THE AMALGAMATION PROPERTY IN EQUATIONAL CLASSES OF MODULAR LATTICES

George Grätzer, Bjarni Jónsson and H. Lakser

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G. Grätzer, B. Jónsson, and H. Lakser*

It is known that the class of all distributive lattices satisfies the Amalgamation Property. It will be shown that this is the only nontrivial equational class of modular lattices for which the Amalgamation Property holds. A second theorem gives further information about the Amalgamation Class of the class of all modular lattices, and of certain other equational classes.

1. **Introduction.** A class K of algebras (or in general, structures) is said to have the *Amalgamation Property* if for A, B_0, B_1 in K and for embeddings $f_i: A \to B_i$, $i = 0, 1$, there exist a C in K and embeddings $g_i: B_i \to C$, $i = 1, 2$, such that $f_0 g_0 = f_1 g_1$ (see Figure 1).

$$
\begin{array}{ccc}
B_0 & \xrightarrow{\quad g_0 \quad} & C \\
\Big\uparrow f_0 & & \Big\uparrow g_1 \\
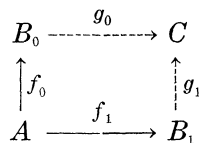A & \xrightarrow{\quad f_1 \quad} & B_1
\end{array}
$$

FIGURE 1.

For the history and importance of the Amalgamation Property we refer the reader to B. Jónsson [8] (see also G. Grätzer [5]).

The class of all lattices and the class of all distributive lattices both have the Amalgamation Property. The problem whether the class M of all modular lattices has the Amalgamation Property has been around for more than a decade.

In January of 1971 B. Jónsson announced [11] that M does not have the Amalgamation Property, in fact, any equational class K of modular lattices having the Amalgamation Property must satisfy the arguesian identity. This was followed by an announcement by G. Grätzer and H. Lakser [7] stating that every member of K can be embedded into the subspace lattice of an infinite dimensional projective geometry. Combining and extending these results, we can now prove the following:

THEOREM 1. *If an equational class K of modular lattices contains a nondistributive lattice, then K does not have the Amalgamation Property.*

If K does not have the Amalgamation Property, then we can use the concept of the Amalgamation Class of K (G. Grätzer and H. Lakser

[6]) to measure to what extent the Amalgamation Property holds. We define A to be in the Amalgamation Class of K ($A \in \text{Amal}(K)$) if and only if for any B_0, $B_1 \in K$ and embeddings $f_i: A \to B_i$, $i = 0, 1$, there exist a $C \in K$ and embeddings $g_i: B_i \to C$, $i = 0, 1$, such that $f_0 g_0 = f_1 g_1$. If K is an equational class of lattices, then it is easily seen that the one-element lattice is always in Amal (K).

THEOREM 2. *No distributive lattice with more than one element belongs to* Amal (M). *In fact, if K is any nontrivial equational class of modular lattices and* Amal (K) *contains a distributive lattice with more than one element, then K satisfies the arguesian identity.*

Some steps in the proofs of Theorems 1 and 2 rely on ideas of R. P. Dilworth and M. Hall [1] even though no result of [1] could be applied directly. The proof of Theorem 1 makes heavy use of some deep results on projective geometries. The proof of Theorem 2 is patterned after the proof of Desargues' theorem for projective geometries of dimension 3 or more.

Theorems 1 and 2 are proved in Sections 2 and 3, respectively. The fourth and final section contains some comments and problems and a result to the effect that a variety generated by a finite non-distributive lattice never has the Amalgamation Property.

2. *Proof of Theorem 1.* In this section let K be an equational class of modular lattices that contains a nondistributive lattice and has the Amalgamation Property. We are going to prove a series of statements about K that will lead to a contradiction, and thus to a proof of Theorem 1.

A *diamond* is the five element modular nondistributive lattice of Figure 2. That the diamond belongs to K follows from the assumptions.
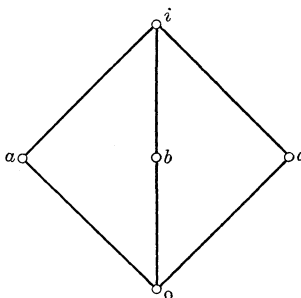

FIGURE 2.

Statement 1. Every member of K can be embedded in a simple

complemented lattice that also belongs to **K**.

Proof. Let L be a lattice in **K**. We can assume that L has 0 and 1 because if a lattice belongs to **K** then that lattice with 0 and 1 adjoined also belongs to **K**. Consider a fixed diamond labelled as in Figure 2. Given $x \in L$, $x \neq 0, 1$, we can take the embeddings

$$f_0: o \rightarrow 0, \ a \rightarrow x, \ i \rightarrow 1$$
$$f_1: o \rightarrow o, \ a \rightarrow a, \ i \rightarrow i$$

of $\{0, a, i\}$ into L and into the diamond, respectively. By the Amalgamation Property there is a lattice L_1 in **K** containing L as a sublattice having the same 0 and 1 as L, such that $\{0, x, 1\}$ is contained in a diamond sublattice of L. Iterating this process, possibly a transfinite number of times (this is possible because **K** is closed under direct limits), we obtain a lattice N that contains L as a sublattice, has the same 0 and 1 as L, and has the property that for every element $x \in L$, $x \neq 0, 1$, there exist y, $z \in N$ such that $\{0, x, y, z, 1\}$ is a diamond.

Repeating this process we obtain an infinite sequence

$$L = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq C_{n+1} \subseteq \cdots$$

of lattices in **K**, all having the same 0 and 1 as L, and having the property that, for all $n < \omega$, every element x of C_n, $x \neq 0, 1$, belongs to a diamond $\{0, x, y, z, 1\}$ in C_{n+1}.

