

Pacific Journal of Mathematics

**ON THE NUMBER OF TYPE- k TRANSLATION-INVARIANT
GROUPS**

JOEL SAUL HILLEL

ON THE NUMBER OF TYPE- k TRANSLATION-INVARIANT GROUPS

J. HILLEL

The concept of a translation-invariant permutation group was introduced in connection with the problem of constructing "algebras of symmetry-classes of tensors". Such a group is of type- k if it has k orbits. In this paper the number of type- k groups is shown to be the same as the number of divisors of $X^k - 1$ over the two-element field.

Let S_∞ be the group of all permutations of finite degree on the set $\{1, 2, 3, \dots\}$. If σ is the permutation given by $(a_1 b_1)(a_2 b_2) \cdots (a_t b_t)$, its *translate* $\sigma^{[1]}$ is defined to be the permutation

$$(a_1 + 1 \ b_1 + 1)(a_2 + 1 \ b_2 + 1) \cdots (a_t + 1 \ b_t + 1).$$

The definition of the translate of σ is independent of the decomposition of σ into a product of transpositions. A subgroup H of S_∞ is said to be *translation-invariant* (briefly, H is a $t - i$ group) if whenever σ is in H so is $\sigma^{[1]}$.

The translation-invariant groups were first introduced in [1] in connection with the problem of generalizing the construction of the Tensor, Grassmann and Symmetric algebras by using symmetry-classes of tensors (see [2]). The following was proven in [1]: if H is a non-trivial $t - i$ group (assume H moves 1), then the orbits for the action of H on $\{1, 2, 3, \dots\}$ are $Z_{i,k} = \{i, i + k, i + 2k, \dots\}$, $1 \leq i \leq k$, for some $k \geq 1$. The number of orbits is called the *type* of H . Let $S_{i,\infty}$ (resp. $A_{i,\infty}$) be the group of all (resp. even) permutations on the set $Z_{i,k}$, $1 \leq i \leq k$, and let $S_\infty(k) = S_{1,\infty} X \cdots X S_{k,\infty}$, $A_\infty(k) = A_{1,\infty} X \cdots X A_{k,\infty}$. For each $k \geq 1$, these are $t - i$ groups and if H is any type- k $t - i$ group, clearly $H < S_\infty(k)$. Moreover, it was proven that a $t - i$ group contains all the even permutations on each of its orbits, i.e.,

THEOREM 1. *If H is a type- k $t - i$ group then $A_\infty(k) < H < S_\infty(k)$.*

In this presentation we are concerned with determining the number of type- k $t - i$ groups for each $k \geq 1$. In [1] it was proven that:

THEOREM 2. *There are $2^n + 1$ $t - i$ groups of type- 2^n , $n \geq 0$.*

The above theorem was proved by looking at some special features of the lattice of the type- k $t - i$ groups. However, here we will show that the number of type- k $t - i$ groups is the same as the number of factors of the polynomial $X^k - 1$ over the two-element field F_2 and

thus is completely known.

2. Let $k \geq 1$ be fixed and let $P(k)$ denote the power set on the set $\{1, 2, \dots, k\}$. Let Δ denote the symmetric-difference of sets, then $\{P(k), \Delta\}$ is an abelian group whose zero element is the empty set ϕ , and every α in $P(k)$ satisfies $\alpha\Delta\alpha = \phi$, i.e., $\{P(k), \Delta\}$ is a k -dimensional vector-space over F_2 and the singleton sets $\{i\}, 1 \leq i \leq k$ form a basis.

