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CENTRAL 2-SYLOW INTERSECTIONS

MARCEL HERZOG

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Let G be a finite group. A subgroup D of G is called a 2-Sylow intersection if there exist distinct Sylow 2-subgroups S_1 and S_2 of G such that $D = S_1 \cap S_2$. An involution of G is called central if it is contained in a center of a Sylow 2-subgroup of G . A 2-Sylow intersection is called central if it contains a central involution. The aim of this work is to determine all non-abelian simple groups G which satisfy the following condition

B: the 2-rank of all central 2-Sylow intersections is not higher than 1, under the additional assumption that the centralizer of a central involution of G is solvable.

In 1964, M. Suzuki [5] determined all simple groups with all 2-Sylow intersections being trivial (i.e. of rank 0). Using a recent fusion theorem by E. Shult [3, p. 62] the author proved [4] that no additional simple groups are involved if Suzuki's condition is weakened to read: all central 2-Sylow intersections are trivial (i.e. no central involution is contained in a 2-Sylow intersection).

This paper is a step toward the characterization of all simple groups G which satisfy Condition B (in short $G \in B$). We will prove the following

THEOREM. *Let G be a non-abelian simple group. Suppose that $G \in B$ and the centralizer of a central involution z in G is solvable. Then G is isomorphic to one of the following groups:*

- (i) $\text{PSL}(2, q)$, $q = 2^n > 2$;
- (ii) $\text{Sz}(q)$, $q = 2^n \geq 8$;
- (iii) $\text{PSU}(3, q)$, $q = 2^n > 2$ and
- (iv) $\text{PSL}(2, q)$, $q \equiv 3$ or $5 \pmod{8}$, $q > 5$.

A finite group G is of 2-rank n if an elementary abelian 2-subgroup of G of maximal order contains 2^n elements. The 2-length of G is denoted by $l_2(G)$. The maximal power of 2 dividing $|G|$ is denoted by $|G|_2$. An involution z of G is called isolated if it belongs to a Sylow 2-subgroup S of G and $z^g \in S$ implies $z^g = z$. The maximal normal subgroup of G of odd order is denoted by $0(G)$. Finally the groups Q_8, S_3 and S_4 are the ordinary quaternion group, the symmetric group on 3 letters and the symmetric group on 4 letters, respectively.

2. Properties of groups satisfying Condition B.

LEMMA 1. *Let $G \in B, H \subseteq G$.*

- (i) *If $|H|_2 = |G|_2$ then $H \in B$.*
- (ii) *If $H \triangleleft G$ and $|G/H|_2 = |G|_2$ then $G/H \in B$.*

Proof. (i) is obvious. If H is a normal subgroup of G of odd order, then the S_2 -subgroups of $\bar{G} = G/H$ are of the form $SH/H = \bar{S} \cong S$, where S is an S_2 -subgroup of G . Let S_1 and S_2 be S_2 -subgroups of G such that $\bar{S}_1 \cap \bar{S}_2$ is a central 2-Sylow intersection of 2-rank at least 2. Since H is of odd order, there exists a 2-subgroup D of G , such that $\bar{S}_1 \cap \bar{S}_2 = DH/H = \bar{D} \cong D$. It is clear that there exist $h_1, h_2 \in H$ such that $D \subseteq S_1^{h_1} \cap S_2^{h_2}$. If zH is a central involution of $SH/H, z \in S$, then $[z, s] \in S \cap H = 1$ for all $s \in S$, hence $z \in Z(S)$. Thus D contains a central involution of G and as $G \in B$ and the 2-rank of D is at least 2, it follows that $S_1^{h_1} = S_2^{h_2}, \bar{S}_1 = \bar{S}_2$ and \bar{D} is not a 2-Sylow intersection of G . Thus $\bar{G} \in B$.

LEMMA 2. *Let $G \in B, H \subseteq G$ and suppose that the following assumptions hold:*

- (i) *H is solvable;*
- (ii) *$|H|_2 = |G|_2$ and*
- (iii) *$O_2(H)$ contains a central involution of G .*

Then $1_2(H) = 1$, unless $O_2(\bar{H}) \cong Q_8$ and $\bar{H}/O_2(\bar{H}) \cong S_3$, where $\bar{H} = H/O(H)$.

Proof. By Lemma 1 H and \bar{H} satisfy Condition B and $O_2(\bar{H})$ obviously contains a central involution of \bar{H} . If $O_2(\bar{H})$ is cyclic or generalized quaternion (but not ordinary quaternion), then $\text{Aut}(O_2(\bar{H}))$ is a 2-group and therefore $\bar{H}/C(O_2(\bar{H}))$ is a 2-group. As \bar{H} is solvable, $C(O_2(\bar{H})) \subseteq O_2(\bar{H})$ and consequently \bar{H} is a 2-group, hence $1_2(H) = 1$.