The union C_∞ of the C_n belongs to **K** and it is obviously complemented. To show that C_∞ is simple, recall that in a complemented modular lattice every congruence Θ is determined by the ideal $[0]\Theta$ of elements congruent to 0 modulo Θ. If $\Theta \neq \omega$ (the trivial congruence), then there exists an $x \in C_\infty$, $x \neq 0$, such that $x \equiv 0(\Theta)$. If $x = 1$, then $\Theta = \iota$ (the largest congruence). If $x < 1$, then for some $n < \omega$, we have $x \in C_n$ and so there exists a diamond $\{0, x, y, z, 1\}$ in C_{n+1}. Since a diamond is a simple lattice, it follows that $0 \equiv 1(\Theta)$, so that $\Theta = \iota$ in C_∞. This shows that C_∞ is a simple lattice, completing the proof of Statement 1.

If P is a projective geometry, let $\mathscr{L}(P)$ denote the lattice of all subspaces of P. We say that P is *degenerate* if some line contains only two points; otherwise P is called nondegenerate.

Statement 2. For every member L of **K** there exists an infinite dimensional nondegenerate projective geometry P such that L can be embedded in $\mathscr{L}(P)$ and $\mathscr{L}(P)$ belongs to **K**.

Proof. We may assume that L has 0 and 1 and that it contains

an infinite chain, because if it does not, adjoin an infinite chain to "the top of" L and the resulting lattice is still in K. By Statement 1, L can be embedded in a simple complemented lattice C of K. By O. Frink [4] C can embedded into some $\mathscr{L}(P)$ where P is a projective geometry; crucial for our argument is the result of B. Jónsson [9] according to which P can be chosen so that $\mathscr{L}(P) \in K$. The lattice $\mathscr{L}(P)$ has a representation

$$\mathscr{L}(P) \cong \Pi(\mathscr{L}(P_i) \mid i \in I)$$

where the P_i, $i \in I$, are nondegenerate projective geometries. Let φ_i denote the embedding of C into $\mathscr{L}(P)$ followed by the ith projection. Since φ_i preserves 0 and 1, it cannot be a constant homomorphism. Thus, since C is simple, φ_i is an embedding of C into $\mathscr{L}(P_i)$. Since C contains an infinite chain, P_i is infinite dimensional, completing the proof of Statement 2.

Let D be a division ring and let V be a vector space over D. Let $\mathscr{L}(V, D)$ denote the lattice of all subspaces of V. We define a projective space P as follows: the points of P are the one-dimensional subspaces of V; if A and B are distinct one-dimensional subspaces of V, then the line $A + B$ through A and B consists of all one-dimensional subspaces C that are contained in the subspace spanned by A and B. Then P is a nondegenerate projective space and

$$\mathscr{L}(V, D) \cong \mathscr{L}(P) .$$

The celebrated coordinatization theorem of projective geometry asserts that if P is a nondegenerate projective geometry and if $\mathscr{L}(P)$ contains a five element chain, then there exist a division ring D and a vector space V over D such that

$$\mathscr{L}(P) \cong \mathscr{L}(V, D) .$$

This division ring, called the coordinate ring of P, is unique up to isomorphism. All the lines of P have the same number of points: this number is $|D|$ if D is infinite and $|D| + 1$ if D is finite.

Statement 3. There exists a projective plane Q such that $\mathscr{L}(Q)$ belongs to K and Q has at least six points on each line.

Proof. By Statement 2, there exist an infinite dimensional nondegenerate projective geometry P such that $\mathscr{L}(P)$ belongs to K. If P has the property that every line has at least six points, then choose as Q any plane in P. Since $\mathscr{L}(Q)$ is a sublattice of $\mathscr{L}(P)$, we conclude that $\mathscr{L}(Q) \in K$ and, obviously, every line of Q has at

least six points. If the lines in P have less than six points, then the coordinate ring of P must be a field F of order 2, 3, or 4. Therefore,

$$\mathscr{L}(P) \cong \mathscr{L}(V, F)$$

for some infinite dimensional vector space V over F. Let us choose a finite field extension K of F of order $k \geq 5$, and let W be a three-dimensional vector space over K. Then $\mathscr{L}(W, K)$ determines a projective plane Q (in the sense that $\mathscr{L}(W, K) \cong \mathscr{L}(Q)$).

$\mathscr{L}(W, K)$ is a sublattice of $\mathscr{L}(W, F)$, and $\mathscr{L}(W, F)$ is isomorphic to a sublattice of $\mathscr{L}(V, F)$, proving that $\mathscr{L}(Q)$ is in **K**. Since every line in Q has at least $k + 1 \geq 6$ points, this completes the proof of Statement 3.

The next two statements contain known facts that apply to any class that has the Amalgamation Property.

Statement 4. Let $C, D \in \mathbf{K}$. Let D be an extension of C and let $\varphi \colon C \to D$ be a one-to-one homomorphism. Then there exists an extension E of D ($E \in \mathbf{K}$) and a one-to-one homomorphism $\beta \colon D \to E$ extending φ.

Proof. $C \subseteqq D$ and $\varphi \colon C \to D$; hence we can apply the Amalgamation Property to get an extension E of D and a one-to-one homomorphism $\beta \colon D \to E$ such that for $c \in C$ we have $c\varphi = c\beta$, completing the proof.

Statement 5. Let $A, B \in \mathbf{K}$. Let B be an extension of A and let α be an automorphism of A. Then there exists an extension $\bar{B} \in \mathbf{K}$ of B and an automorphism $\bar{\alpha}$ of $\bar{B}$ extending α.

