MAXIMAL SUBFIELDS OF TENSOR PRODUCTS

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Let $D_1$ and $D_2$ be finite-dimensional division rings with center $K$ such that $D_1 \otimes_K D_2$ is a division ring. If $L_1$ and $L_2$ are maximal subfields of $D_1$ and $D_2$, respectively, then clearly $L_1 \otimes_K L_2$ is a maximal subfield of $D_1 \otimes_K D_2$. In this note the converse question is considered: does there exist a maximal subfield $L$ of $D_1 \otimes_K D_2$ which is not isomorphic to $L_1 \otimes_K L_2$ for maximal subfields $L_1$ and $L_2$ of $D_1$ and $D_2$? Examples are given to show that such noncomposite $L$ may fail to exist even when $K$ is a local field. For $K$ an algebraic number field, however, it is shown that infinitely many non-composite $L$ always exist.

We say that a division algebra with center a field $K$ is a $K$-division ring if it is finite-dimensional over $K$. Throughout this note $D_1$ and $D_2$ will denote $K$-division rings such that $D_1 \otimes_K D_2$ is a $K$-division ring. We say that a maximal subfield $L$ of $D_1 \otimes_K D_2$ is a composite if $L \cong L_1 \otimes_K L_2$ where $L_1$ and $L_2$ are maximal subfields of $D_1$ and $D_2$, respectively.

A sufficient condition for $D_1 \otimes_K D_2$ to be a division ring is for $([D_1: K], [D_2: K]) = 1$ [2, Theorem 10, p. 52]. This condition is necessary if $K$ is either an algebraic number field or a local field since for these $K$ the exponent of a $K$-division ring equals its index $[2, \text{Theorem 25, p. 144, and Theorem 32, p. 149}]$. This condition is not, however, necessary for $K$ arbitrary, as is shown in [1]. We begin by determining, for the case when $([D_1: K], [D_2: K]) = 1$ necessary and sufficient conditions for a maximal subfield of $D_1 \otimes_K D_2$ to be a composite.

**THEOREM 1.** Let $D_1$ and $D_2$ be $K$-division rings such that $([D_1: K], [D_2: K]) = 1$, and let $L$ be a maximal subfield of $D_1 \otimes_K D_2$. Then $L$ is a composite if and only if $L$ has subfields $L_1$ and $L_2$ with $[L_1: K] = [D_1: K]$ and $[L_2: K] = [D_2: K]$.

**Proof.** Let $n_i = [D_i: K]^{1/2}$, $i = 1, 2$. If $L_i$ is a maximal subfield of $D_i$ then $[L_i: K] = n_i$, $i = 1, 2$. It follows that if $L = L_1 \otimes_K L_2$ is a composite with $L_i$, a maximal subfield of $D_i$, then $[L_i: K] = n_i$, $i = 1, 2$. This establishes one direction of the Theorem.

Suppose now that $L$ has subfields $L_1$ and $L_2$ with $[L_i: K] = n_i$, $i = 1, 2$. Since $L$ is a maximal subfield of $D_1 \otimes_K D_2$ we have $[L: K] = n_1 n_2$. As $(n_1, n_2) = 1$, it follows that $L \cong L_1 \otimes_K L_2$. Thus to conclude $L$ is a composite we need only show that $L_i$ splits $D_i$,
We have \((A \otimes A) \otimes A \cong [(D_1 \otimes_K L_1) \otimes_{L_1} L] \otimes_L [(D_2 \otimes_K L_1) \otimes_{L_1} L].\) Since \(L\) splits \(D_1 \otimes_K D_2,\)
\((D_1 \otimes_K L_1) \otimes_{L_1} L = A_1\) is in the class of the opposite algebra of \(A_2 = (D_2 \otimes_K L_1) \otimes_{L_1} L\) in the Brauer group of \(L\). In particular, these algebras have the same exponent. Since \((n_1, n_2) = 1\) and the exponent of \(A_i\) divides \(n_i\), it follows that \(A_1\) and \(A_2\) are complete matrix algebras. Thus \(L\) splits \(D_1 \otimes_K L_1\). Since \(n_1\) is prime to \([L : L_1] = n_2\), \(L_1\) splits \(D_1\). Similarly, \(L_2\) splits \(D_2\), proving the proposition.

**Corollary 2.** Let \(D_1\) and \(D_2\) be \(K\)-division rings such that \([(D_1 : K), [D_2 : K]] = 1\) and let \(L\) be a maximal subfield of \(D_1 \otimes_K D_2\). If \(L\) is Galois over \(K\) with solvable Galois group, then \(L\) is a composite. In particular, if \(K\) is a local field and \(L\) is Galois over \(K\), then \(L\) is a composite.

**Proof.** Take \(G_i\) to be a Hall subgroup of order \([D_i : K]^{1/2}\) of the Galois group of \(L\) over \(K\). Let \(L_1\) and \(L_2\) be the fixed fields of \(G_2\) and \(G_1\), respectively. Then \(L \cong L_1 \otimes_K L_2\), and \(L\) is composite by Theorem 1. The final assertion of the corollary follows from the result that Galois groups over local fields are solvable [6, Proposition 3.6.6, p. 101].

