

# Pacific Journal of Mathematics

## ON THE NUMBER OF TYPE- $k$ TRANSLATION-INVARIANT GROUPS

JOEL SAUL HILLEL

## ON THE NUMBER OF TYPE- $k$ TRANSLATION-INVARIANT GROUPS

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**The concept of a translation-invariant permutation group was introduced in connection with the problem of constructing "algebras of symmetry-classes of tensors". Such a group is of type- $k$  if it has  $k$  orbits. In this paper the number of type- $k$  groups is shown to be the same as the number of divisors of  $X^k - 1$  over the two-element field.**

Let  $S_\infty$  be the group of all permutations of finite degree on the set  $\{1, 2, 3, \dots\}$ . If  $\sigma$  is the permutation given by  $(a_1 b_1)(a_2 b_2) \dots (a_t b_t)$ , its *translate*  $\sigma^{[1]}$  is defined to be the permutation

$$(a_1 + 1 \ b_1 + 1)(a_2 + 1 \ b_2 + 1) \dots (a_t + 1 \ b_t + 1).$$

The definition of the translate of  $\sigma$  is independent of the decomposition of  $\sigma$  into a product of transpositions. A subgroup  $H$  of  $S_\infty$  is said to be *translation-invariant* (briefly,  $H$  is a  $t - i$  group) if whenever  $\sigma$  is in  $H$  so is  $\sigma^{[1]}$ .

The translation-invariant groups were first introduced in [1] in connection with the problem of generalizing the construction of the Tensor, Grassmann and Symmetric algebras by using symmetry-classes of tensors (see [2]). The following was proven in [1]: if  $H$  is a non-trivial  $t - i$  group (assume  $H$  moves 1), then the orbits for the action of  $H$  on  $\{1, 2, 3, \dots\}$  are  $Z_{i,k} = \{i, i + k, i + 2k, \dots\}$ ,  $1 \leq i \leq k$ , for some  $k \geq 1$ . The number of orbits is called the *type* of  $H$ . Let  $S_{i,\infty}$  (resp.  $A_{i,\infty}$ ) be the group of all (resp. even) permutations on the set  $Z_{i,k}$ ,  $1 \leq i \leq k$ , and let  $S_\infty(k) = S_{1,\infty} X \dots X S_{k,\infty}$ ,  $A_\infty(k) = A_{1,\infty} X \dots X A_{k,\infty}$ . For each  $k \geq 1$ , these are  $t - i$  groups and if  $H$  is any type- $k$   $t - i$  group, clearly  $H < S_\infty(k)$ . Moreover, it was proven that a  $t - i$  group contains all the even permutations on each of its orbits, i.e.,

**THEOREM 1.** *If  $H$  is a type- $k$   $t - i$  group then  $A_\infty(k) < H < S_\infty(k)$ .*

In this presentation we are concerned with determining the number of type- $k$   $t - i$  groups for each  $k \geq 1$ . In [1] it was proven that:

**THEOREM 2.** *There are  $2^n + 1$   $t - i$  groups of type- $2^n$ ,  $n \geq 0$ .*

The above theorem was proved by looking at some special features of the lattice of the type- $k$   $t - i$  groups. However, here we will show that the number of type- $k$   $t - i$  groups is the same as the number of factors of the polynomial  $X^k - 1$  over the two-element field  $F_2$  and

thus is completely known.

2. Let  $k \geq 1$  be fixed and let  $P(k)$  denote the power set on the set  $\{1, 2, \dots, k\}$ . Let  $\Delta$  denote the symmetric-difference of sets, then  $\{P(k), \Delta\}$  is an abelian group whose zero element is the empty set  $\phi$ , and every  $\alpha$  in  $P(k)$  satisfies  $\alpha\Delta\alpha = \phi$ , i.e.,  $\{P(k), \Delta\}$  is a  $k$ -dimensional vector-space over  $F_2$  and the singleton sets  $\{i\}, 1 \leq i \leq k$  form a basis.

Any permutation  $\sigma$  in  $S_\infty(k)$  can be written as a product  $\sigma_1\sigma_2 \dots \sigma_k$  where  $\sigma_i$  is a permutation on the orbit  $Z_{i,k}, 1 \leq i \leq k$ . Define  $F(\sigma)$  to be  $\{i_1, \dots, i_t\}$  where  $\sigma_{i_1}, \dots, \sigma_{i_t}$  are those permutations among  $\sigma_1, \dots, \sigma_k$  which have odd parity. The map  $F: S_\infty(k) \rightarrow P(k)$  satisfies  $F(\sigma\tau) = F(\sigma)\Delta F(\tau)$  for every  $\sigma$  and  $\tau$  in  $S_\infty(K)$ , i.e.,  $F$  is a group homomorphism with  $\text{Ker}(F) = A_\infty(k)$ . By Theorem 1, the usual correspondence between subgroups of  $S_\infty(k)$  which contain  $A_\infty(k)$  and the subgroups of  $P(k)$  sets a one-to-one correspondence between the type- $k$   $t - i$  groups and a certain subfamily of subgroups of  $P(k)$  (the  $t - i \pmod k$  subgroups in [1]).

Consider the basis  $C_k = \{\{1\}, \dots, \{k\}\}$  of the vector-space  $P(k)$  and define a multiplication on  $C_k$  by  $\{i\} \cdot \{j\} = \{(i + j - 1) \pmod k\}$  for  $1 \leq i \leq k, 1 \leq j \leq k$ .  $C_k$  thus becomes a cyclic group and the multiplication is uniquely extendable to all of  $P(k)$ , i.e.,

$$\{i_1, \dots, i_m\} \cdot \{j_1, \dots, j_n\} = \bigtriangleup_{\substack{1 \leq r \leq m \\ 1 \leq s \leq n}} \{i_r\} \cdot \{j_s\}.$$

This multiplication endows  $P(k)$  with a commutative ring structure. In fact,  $P(k)$  is the group-ring  $F_2(C_k)$ . We note that as  $\{2\}$  is a generator of the group  $C_k$ , it is also a generator (in the algebraic sense) of  $P(k)$ .