Any permutation σ in $S_\infty(k)$ can be written as a product $\sigma_1\sigma_2 \cdots \sigma_k$ where σ_i is a permutation on the orbit $Z_{i,k}, 1 \leq i \leq k$. Define $F(\sigma)$ to be $\{i_1, \dots, i_t\}$ where $\sigma_{i_1}, \dots, \sigma_{i_t}$ are those permutations among $\sigma_1, \dots, \sigma_k$ which have odd parity. The map $F: S_\infty(k) \rightarrow P(k)$ satisfies $F(\sigma\tau) = F(\sigma)\Delta F(\tau)$ for every σ and τ in $S_\infty(k)$, i.e., F is a group homomorphism with $\text{Ker}(F) = A_\infty(k)$. By Theorem 1, the usual correspondence between subgroups of $S_\infty(k)$ which contain $A_\infty(k)$ and the subgroups of $P(k)$ sets a one-to-one correspondence between the type- k $t - i$ groups and a certain subfamily of subgroups of $P(k)$ (the $t - i \pmod k$ subgroups in [1]).

Consider the basis $C_k = \{\{1\}, \dots, \{k\}\}$ of the vector-space $P(k)$ and define a multiplication on C_k by $\{i\} \cdot \{j\} = \{(i + j - 1) \pmod k\}$ for $1 \leq i \leq k, 1 \leq j \leq k$. C_k thus becomes a cyclic group and the multiplication is uniquely extendable to all of $P(k)$, i.e.,

$$\{i_1, \dots, i_m\} \cdot \{j_1, \dots, j_n\} = \bigtriangleup_{\substack{1 \leq r \leq m \\ 1 \leq s \leq n}} \{i_r\} \cdot \{j_s\}.$$

This multiplication endows $P(k)$ with a commutative ring structure. In fact, $P(k)$ is the group-ring $F_2(C_k)$. We note that as $\{2\}$ is a generator of the group C_k , it is also a generator (in the algebraic sense) of $P(k)$.

PROPOSITION. *The type- k $t - i$ groups are in one-to-one correspondence with the ideals of the ring $P(k)$.*

Proof. Let I be a nontrivial subgroup of $P(k)$ which corresponds to a $t - i$ group H under the homomorphism F defined above. Suppose $\alpha = \{i_1, \dots, i_t\}$ is in I , then $F(\sigma) = \alpha$ for some σ in H , i.e., $\sigma = \sigma_1 \cdots \sigma_k$ where σ_i acts on the orbit $Z_{i,k}$ and $\sigma_{i_1}, \dots, \sigma_{i_t}$ are the permutations of odd parity. Since H is a $t - i$ group, $\tau = \sigma^{[1]}$ is in H and $F(\tau)$ is in I . Writing τ as a product $\tau_1 \cdots \tau_k$ where τ_i acts on $Z_{i,k}$, it is easily seen that $\tau_{i+1} = \sigma_i, 1 \leq i < k$ and $\tau_1 = \sigma_k^{[1]}$. Hence $F(\tau) = \{(i_1 + 1) \pmod k, \dots, (i_t + 1) \pmod k\} = \{i_1, \dots, i_t\} \cdot \{2\}$, i.e., $\alpha \cdot \{2\}$ is in I whenever α is in I . As $\{2\}$ generates the whole ring, it follows that I is an ideal.

Conversely, if I is an ideal of $P(k)$ it is immediate that $F^{-1}(I)$ is a $t - i$ group.

The group-ring $P(k)$ is isomorphic to $F_2[X]/(X^k - 1)$ hence the ideals in $P(k)$ correspond to the divisors of $X^k - 1$ in $F_2[X]$. Let $k = 2^n r$ where $(2, r) = 1$, then $X^k - 1 = (X^r - 1)^{2^n}$. Now $X^r - 1 = \prod_{d|r} \phi_d(X)$ where $\phi_d(X)$ are the cyclotomic polynomials. Furthermore (see [3], Theorem 7-2-4), $\phi_d(X)$ is a product of the irreducible polynomials $P_1(X) \cdots P_{m_d}(X)$, $m_d = \varphi(d)/f_d$, where φ is the Euler function and f_d is the smallest integer f such that $2^f \equiv 1 \pmod{d}$. Thus, if s_r is the number of irreducible divisors of $X^r - 1$, then $s_r = \sum_{d|r} \varphi(d)/f_d$. Letting $s_1 = 1$, we conclude:

THEOREM 3. *Let $k = 2^n r$ where $(2, r) = 1$, then there are $(2^n + 1)^{s_r}$ translation-invariant groups of type- k .*

REFERENCES

1. J. Hillel, *Algebras of symmetry classes of tensors*, J. Algebra, **23** (1972), 215-227.
2. M. Marcus and H. Minc, *Permutations on symmetry classes*, J. Algebra, **5** (1967), 59-71.
3. E. Weiss, *Algebraic Number Theory*, McGraw-Hill, 1963.

Received November 14, 1971.

SIR GEORGE WILLIAMS UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON

Stanford University
Stanford, California 94305

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY

University of Washington
Seattle, Washington 98105

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 45, No. 2

October, 1973

Kenneth Paul Baclawski and Kenneth Kapp, <i>Induced topologies for quasigroups and loops</i>	393
D. G. Bourgin, <i>Fixed point and min – max theorems</i>	403
J. L. Brenner, <i>Zolotarev's theorem on the Legendre symbol</i>	413
Jospeh Atkins Childress, Jr., <i>Restricting isotopies of spheres</i>	415
John Edward Coury, <i>Some results on lacunary Walsh series</i>	419
James B. Derr and N. P. Mukherjee, <i>Generalized Sylow tower groups. II</i>	427
Paul Frazier Duvall, Jr., Peter Fletcher and Robert Allen McCoy, <i>Isotopy Galois spaces</i>	435
Mary Rodriguez Embry, <i>Strictly cyclic operator algebras on a Banach space</i>	443
Abi (Abiadbollah) Fattahi, <i>On generalizations of Sylow tower groups</i>	453
Burton I. Fein and Murray M. Schacher, <i>Maximal subfields of tensor products</i> ...	479
Ervin Fried and J. Sichler, <i>Homomorphisms of commutative rings with unit element</i>	485
Kenneth R. Goodearl, <i>Essential products of nonsingular rings</i>	493
George Grätzer, Bjarni Jónsson and H. Lakser, <i>The amalgamation property in equational classes of modular lattices</i>	507
H. Groemer, <i>On some mean values associated with a randomly selected simplex in a convex set</i>	525
Marcel Herzog, <i>Central 2-Sylow intersections</i>	535
Joel Saul Hillel, <i>On the number of type-k translation-invariant groups</i>	539
Ronald Brian Kirk, <i>A note on the Mackey topology for $(C^b(X))^*$, $C^b(X)$</i>	543
J. W. Lea, <i>The peripherality of irreducible elements of lattice</i>	555
John Stewart Locker, <i>Self-adjointness for multi-point differential operators</i>	561
Robert Patrick Martineau, <i>Splitting of group representations</i>	571
Robert Massagli, <i>On a new radical in a topological ring</i>	577
James Murdoch McPherson, <i>Wild arcs in three-space. I. Families of Fox-Artin arcs</i>	585
James Murdoch McPherson, <i>Wild arcs in three-space. III. An invariant of oriented local type for exceptional arcs</i>	599
Fred Richman, <i>The constructive theory of countable abelian p-groups</i>	621
Edward Barry Saff and J. L. Walsh, <i>On the convergence of rational functions which interpolate in the roots of unity</i>	639
Harold Eugene Schlais, <i>Non-aposyndesis and non-hereditary decomposability</i>	643
Mark Lawrence Teply, <i>A class of divisible modules</i>	653
Edward Joseph Tully, Jr., <i>\mathcal{H}-commutative semigroups in which each homomorphism is uniquely determined by its kernel</i>	669
Garth William Warner, Jr., <i>Zeta functions on the real general linear group</i>	681
Keith Yale, <i>Cocycles with range $\{\pm 1\}$</i>	693
Chi-Lin Yen, <i>On the rest points of a nonlinear nonexpansive semigroup</i>	699