Corollary 2 is false without the restriction that \(L\) have a solvable Galois group. By [5, Theorem 9.1, p. 472] there is a field \(K\), a \(K\)-division ring \(D\), and a maximal subfield \(L\) of \(D\) such that \(L\) is a Galois extension of \(K\) with group \(A_5\). By [2, Theorem 18, p. 77], \(D \cong D_1 \otimes_K D_2\) where \(D_1\) and \(D_2\) are \(K\)-division rings with \(D_1\) of index 20 and \(D_2\) of index 3. However, \(L\) clearly has no subfield \(L_2\) with \([L_2 : K] = 3\), since \(A_5\) has no subgroup of order 20.

Theorem 1 is false without the assumption that \([(D_1 : K), [D_2 : K]] = 1\). In [1] an example is presented of two quaternion algebras \(D_1\) and \(D_2\) central over a field \(K\) such that \(D_1 \otimes_K D_2\) is a cyclic division algebra. If \(L\) is a maximal subfield of \(D_1 \otimes_K D_2\) with \(L | K\) cyclic, then \(L\) contains a subfield of degree two over \(K\) but is not a composite as composites would have Galois group \(Z_2 \times Z_2\).

While one might expect that there should always exist maximal subfields of \(D_1 \otimes_K D_2\) which are not composites, this is not the case even when \(K\) is a local field. Our next result treats the case when \(K\) is local and \([(D_1 \otimes_K D_2 : K)]^{1/2}\) is a product of two primes. The general case may be expected to be much more complicated.

**Theorem 3.** Let \(p\) and \(r\) be distinct primes, \(p < r\), and let \(K\) be a local field with residue class field \(GF(q)\) where \(p \nmid q\). Let \(D_1\) and \(D_2\) be \(K\)-division rings of indices \(p\) and \(r\) respectively. If either \(p \nmid r - 1\) or \(q \equiv 1 (mod \, pr)\), then every maximal subfield of
$D_1 \otimes_K D_2$ is a composite. If $p \mid r - 1$ there are infinitely many primes $q$ and $Q_q$-division rings $D_1$ and $D_2$ (where $Q_q$ is the $q$-adic field) of indices $p$ and $r$, respectively, having maximal subfields which are not composites.

Proof. Suppose $p \nmid r - 1$ or $q \equiv 1 \pmod{pr}$. Let $L$ be a maximal subfield of $D_1 \otimes_K D_2$. Then $[L: K] = pr$. Since $p \nmid q$, $r \nmid q$, $L$ is tamely ramified over $K$. $L$ will have subfields of degrees $p$ and $r$ over $K$ if $L$ is either unramified or totally ramified over $K$. From Corollary 2 we also see that $L$ will be a composite if $L$ is Galois over $K$. Let $e$ and $f$ be, respectively, the ramification and residue class degrees of $L$ over $K$. Thus $ef = pr$ and we may assume that $e > 1$ and $f > 1$. If $q \equiv 1 \pmod{e}$ then $L$ is normal over $K$ [3, Theorem 6, p. 680]. Thus $L$ is a composite if $q \equiv 1 \pmod{pr}$, so we assume that $p \nmid r - 1$ and $e \nmid q - 1$. By [3, Theorem 2, p. 678], we may assume that $L = K(\zeta, \alpha)$, where $\zeta$ is a primitive $(q^f - 1)$th root of unity, $\alpha^e = \zeta^i \pi$, $i$ is an integer, and $\pi$ is a prime element of $K$. Let $q_f - 1 = (q - 1)t$. If $e$ divided $t$, then $q_f \equiv 1 \pmod{e}$. But $(f, e - 1) = 1$ since $p \nmid r - 1$ and $p < r$. Thus $q \equiv 1 \pmod{e}$, against our assumption. Thus $(e, t) = 1$ so there is an integer $j$ with $jt \equiv i \pmod{e}$. Let $\beta$ be any root of $x^e - \zeta^{jt} \pi$ in an algebraic closure of $K$. Then $K(\zeta, \beta)$ is isomorphic to $L$ by [3, Theorem 3, p. 679]. But $\zeta^i \in K$ since $K$ contains all $(q - 1)$th roots of unity, so $[K(\beta): K] = e$. Thus $L$ has a subfield isomorphic to $K(\beta)$ which is of degree $e$ over $K$. Since $L$ also contains an unramified extension of degree $f$ over $K$, Theorem 1 shows $L$ is a composite.

