We remark that there are other examples where every maximal subfield of $D_1 \otimes_K D_2$ is a composite. In [4] an example is constructed of a field K and two quaternions D_1 and D_2 over K such that every maximal subfield of $D_1 \otimes_K D_2$ (which is a division ring) is a composite.

Our final result shows that over number fields it is never the

case that every maximal subfield of a tensor product is a composite. We use freely the classification of rational division algebras by means of Hasse invariants [2, Chapter 9].

THEOREM 4. *Let K be an algebraic number field, D_1 and D_2 K -division rings such that $D_1 \otimes_K D_2$ is a division ring. Then there are infinitely many maximal subfields of $D_1 \otimes_K D_2$ which are not composites.*

Proof. Suppose that $[D_1: K] = n^2$, $[D_2: K] = m^2$ and $m < n$. Let $\{\mathcal{P}_1, \dots, \mathcal{P}_m\}$ be the set of finite primes of K for which the Hasse invariants of $D_1 \otimes_K D_2$ are nonzero. Let \mathcal{P} be a finite prime of K , $\mathcal{P} \notin \{\mathcal{P}_1, \dots, \mathcal{P}_m\}$. Let K_i be the completion of K at \mathcal{P}_i , $K_{\mathcal{P}}$ the completion of K at \mathcal{P} . Let $K_i(\alpha_i)$ have degree mn over K_i and $K_{\mathcal{P}}(\alpha)$ have degree n over $K_{\mathcal{P}}$. We write $f_i(x)$ for the monic minimal polynomial of α_i over K_i and $f(x)$ for the monic minimal polynomial of α over $K_{\mathcal{P}}$. Let $g(x)$ be monic in $K[x]$ of degree nm “sufficiently close” to $f_i(x)$ in the \mathcal{P}_i -topology, $i = 1, \dots, m$, and “sufficiently close” to $(x-1)^{nm-n}f(x)$ in the \mathcal{P} -topology. If nm is even, take $g(x)$ also “sufficiently close” to $(x^2+1)^{nm/2}$ at all infinite primes of K . Here “sufficiently close” means close enough to guarantee

- (1) $g(x)$ is irreducible over K
- (2) For any root β of $g(x)$, the field $L = K(\beta)$ has local degree nm at \mathcal{P}_i , $i = 1, \dots, m$, and \mathcal{P} splits into $n(m-1)$ primes of degree one and one prime of degree n in L .
- (3) If nm is even, L is totally imaginary.

This is possible by [6, Ex. 3.2, p. 116].

It follows from the theory of Hasse invariants that L splits $D_1 \otimes_K D_2$. Since $[L: K] = nm$, L is a maximal subfield of $D_1 \otimes_K D_2$. Suppose there were a field E , $L \supset E \supset K$, $[E: K] = n$. If π is a prime of E dividing \mathcal{P} of degree greater than one, then π must remain irreducible in L since otherwise L would have two primes of degree > 1 dividing \mathcal{P} . But then if γ is the prime of L extending π , the local degree of γ over \mathcal{P} is divisible by $[L: E] = m$. Thus m would divide n which is not the case since $D_1 \otimes_K D_2$ is a division ring. This shows that \mathcal{P} splits completely in E . But then the local degree of any prime of L dividing \mathcal{P} is at most $[L: E] = m < n$. This proves that E can not exist and so L is not a composite. Since there are infinitely many choices for \mathcal{P} , there are infinitely many such L .

REFERENCES

1. A. A. Albert, *A note on cyclic algebras of order 16*, Bull. Amer. Math. Soc., **37** (1931), 727-30.
2. ———, *Structure of algebras*, Amer. Math. Soc. Colloq. Publ., **24**, Amer. Math. Soc., Providence, R.I., 1939.
3. ———, *On p -adic fields and rational division algebras*, Ann. Math., **41** (1940), 674-693.
4. S. Amitsur, *On central division algebras*, to appear Israel J. Math., **12** (1972), 408-421.
5. M. Schacher, *Subfields of division rings*, I, J. of Algebra, **9** (1968), 451-477.
6. E. Weiss, *Algebraic number theory*, McGraw-Hill, New York, 1963.

Received November 11, 1971. This research was supported in part by National Science Foundation grants GP-29068 and GP-28696.

OREGON STATE UNIVERSITY

AND

UNIVERSITY OF CALIFORNIA, LOS ANGELES

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California 94305

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY

University of Washington
Seattle, Washington 98105

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 45, No. 2

October, 1973

Kenneth Paul Baclawski and Kenneth Kapp, <i>Induced topologies for quasigroups and loops</i>	393
D. G. Bourgin, <i>Fixed point and min – max theorems</i>	403
J. L. Brenner, <i>Zolotarev's theorem on the Legendre symbol</i>	413
Jospeh Atkins Childress, Jr., <i>Restricting isotopies of spheres</i>	415
John Edward Coury, <i>Some results on lacunary Walsh series</i>	419
James B. Derr and N. P. Mukherjee, <i>Generalized Sylow tower groups. II</i>	427
Paul Frazier Duvall, Jr., Peter Fletcher and Robert Allen McCoy, <i>Isotopy Galois spaces</i>	435
Mary Rodriguez Embry, <i>Strictly cyclic operator algebras on a Banach space</i>	443
Abi (Abiadbollah) Fattahi, <i>On generalizations of Sylow tower groups</i>	453
Burton I. Fein and Murray M. Schacher, <i>Maximal subfields of tensor products</i> ...	479
Ervin Fried and J. Sichler, <i>Homomorphisms of commutative rings with unit element</i>	485
Kenneth R. Goodearl, <i>Essential products of nonsingular rings</i>	493
George Grätzer, Bjarni Jónsson and H. Lakser, <i>The amalgamation property in equational classes of modular lattices</i>	507
H. Groemer, <i>On some mean values associated with a randomly selected simplex in a convex set</i>	525
Marcel Herzog, <i>Central 2-Sylow intersections</i>	535
Joel Saul Hillel, <i>On the number of type-k translation-invariant groups</i>	539
Ronald Brian Kirk, <i>A note on the Mackey topology for $(C^b(X))^*$, $C^b(X)$</i>	543
J. W. Lea, <i>The peripherality of irreducible elements of lattice</i>	555
John Stewart Locker, <i>Self-adjointness for multi-point differential operators</i>	561
Robert Patrick Martineau, <i>Splitting of group representations</i>	571
Robert Massagli, <i>On a new radical in a topological ring</i>	577
James Murdoch McPherson, <i>Wild arcs in three-space. I. Families of Fox-Artin arcs</i>	585
James Murdoch McPherson, <i>Wild arcs in three-space. III. An invariant of oriented local type for exceptional arcs</i>	599
Fred Richman, <i>The constructive theory of countable abelian p-groups</i>	621
Edward Barry Saff and J. L. Walsh, <i>On the convergence of rational functions which interpolate in the roots of unity</i>	639
Harold Eugene Schlais, <i>Non-aposyndesis and non-hereditary decomposability</i>	643
Mark Lawrence Teply, <i>A class of divisible modules</i>	653
Edward Joseph Tully, Jr., <i>\mathcal{H}-commutative semigroups in which each homomorphism is uniquely determined by its kernel</i>	669
Garth William Warner, Jr., <i>Zeta functions on the real general linear group</i>	681
Keith Yale, <i>Cocycles with range $\{\pm 1\}$</i>	693
Chi-Lin Yen, <i>On the rest points of a nonlinear nonexpansive semigroup</i>	699