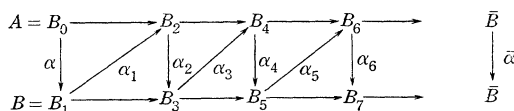*Proof.* We define (see Figure 3) a sequence of extensions of A:

FIGURE 3.

$$A = B_0 \subseteqq B = B_1 \subseteqq B_2 \subseteqq \cdots \subseteqq B_n \subseteqq B_{n+1} \subseteqq \cdots$$

and a sequence of one-to-one homomorphisms $\alpha_n \colon B_n \to B_{n+1}$ such that $\alpha_0 = \alpha$ and α_{n+1} extends α_n^{-1} (on $\mathrm{Im}\,(\alpha_n)$) for all n. Indeed, if we have B_n, B_{n+1} and α_n, we apply Statement 4 to $C = \mathrm{Im}\,(\alpha_n)$, $D = B_{n+1}$, $\varphi = \alpha_n^{-1}$ and we get B_{n+2} an extension of B_{n+1} and $\alpha_{n+1} \colon B_{n+1} \to B_{n+2}$ extending $\alpha_n^{-1} \colon \mathrm{Im}\,(\alpha_n) \to B_{n+1}$. We then define

$$\bar{B} = \bigcup (B_n \mid n < \omega), \quad \bar{\alpha} = \bigcup (\alpha_{2n} \mid n < \omega) .$$

Obviously $\bar{\alpha}$ is a one-to-one homomorphism. Let $y \in \bar{B}$. Then there is an n such that $y \in B_{2n+1}$. Let $x = y\alpha_{2n+1}$; since α_{2n+2} is an extension of α_{2n+1}^{-1}, $x\bar{\alpha} = x\alpha_{2n+2} = y$. Hence $\bar{\alpha}$ is an automorphism of $\bar{B}$ extending α, completing the proof of Statement 5.

We are now ready to start the final argument leading to the desired contradiction.

By Statement 3 there exists a projective plane Q with at least six points on each line that $\mathscr{L}(Q) \in K$. By Statement 2 there exists an infinite dimensional nondegenerate projective geometry P such that $\mathscr{L}(P) \in K$ and $\mathscr{L}(Q)$ is isomorphic to a sublattice of $\mathscr{L}(P)$. By the coordinatization theorem, there exists a vector space V over a division ring D such that $\mathscr{L}(P) \cong \mathscr{L}(V, D)$. Consequently there exists a one-to-one homomorphism of $\mathscr{L}(Q)$ into $\mathscr{L}(V, D)$.

The fact that $\mathscr{L}(Q)$ is isomorphic to a sublattice of the lattice of all subgroups of an abelian group,—the group V,—implies that the arguesian identity holds in Q. Therefore coordinates can be introduced in Q in the usual manner, and we shall actually have to make use of part of that construction. Choose a line m in Q and two distinct points a_0 and a_∞ on m, and let $K = m - \{a_\infty\}$. Choose two distinct points p and q of Q that are collinear with a_0 but are not on m. Given $x, y \in K$, let

(1) $$u = (x \vee p) \wedge (q \vee a_\infty), \quad v = (y \vee q) \wedge (p \vee a_\infty) ,$$

(2) $$x + y = (u \vee v) \wedge m .$$

K is of course an abelian group under this operation, and the operation is independent of the points p and q.

Any permutation of the points of m induces an automorphism of the interval $[0, m]$. Using the fact that m has at least six points, we can choose $x, y \in K - \{a_0\}$ so that $x + y \neq a_0$, and we can then find an automorphism α of the interval $[0, m]$ that keeps a_0, a_∞, x and y fixed but maps $x + y$ onto a point $z \neq x + y$. By Statement 5 there exists an extension L of $\mathscr{L}(Q)$ such that α extends to an automorphism β of L and, as above, by Statement 2 there exists an embedding F of L into the lattice of all subgroups of an abelian group A. We claim that $F(x + y)$ is the set of all elements $a \in A$ such that, for some $b, c \in A$

(3) $$b \in F(x), \ c \in F(y), \ a - b \in F(a_\infty), \ a - c \in F(a_\infty), \ a - b - c \in F(a_0) .$$

Let u and v be as in (1). First assume that $a \in F(x + y)$. Since $x + y \leqq u \vee v$, we have $F(x + y) \subseteqq F(u \vee v) = F(u) + F(v)$, and so there exists a $d \in A$ satisfying

(4) $$d \in F(u), \ a - d \in F(v) \ .$$

Since $u \leqq x \vee p$ and $v \leqq y \vee q$, it follows that there exist $b, c \in A$ such that

(5) $$b \in F(x), \ d - b \in F(p), \ c \in F(y), \ a - d - c \in F(q) \ .$$

The element $a - b$ belongs to each of the groups $F(x + y) + F(x)$ and $F(v) + F(p)$. Since

$$((x + y) \vee x) \wedge (v \vee p) \leqq m \wedge (v \vee p) \leqq a_\infty \ ,$$

it follows that $a - b \in F(a_\infty)$. Similarly, $a - c \in F(a_\infty)$ and $a - b - c \in F(a_0)$, verifying (3).

Conversely, assume that (3) holds. Since $a_0 \leqq p \vee q$, there exists an $n \in A$ such that

$$n \in F(p), \ a - b - c - n \in F(q) \ .$$

Using (1) we infer that

$$b + c \in F(u) \quad \text{and} \quad a - b - c \in F(v) \ ,$$

and so (2) implies that $a \in F(x + y)$.

In this argument we can replace $F(t)$ by

$$F'(t) = F(t\beta) \ .$$

Since F and F' agree on a_0, a_∞, x, and y, it follows from the description (3) that

$$F(x + y) = F'(x + y) \ .$$

This, however, is impossible since $F'(x + y) = F(z)$ and $z \neq x + y$. This contradiction proves the theorem.

It should be observed that if we only want to show that M fails to have the Amalgamation Property, then it suffices to combine Statement 1 with the example of R. P. Dilworth and M. Hall [1] of a modular lattice that cannot be embedded in a complemented modular lattice.