Now suppose $p \mid r - 1$. Let $b$ be an integer, $b \equiv 1 \pmod{r}$, $b^p \equiv 1 \pmod{r}$. Take $q$ a prime, $q \equiv b \pmod{r}$. There are infinitely many such $q$ by Dirichlet's theorem. If $q^p - 1 = (q - 1)t$, then $r$ divides $t$. Let $D_1$ and $D_2$ be $Q_q$-division rings of indices $p$ and $r$ respectively. Let $\zeta$ be a primitive $(q^p - 1)$th root of unity and let $\alpha^r = \zeta q$. Since $[Q_q(\zeta, \alpha): Q_q] = pr$, $Q_q(\zeta, \alpha)$ is a maximal subfield of $D_1 \otimes_K D_2$ [2, Theorem 23, p. 144]. If $Q_q(\zeta, \alpha)$ were a composite, it would have a subfield $E$ with $[E: Q_q] = r$. $E$ would be totally and tamely ramified over $Q_q$, and so $E \cong Q_q(\beta)$ where $\beta^r = \zeta^i \pi$ for some integer $j$. Thus $Q_q(\zeta, \alpha) \cong Q_q(\zeta, \beta)$ and $1 \equiv jt \pmod{d}$ where $d = (r, q^p - 1)$ by [3, Theorem 3, p. 678]. Since $d = r$, we have $jt \equiv 1 \pmod{r}$. But $r \mid t$, a contradiction.

We remark that there are other examples where every maximal subfield of $D_1 \otimes_K D_2$ is a composite. In [4] an example is constructed of a field $K$ and two quaternions $D_1$ and $D_2$ over $K$ such that every maximal subfield of $D_1 \otimes_K D_2$ (which is a division ring) is a composite.

Our final result shows that over number fields it is never the
case that every maximal subfield of a tensor product is a composite. We use freely the classification of rational division algebras by means of Hasse invariants [2, Chapter 9].

**Theorem 4.** Let $K$ be an algebraic number field, $D_1$ and $D_2$ $K$-division rings such that $D_1 \otimes_K D_2$ is a division ring. Then there are infinitely many maximal subfields of $D_1 \otimes_K D_2$ which are not composites.

**Proof.** Suppose that $[D_1: K] = n^2$, $[D_2: K] = m^2$ and $m < n$. Let $\mathcal{P}_1, \cdots, \mathcal{P}_m$ be the set of finite primes of $K$ for which the Hasse invariants of $D_1 \otimes_K D_2$ are nonzero. Let $\mathcal{P}$ be a finite prime of $K$, $\mathcal{P} \in \{\mathcal{P}_1, \cdots, \mathcal{P}_m\}$. Let $K_i$ be the completion of $K$ at $\mathcal{P}_i$, $K_\mathcal{P}$ the completion of $K$ at $\mathcal{P}$. Let $K_i(\alpha_i)$ have degree $mn$ over $K_i$ and $K_\mathcal{P}(\alpha)$ have degree $n$ over $K_\mathcal{P}$. We write $f_i(x)$ for the monic minimal polynomial of $\alpha_i$ over $K_i$ and $f(x)$ for the monic minimal polynomial of $\alpha$ over $K_\mathcal{P}$. Let $g(x)$ be monic in $K[x]$ of degree $nm$ "sufficiently close" to $f_i(x)$ in the $\mathcal{P}_i$-topology, $i = 1, \cdots, m$, and "sufficiently close" to $(x - 1)^{m-1}f(x)$ in the $\mathcal{P}$-topology. If $nm$ is even, take $g(x)$ also "sufficiently close" to $(x^2 + 1)^{m/2}$ at all infinite primes of $K$.

Here "sufficiently close" means close enough to guarantee

1. $g(x)$ is irreducible over $K$
2. For any root $\beta$ of $g(x)$, the field $L = K(\beta)$ has local degree $nm$ at $\mathcal{P}_i$, $i = 1, \cdots, m$, and $\mathcal{P}$ splits into $n(m-1)$ primes of degree one and one prime of degree $n$ in $L$.
3. If $nm$ is even, $L$ is totally imaginary.

This is possible by [6, Ex. 3.2, p. 116].

It follows from the theory of Hasse invariants that $L$ splits $D_1 \otimes_K D_2$. Since $[L: K] = nm$, $L$ is a maximal subfield of $D_1 \otimes_K D_2$. Suppose there were a field $E$, $L \supset E \supset K$, $[E: K] = n$. If $\pi$ is a prime of $E$ dividing $\mathcal{P}$ of degree greater than one, then $\pi$ must remain irreducible in $L$ since otherwise $L$ would have two primes of degree $\geq 1$ dividing $\mathcal{P}$. But then if $\gamma$ is the prime of $L$ extending $\pi$, the local degree of $\gamma$ over $\mathcal{P}$ is divisible by $[L: E] = m$. Thus $m$ would divide $n$ which is not the case since $D_1 \otimes_K D_2$ is a division ring. This shows that $\mathcal{P}$ splits completely in $E$. But then the local degree of any prime of $L$ dividing $\mathcal{P}$ is at most $[L: E] = m < n$. This proves that $E$ can not exist and so $L$ is not a composite. Since there are infinitely many choices for $\mathcal{P}$, there are infinitely many such $L$. 
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