If $O_2(\bar{H})$ is of 2-rank at least 2, then $\bar{H} \in B$ forces \bar{H} to be 2-closed, hence $1_2(H) = 1$.

Suppose, finally, that $O_2(\bar{H}) \cong Q_8$. Then $\bar{H}/C(O_2(\bar{H}))$ is isomorphic to a subgroup of S_4 and if \bar{H} is not 2-closed then obviously 24 divides the order of $\bar{H}/C(O_2(\bar{H}))$. Thus $\bar{H}/C(O_2(\bar{H})) \cong S_4$ and $\bar{H}/O_2(\bar{H}) \cong S_3$.

LEMMA 3. *Let $G \in B$ and suppose that S and S_1 are S_2 -subgroups of G . Let $z \in Z(S)$ be an involution, $g \in G$, and suppose that $z^g \in S_1$. Then $z^g \in Z(S_1)$.*

Proof. Suppose that z^g is not central in S_1 . Then $S_1 \cap C_G(z^g)$ contains z^g and a central involution of S_1 . Let T be an S_2 -subgroup of $C_G(z^g)$ containing $S_1 \cap C_G(z^g)$; as $C_G(z^g) \cong S^g$, T is an S_2 -subgroup of G . Since the 2-rank of $D = S_1 \cap T$ is at least 2 and D contains a

central involution of G , it follows from our assumptions that $S_1 = T$, hence $z^g \in Z(S_1)$, a contradiction.

LEMMA 4. *Let $G \in B$ and suppose that $|\Omega_1(Z(S))| = 2$, where S is an S_2 -subgroup of G . Then $\Omega_1(Z(S)) \cong Z^*(G)$, where $Z^*(G)/O(G) = Z(G/O(G))$.*

Proof. Let $z \in \Omega_1(Z(S))$; then by Lemma 3 z is an isolated involution in G . It follows then by the Z^* -theorem of Glauberman [2] that $\Omega_1(Z(S)) \cong Z^*(G)$.

LEMMA 5. *Let $G \in B$, S be an S_2 -subgroup of G and $G = O(G)S$. Suppose that $|\Omega_1(Z(S))| > 2$ and S is not normal in G . Then the 2-rank of G is at most 2.*

Proof. Let G be a counterexample of minimal order. Then S contains an elementary abelian subgroup A of order 8 such that $|A : Z(S) \cap A| \leq 2$. Let $H = O(G)$ and $C = C_S(H)$. Then $C \triangleleft S$ and consequently $C \triangleleft SH = G$. As G is not 2-closed and $G \in B$, we have $A \not\subset C$. Consider AH ; A is not normal in AH and $|A \cap A^h| \leq 2$ for all $h \in H - N(A)$, as otherwise $G \in B$. Thus AH is a counterexample and by the minimality of G , $G = AH$.

Let P be a Sylow p -subgroup of H , such that $A \subseteq N(P)$ and $A \not\subset C(P)$; then again by the minimality of G , $G = AP$. As by a theorem of Burnside A does not centralize $P/\Phi(P)$, it follows by Lemma 1 (ii) and the minimality G that $\Phi(P) = 1$, P is elementary abelian. Since A acts on P in a completely reducible way, it follows again by the minimality of G that A acts irreducibly on P and $A/C_A(P)$ acts faithfully and irreducibly on P . Thus $A/C_A(P)$ is a cyclic group and $C_A(P)$ is a normal subgroup of G of 2-rank 2. As $C_A(P)$ contains a central involution and $G \in B$, it follows that G is 2-closed, a contradiction.

3. **Proof of the theorem.** Let $H = C_G(z)$. If H is 2-closed then by Lemma 3 z belongs to a unique Sylow 2-subgroup of G . Therefore by Theorem C of [4] G is isomorphic to one of the groups in (i)-(iii).

Suppose now that H is not 2-closed. Let $\bar{H} = H/O(H)$ and suppose that $O_2(\bar{H}) \cong Q_8$ and $\bar{H}/O_2(\bar{H}) \cong S_3$. Then obviously

$$(*) \quad \text{2-rank } H = \text{2-rank } G = 2.$$

Otherwise it follows by Lemma 2 that $1_2(H) = 1$, hence $O_{2',2}(H) = SL$, where $L = O_2(H)$ and S is an S_2 -subgroup of G . Since H is not 2-closed, S is not normal in $O_{2',2}(H)$. As G is simple, it follows by Lemma 4 that $|\Omega_1(Z(S))| > 2$ and Lemma 5 then yields (*) again.

Thus in all cases 2-rank $G = 2$ and by the classification theorem of Alperin, Brauer and Gorenstein [1] only three types of 2-groups could occur as a Sylow subgroup S of a group not mentioned in (i)-(iv):

- (a) dihedral of order 8 at least,
- (b) quasi-dihedral, or
- (c) wreathed.

In all of these cases $Z(S)$ is cyclic, hence by Lemma 4 G is non-simple, a contradiction. The proof of the theorem is complete.

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