A naive first approach to Theorem 1 would be to find three lattices A, B_0, B_1, A a sublattice of B_0 and B_1, in the equational class generated by the diamond that cannot be amalgamated in the class of all modular lattices. However, it follows from results in [3] that such an amalgamation can always be effected.

3. *Proof of Theorem* 2. In preparation for the proof of Theorem 2, we establish three lemmas that are of some independent interest.

We define a triangle in a lattice L to be an ordered triple of elements of L. We shall have frequent occasions to consider two triangles in L, $a = \langle a_0, a_1, a_2 \rangle$ and $b = \langle b_0, b_1, b_2 \rangle$, and the following associated elements:

$$p = (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \,,$$
$$c_0 = (a_1 \vee a_2) \wedge (b_1 \vee b_2) \,, \qquad c_1 = (a_2 \vee a_0) \wedge (b_2 \vee b_0) \,,$$
$$c_2 = (a_0 \vee a_1) \wedge (b_0 \vee b_1) \,, \qquad m = c_2 \wedge (c_0 \vee c_1) \,.$$

The symbols p, c_0, c_1, c_2 and m will always be used in this sense.

Two triangles a and b in L are said to be centrally perspective provided $p = (a_0 \vee b_0) \wedge (a_1 \vee b_1)$ or, equivalently,

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \leqq a_2 \vee b_2 \,,$$

and p is then called the center of perspectivity of a and b. The triangles a and b are said to be axially perspective provided $m = c_2$, or, equivalently,

$$c_2 \leqq c_0 \vee c_1 \,.$$

The lattice L is said to be arguesian provided, for any triangles a and b in L,

$$p \leqq (a_0 \wedge (a_1 \vee m)) \vee b_0 \,.$$

We could have followed more closely the geometric theory and called a lattice arguesian provided it satisfies the condition $D(L)$: Any two triangles in L that are centrally perspective are also axially perspective. However, unlike this implication, the above definition is obviously equivalent to an identity. In any case, the two are known to be equivalent in the case of projective spaces, and in B. Jónsson and G. S. Monk [12], Theorem 3.4, it is shown that every arguesian lattice satisfies the condition $D(L)$. We shall now further justify the terminology by proving the converse of that assertion.

LEMMA 1. *For any modular lattice L, the following conditions are equivalent:*

(i)   *L is arguesian.*

(ii)   *Any two triangles in L that are centrally perspective are also axially perspective.*

(iii)   *For any triangles $a = \langle a_0, a_1, a_2 \rangle$ and $b = \langle b_0, b_1, b_2 \rangle$ in L that are centrally perspective, if $a_0 \wedge a_1 \leqq a_2$ and $b_0 \wedge b_1 \leqq b_2$, and if the center of perspectivity, p, satisfies the conditions*

$$a_i \vee p = b_i \vee p = a_i \vee b_i \quad for \quad i = 0, 1, 2 \,,$$

then a and b are axially perspective.

Proof. Since (i) is known to imply (ii), and (iii) is but a special case of (ii), we need only show that (iii) implies (i). Thus, assuming (iii), we consider two triangle a and b in L, and using the notation introduced earlier we wish to prove that

(1) $$p \leqq (a_0 \wedge (a_1 \vee m)) \vee b_0 .$$

We first reduce the problem to the special case in which

(2) $$p \vee a_i = p \vee b_i = a_i \vee b_i \quad \text{for} \quad i = 0, 1, 2 .$$

Assuming that (1) holds under this additional hypothesis, we consider two arbitrary triangles a and b in L, and we let

$$a_i' = a_i \wedge (b_i \vee p) , \ b_i' = b_i \wedge (a_i \vee p) \quad \text{for} \quad i = 0, 1, 2 .$$

Then

$$p \vee a_i' = p \vee b_i' = a_i' \vee b_i' \quad \text{for} \quad i = 0, 1, 2 ,$$

whence it readily follows that

$$p = (a_0' \vee b_0') \wedge (a_1' \vee b_1') \wedge (a_2' \vee b_2') .$$

Defining c_i' and m' in the same manner as c_i and m were defined, we therefore have

$$p \leqq (a_0' \wedge (a_1' \vee m')) \vee b_0' .$$

Inasmuch as $a_i' \leqq a_i$, $b_i' \leqq b_i$, and $m' \leqq m$, it follows that (1) holds.
    We henceforth assume that (2) holds, and we shall prove that in this case $a_0 \leqq a_1 \vee m$, whence (1) certainly follows. Let

$$q = (a_0 \vee b_0) \wedge (a_1 \vee b_1) .$$

Then

$$a_0 \vee q = (a_0 \vee b_0) \wedge (a_0 \vee a_1 \vee b_1) .$$

Since $p \leqq a_1 \vee b_1$ and, by (2), $b_0 \leqq a_0 \vee p$, this yields

$$a_0 \vee q = a_0 \vee b_0 .$$

Similarly,

$$b_0 \vee q = a_0 \vee b_0, \ a_1 \vee q = b_1 \vee q = a_1 \vee b_1 .$$

Furthermore

$$q = (a_0 \vee p) \wedge (a_1 \vee b_1) = p \vee (a_0 \wedge (a_1 \vee b_1)) ,$$
$$q = (b_0 \vee p) \wedge (a_1 \vee b_1) = p \vee (b_0 \wedge (a_1 \vee b_1)) .$$

Therefore, if we let

$$a_2^* = a_2 \vee (a_0 \wedge a_1) \,, \quad b_2^* = b_2 \vee (b_0 \wedge (a_1 \vee b_1)) \,,$$

then

$$a_2^* \vee b_2^* = a_2^* \vee q = b_2^* \vee q \,,$$
$$a_0 \wedge a_1 \leqq a_2^* \,, \quad b_0 \wedge b_1 \leqq b_2^* \,,$$

and we can therefore apply (iii) to the triangles $\langle a_0, a_1, a_2^* \rangle$ and $\langle b_0, b_1, b_2^* \rangle$. Letting

$$c_0^* = (a_1 \vee a_2^*) \wedge (b_1 \vee b_2^*) \,, \quad c_1^* = (a_2^* \vee a_0) \wedge (b_2^* \vee b_0) \,,$$

we thus have

$$c_2 \leqq c_0^* \vee c_1^* \,.$$

We now compute

$$
\begin{aligned}
c_0^* &= (a_1 \vee a_2) \wedge (b_1 \vee b_2 \vee (b_0 \wedge (a_1 \vee b_1))) \\
&= (a_1 \vee a_2) \wedge (b_2 \vee ((b_0 \vee b_1) \vee (a_1 \vee b_1))) \\
&= (a_1 \vee a_2) \wedge (b_1 \vee b_2 \vee (a_1 \wedge (b_0 \vee b_1))) \\
&= ((a_1 \vee a_2) \wedge (b_1 \vee b_2)) \vee (a_1 \wedge (b_0 \vee b_1)) \\
&= c_0 \vee (a_1 \wedge (b_0 \vee b_1)) \,.
\end{aligned}
$$

On the other hand it is easy to see that $c_1^* = c_1$. Thus

$$c_2 \leqq c_0 \vee c_1 \vee (a_1 \wedge (b_0 \vee b_1)) \,,$$
$$c_2 = (c_2 \wedge (c_0 \vee c_1)) \vee (a_1 \wedge (b_0 \vee b_1)) = m \vee (a_1 \wedge (b_0 \vee b_1)) \,,$$
$$a_1 \vee m = a_1 \vee c_2 = a_0 \vee a_1 \geqq a_0 \,,$$

and the proof of Lemma 1 is complete.

LEMMA 2. *Suppose L is a modular lattice, u, $v \in L$, and $v \leqq u$. If there exist elements s, $t \in L$ such that*

$$u \wedge s = u \wedge t = s \wedge t = v \,, \quad u \vee s = u \vee t = s \vee t \,,$$

then the interval $[v, u]$ is an arguesian lattice.

Proof. Suppose the triangles $a = \langle a_0, a_1, a_2 \rangle$ and $b = \langle b_0, b_1, b_2 \rangle$ in $[v, u]$ satisfy the conditions of (iii) in Lemma 1. Let $p, c_0, c_1,$ and c_2 be defined as usual, and let

$$q = s \wedge (p \vee t) \,, \quad r = t \wedge (p \vee s) \,,$$
$$d_i = (a_i \vee q) \wedge (b_i \vee r) \quad \text{for} \quad i = 0, 1, 2 \,.$$

(See Figure 4.) Then
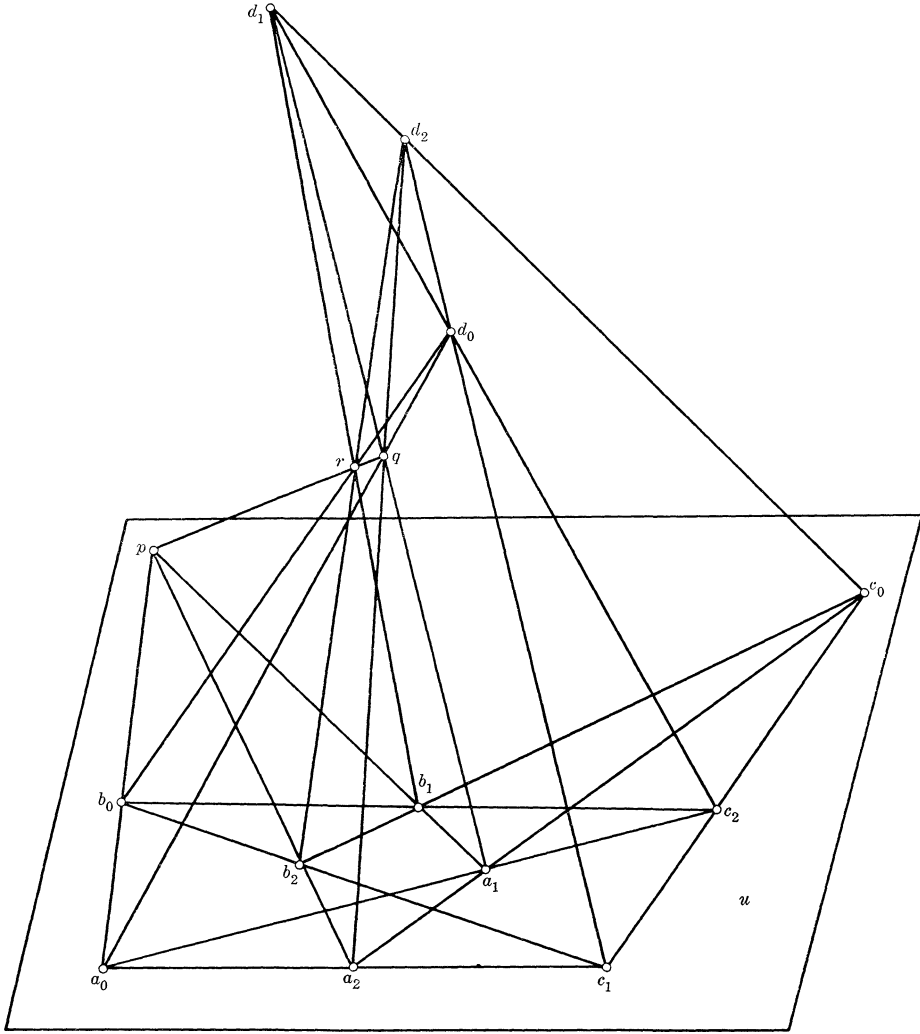
FIGURE 4.

$$p \vee q = q \vee r = r \vee p = (p \vee s) \wedge (p \vee t) \, ,$$
$$p \wedge q = q \wedge r = r \wedge p = v \, ,$$
$$a_i \vee d_i = (a_i \vee q) \wedge (a_i \vee b_i \vee r) = a_i \vee q \, ,$$

because $p \leqq a_i \vee b_i$ and $q \leqq p \vee r$.   Also,

$$d_i \vee q = (a_i \vee q) \wedge (b_i \vee q \vee r) = a_i \vee q \, ,$$

because $p \leqq q \vee r$ and $a_i \leqq b_i \vee p$.

Observe that for any elements $x, y \in [v, u]$, the sets $\{x, y, q\}$ and $\{x, y, r\}$ generate a distributive lattice, because

$$(x \vee y) \wedge q \leqq u \wedge q = v = (x \wedge q) \vee (y \vee q) \, ,$$

and similarly with q replaced by r. In particular, it follows from this that

$$(x \vee q) \wedge y = x \wedge y \,, \quad (x \vee q) \wedge (y \vee q) = (x \wedge y) \vee q.$$

These observations will be used several times below. We could use them at this point to prove that the triangle $d = \langle d_0, d_1, d_2 \rangle$ is centrally perspective with each of the triangles a and b, but no use will be made of that fact.

We shall presently need the formula

$$(3) \qquad\qquad a_0 \wedge (b_0 \vee b_1) \leqq d_0 \vee d_1 \,.$$

To verify this inequality we compute

$$
\begin{aligned}
d_0 \vee d_1 &\geqq ((a_0 \vee q) \wedge (b_0 \vee r)) \vee (q \wedge (b_1 \vee r)) \\
&= (a_0 \vee q) \wedge (b_0 \vee r \vee (q \wedge (b_1 \vee r))) \\
&= (a_0 \vee q) \wedge (b_0 \vee ((q \vee r) \wedge (b_1 \vee r))) \\
&= (a_0 \vee q) \wedge (b_0 \vee r \vee (b_1 \wedge (q \vee r))) \\
&\geqq a_0 \wedge (b_0 \vee (b_1 \wedge p)) = a_0 \wedge (b_0 \vee (b_1 \wedge (a_0 \vee b_0))) \\
&= a_0 \wedge (b_0 \vee b_1) \,.
\end{aligned}
$$

We now wish to prove the inequalities

$$(4) \qquad\qquad c_2 \leqq d_0 \vee d_1 \,,$$

$$(5) \qquad\qquad d_1 \leqq c_0 \vee d_2 \,, \quad d_0 \leqq c_1 \vee d_2 \,.$$

For the proof of (4) we first note that

$$(6) \qquad\qquad (a_0 \vee q) \wedge (b_0 \vee b_1 \vee r) \leqq d_0 \vee d_1 \,.$$

In fact,

$$
\begin{aligned}
(a_0 \vee q) \wedge (b_0 \vee b_1 \vee r) &\leqq (a_0 \vee p \vee q) \wedge (b_0 \vee b_1 \vee r) \\
&= ((a_0 \vee p \vee q) \wedge (b_0 \vee b_1)) \vee r \\
&= ((a_0 \vee b_0) \wedge (b_0 \vee b_1)) \vee r \\
&= (a_0 \wedge (b_0 \vee b_1)) \vee b_0 \vee r \,, \\
(a_0 \vee q) \wedge (b_0 \vee b_1 \vee r) &\leqq (a_0 \vee q) \wedge ((a_0 \wedge (b_0 \vee b_1)) \vee b_0 \vee r) \\
&= (a_0 \wedge (b_0 \vee b_1)) \vee ((a_0 \vee q) \wedge (b_0 \vee r)) \\
&= (a_0 \wedge (b_0 \vee b_1)) \vee d_1 \,.
\end{aligned}
$$

In view of (3), this implies (6). Using (6) we compute

$$
\begin{aligned}
d_0 \vee d_1 &\geqq ((a_0 \vee q) \wedge (b_0 \vee b_1 \vee r)) \vee d_1 \\
&= (a_0 \vee q \vee d_1) \wedge (b_0 \vee b_1 \vee r) \\
&= (a_0 \vee a_1 \vee q) \wedge (b_0 \vee b_1 \vee r) \geqq c_2 \,.
\end{aligned}
$$

To prove the first inequality in (5) we first show that

$$(7) \qquad (a_1 \vee a_2 \vee q) \wedge (b_2 \vee r) \leqq c_0 \vee d_2 \, .$$

In fact,

$$\begin{aligned}
(a_1 \vee a_2 \vee q) \wedge (b_2 \vee r) &\leqq (a_1 \vee a_2 \vee q) \wedge (b_2 \vee p \vee r) \\
&\leqq q \vee ((a_1 \vee a_2) \wedge (b_2 \vee p \vee r)) \\
&= q \vee ((a_1 \vee a_2) \wedge (a_2 \vee b_2)) \\
&= q \vee a_2 \vee (b_2 \wedge (a_1 \vee a_2)) \, , \\
(a_1 \vee a_2 \vee q) \wedge (b_2 \vee r) &\leqq (q \vee a_2 \vee (b_2 \wedge (a_1 \vee a_2))) \wedge (b_2 \vee r) \\
&= ((a_2 \vee q) \wedge (b_2 \vee r)) \vee (b_2 \wedge (a_1 \vee a_2)) \\
&= d_2 \vee (b_2 \wedge (a_1 \vee a_2)) \leqq c_0 \vee d_2 \, .
\end{aligned}$$

With the aid of (7) we find that

$$\begin{aligned}
c_0 \vee d_2 &\geqq ((a_1 \vee a_2 \vee q) \wedge (b_2 \vee r)) \vee c_0 \\
&= (a_1 \vee a_2 \vee q) \wedge (b_2 \vee c_0 \vee r) \\
&= (a_1 \vee a_2 \vee q) \wedge (b_1 \vee b_2 \vee r) \geqq d_1 \, .
\end{aligned}$$

The first formula in (5) therefore holds, and the second follows by symmetry.

Using (4) and (5) we find that

$$\begin{aligned}
c_2 \leqq (d_0 \vee d_1) \wedge u &\leqq (c_0 \vee c_1 \vee d_2) \wedge u \\
&= c_0 \vee c_1 \vee (d_2 \wedge u) \, .
\end{aligned}$$

Since $d_2 \wedge u = (a_2 \vee q) \wedge u \wedge (b_2 \vee r) \wedge u = a_2 \wedge b_2$, we conclude that

$$c_2 \leqq c_0 \vee c_1 \, .$$

The proof of Lemma 2 is now complete.

LEMMA 3. *If K is any equational class of lattices, and if* Amal(K) *contains a distributive lattice with more than one element, then the two-element chain belongs to* Amal (K).

Proof. Let $C_2 = \{0, 1\}$ be the two-element chain, let B be a distributive lattice, and let $C \in K$. Let $f_0: C_2 \to B$, $f_1: C_2 \to C$ be one-to-one homomorphisms. We show that this amalgamation can be effected in K. Since B is distributive there is a retraction $\rho: B \to C_2$ such that $f_0\rho = 1_{C_2}$. Given any $x, y \in C$, $x \neq y$, define $g_1: C \to C$ as the identity and define $g_0: B \to C$ by $g_0 = \rho f_1$. Then $f_0 g_0 = f_1 g_1$ and $xg_1 \neq yg_1$. For any $x, y \in B$, $x \neq y$, there is a homomorphism $\bar{h}: B \to C_2$ such that $x\bar{h} \neq y\bar{h}$ since B is distributive. Define $h_0: B \to C$ by

$h_0 = \bar{h}f_1$. If $0f_0\bar{h} = 1f_0\bar{h}$ define $h_1\colon C \to C$ by $zh_1 = 0f_0h_0$. If $0f_0\bar{h} \neq 1f_0\bar{h}$ define $h_1\colon C \to C$ as the identity. In either case $f_0h_0 = f_1h_1$ and $xh_0 \neq yh_0$. It then follows (see Theorem 13.18 of [5]) that there is a lattice $D \in K$ and embeddings $g_0\colon B \to D$, $g_1\colon C \to D$ such that $f_0g_0 = f_1g_1$.

Now suppose B, $C \in K$, and suppose $f_0\colon C_2 \to B$ and $f_1\colon C_2 \to C$ are embeddings. By hypothesis there exists a distributive lattice $A \in \text{Amal}(K)$ with more than one element. Choosing an embedding $\varphi\colon C_2 \to A$, we effect three amalgamations as indicated in Figure 5, using Lemma 3
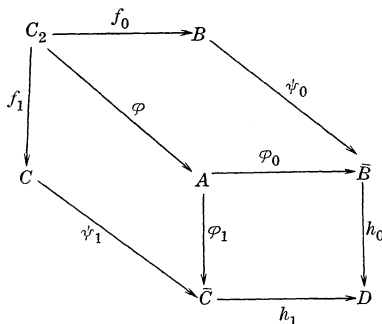
FIGURE 5.

twice to obtain the pairs φ_0, ψ_0 and φ_1, ψ_1, and finally using the fact that A is in the Amalgamation Class of K to obtain h_0 and h_1. Then $g_0 = \psi_0h_0$ and $g_1 = \psi_1h_1$ are embeddings of B and C into D with $f_0g_0 = f_1g_1$. Thus $C_2 \in \text{Amal}(K)$.

Now we are ready to prove Theorem 2. Assume that K is an equational class of modular lattices and that $\text{Amal}(K)$ contains a distributive lattice with more than one element. We have to prove that every $L \in K$ is arguesian. It is sufficient to prove this for finitely generated L, and so L has a 0 and a 1. Since every distributive lattice is arguesian we may assume that K contains a non-distributive lattice, and hence it contains the diamond. Let $C_2 = \{0, 1\}$ be the two-element chain, f_0 the embedding of C_2 into L defined by

$$0f_0 = 0 \quad \text{and} \quad 1f_0 = 1 .$$

Let f_1 embed C_2 into the diamond (see Figure 2) by

$$0f_1 = o \quad \text{and} \quad 1f_1 = a .$$

Since $C_2 \in \text{Amal}(K)$, there exist $C \in K$ and embeddings g_0, g_1 into C of L and of the diamond respectively, such that $f_0g_0 = f_1g_1$. By Lemma 2, the interval $[og_0, ag_0]$ is arguesian. Since L is isomorphic to Lg_0

and $Lg_0 \subseteqq [og_1, ag_1]$, we conclude that L is arguesian, completing the proof of the second statement of Theorem 2.

If P is a projective plane that does not satisfy Desargues' theorem, then $\mathscr{L}(P)$ is not arguesian, hence the second statement in Theorem 2 implies the first one.

4. Concluding remarks. Lemma 3 above provides some information about the amalgamation classes of nonmodular equational classes. The following simple theorem, essentially obtained by A. Day by using and extending earlier ideas due to S. Comer and S. Fajtlowicz, contains further information.

THEOREM 3. *Let* K *be an equational class of lattices generated by a finite nondistributive lattice. Then* K *does not have the Amalgamation Property. In fact, the two-element chain does not belong to* Amal (K).

Proof. By a result of B. Jónsson [10] there exists an integer n such that every subdirectly irreducible lattice in K has not more than n elements.

Now assume that $C_2 \in \mathrm{Amal}\,(K)$. Since K is nondistributive K contains either the diamond (Figure 2) or N_5 (Figure 6). In the first case K has to contain all the lattices A_k of Figure 7. Indeed, A_1 is the diamond, so $A_1 \in K$ by assumption. If $A_k \in K$, then consider the embedding f_0 of C_2 into the diamond defined by $0f_0 = a$,
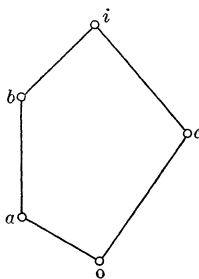
FIGURE 6.

$1f_0 = i$ and the embedding $f_1 \colon C_2 \to A_k$ given by $0f_1 = x$, $1f_1 = i$. Let g_0, g_1, L amalgamate this in K. Then Image $(f_0 g_0) \cup$ Image $(f_1 g_1)$ is a sublattice of L isomorphic to A_{k+1}. Hence $A_k \in K$ for all k. This contradicts the fact that $|A_k| > n$ for some k.

In the second case (i.e., $N_5 \in K$) let A be any finite subdirectly irreducible lattice and consider $f_0 \colon C_2 \to A$, $f_1 \colon C_2 \to N_5$ defined by

$$0f_0 = 0, 1f_0 = 1 \; ; \quad 0f_1 = a, 1f_1 = b \; .$$
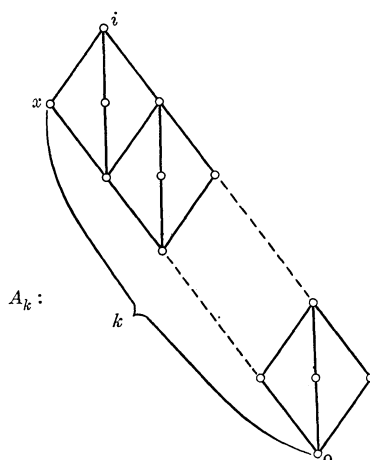
FIGURE 7.

Let g_0, g_1, B amalgamate these in K.   The Image $(f_0g_0) \cup$ Image $(f_1g_1) = A_N$ is a sublattice of B; hence the A_N of Figure 8 is in K.   Obviously
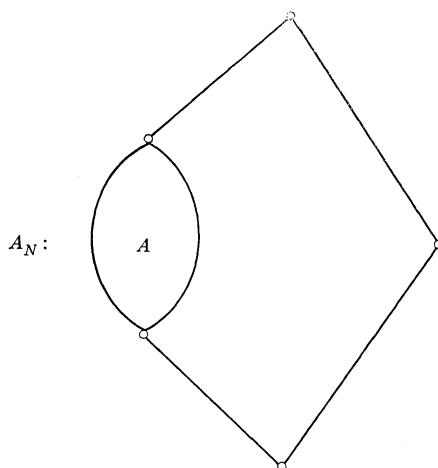
FIGURE 8.

A_N is subdirectly irreducible and $|A_N| = |A| + 3$.   Repeating this sufficiently many times we construct a subdirectly irreducible lattice in K with more than n elements.   This contradiction completes the proof of this theorem.

The results of this paper show that equational classes of lattices having the Amalgamation Property are hard to find.

We do not know of any lattice with more than one element that belongs to Amal (M).   Is there any?

For a division ring D, let $K(D)$ denote the class of all lattices

embeddable in some $\mathscr{L}(V, D)$. E. Fried [2[has shown that all finite distributive lattices and also the diamond belong to Amal $(K(D))$.

Theorem 2 is much weaker than Theorem 1 in that it only gives a necessary condition for $C_2 \in$ Amal (K), where K is an equational class of modular lattices. We do not know whether such classes exist excepting of course the class of distributive lattices and the trivial equational class.

Our information about amalgamation classes in general is still very limited. The following two observations are valid for any equational class K of algebras:

Suppose $A' \in$ Amal (K), and $\varphi: A \to A'$ is an embedding. If, for every $B \in K$, and every embedding $\psi: A \to B$, the amalgam $\langle A, \varphi, A', \psi, B \rangle$ can be amalgamated in K, then $A \in$ Amal (K).

If $A \in K$ and A is an absolute retract in K, then $A \in$ Amal (K).

A proof of the first observation is essentially contained in the proof of Lemma 3 in § 3. To prove the second, suppose B, $C \in K$, and suppose $f_0: A \to B$ and $f_1: A \to C$ are embeddings. Since A is an absolute retract in K, there exist onto homomorphisms $\varphi_0: B \to A$ and $\varphi_1: C \to A$ such that $b\varphi_0 f_0 = b$ for all $b \in B$ and $c\varphi_1 f_1 = c$ for all $c \in C$. Let $D = B \times C$, and for $b \in B$ and $c \in C$ let

$$bg_0 = \langle b, b\varphi_0 f_1 \rangle , \quad cg_1 = \langle c\varphi_1 f_0, c \rangle .$$

Then $g_0: B \to D$ and $g_1: C \to D$ are embeddings, and $f_0 g_0 = f_1 g_1$.

From these two observations one readily obtains:

If $A \times A' \in$ Amal (K), and if A' has a one-element subalgebra, then $A \in$ Amal (K).

If some reduced power of A belongs to Amal (K), then $A \in$ Amal (K).

If $A \in K$, A is subdirectly irreducible and of finite order n, and K has no subdirectly irreducible member whose order is larger than n, then $A \in$ Amal (K).

For the case when K is an equational class of lattices we also have:

If the lattice of all ideals of L belongs to Amal (K), then $L \in$ Amal (K).

If an ordered sum of two or more lattices belongs to Amal (K), then each summand belongs to Amal (K).

If K is generated by a finite, subdirectly irreducible lattice L, then $L \in$ Amal (K).

For certain equational classes K of modular lattices, specifically those generated by lattices of length two, the finite members of Amal (K) are determined in E. Fried, G. Grätzer, and H. Lakser [3].

Remark added in proof. It has been proved by M. Yashuhara that if K is an equational class, then every member of K can be extended to a member of Amal(K). This shows in particular that Amal(M) is very big. However, no lattice with more than one element is known to belong to Amal(M).

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