**PROPOSITION.** *The type- $k$   $t - i$  groups are in one-to-one correspondence with the ideals of the ring  $P(k)$ .*

*Proof.* Let  $I$  be a nontrivial subgroup of  $P(k)$  which corresponds to a  $t - i$  group  $H$  under the homomorphism  $F$  defined above. Suppose  $\alpha = \{i_1, \dots, i_t\}$  is in  $I$ , then  $F(\sigma) = \alpha$  for some  $\sigma$  in  $H$ , i.e.,  $\sigma = \sigma_1 \dots \sigma_k$  where  $\sigma_i$  acts on the orbit  $Z_{i,k}$  and  $\sigma_{i_1}, \dots, \sigma_{i_t}$  are the permutations of odd parity. Since  $H$  is a  $t - i$  group,  $\tau = \sigma^{[1]}$  is in  $H$  and  $F(\tau)$  is in  $I$ . Writing  $\tau$  as a product  $\tau_1 \dots \tau_k$  where  $\tau_i$  acts on  $Z_{i,k}$ , it is easily seen that  $\tau_{i+1} = \sigma_i, 1 \leq i < k$  and  $\tau_1 = \sigma_k^{[1]}$ . Hence  $F(\tau) = \{(i_1 + 1) \pmod k, \dots, (i_t + 1) \pmod k\} = \{i_1, \dots, i_t\} \cdot \{2\}$ , i.e.,  $\alpha \cdot \{2\}$  is in  $I$  whenever  $\alpha$  is in  $I$ . As  $\{2\}$  generates the whole ring, it follows that  $I$  is an ideal.

Conversely, if  $I$  is an ideal of  $P(k)$  it is immediate that  $F^{-1}(I)$  is a  $t - i$  group.

The group-ring  $P(k)$  is isomorphic to  $F_2[X]/(X^k - 1)$  hence the ideals in  $P(k)$  correspond to the divisors of  $X^k - 1$  in  $F_2[X]$ . Let  $k = 2^n r$  where  $(2, r) = 1$ , then  $X^k - 1 = (X^r - 1)^{2^n}$ . Now  $X^r - 1 = \prod_{d|r} \phi_d(X)$  where  $\phi_d(X)$  are the cyclotomic polynomials. Furthermore (see [3], Theorem 7-2-4),  $\phi_d(X)$  is a product of the irreducible polynomials  $P_1(X) \cdots P_{m_d}(X)$ ,  $m_d = \varphi(d)/f_d$ , where  $\varphi$  is the Euler function and  $f_d$  is the smallest integer  $f$  such that  $2^f \equiv 1 \pmod{d}$ . Thus, if  $s_r$  is the number of irreducible divisors of  $X^r - 1$ , then  $s_r = \sum_{d|r} \varphi(d)/f_d$ . Letting  $s_1 = 1$ , we conclude:

**THEOREM 3.** *Let  $k = 2^n r$  where  $(2, r) = 1$ , then there are  $(2^n + 1)^{s_r}$  translation-invariant groups of type- $k$ .*

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Kenneth Paul Baclawski and Kenneth Kapp, <i>Induced topologies for quasigroups and loops</i> .....	393
D. G. Bourgin, <i>Fixed point and min – max theorems</i> .....	403
J. L. Brenner, <i>Zolotarev's theorem on the Legendre symbol</i> .....	413
Joseph Atkins Childress, Jr., <i>Restricting isotopies of spheres</i> .....	415
John Edward Coury, <i>Some results on lacunary Walsh series</i> .....	419
James B. Derr and N. P. Mukherjee, <i>Generalized Sylow tower groups. II</i> .....	427
Paul Frazier Duvall, Jr., Peter Fletcher and Robert Allen McCoy, <i>Isotopy Galois spaces</i> .....	435
Mary Rodriguez Embry, <i>Strictly cyclic operator algebras on a Banach space</i> .....	443
Abi (Abiadbollah) Fattahi, <i>On generalizations of Sylow tower groups</i> .....	453
Burton I. Fein and Murray M. Schacher, <i>Maximal subfields of tensor products</i> .....	479
Ervin Fried and J. Sichler, <i>Homomorphisms of commutative rings with unit element</i> .....	485
Kenneth R. Goodearl, <i>Essential products of nonsingular rings</i> .....	493
George Grätzer, Bjarni Jónsson and H. Lakser, <i>The amalgamation property in equational classes of modular lattices</i> .....	507
H. Groemer, <i>On some mean values associated with a randomly selected simplex in a convex set</i> .....	525
Marcel Herzog, <i>Central 2-Sylow intersections</i> .....	535
Joel Saul Hillel, <i>On the number of type-k translation-invariant groups</i> .....	539
Ronald Brian Kirk, <i>A note on the Mackey topology for <math>(C^b(X))^*</math>, <math>C^b(X)</math></i> .....	543
J. W. Lea, <i>The peripherality of irreducible elements of lattice</i> .....	555
John Stewart Locker, <i>Self-adjointness for multi-point differential operators</i> .....	561
Robert Patrick Martineau, <i>Splitting of group representations</i> .....	571
Robert Massagli, <i>On a new radical in a topological ring</i> .....	577
James Murdoch McPherson, <i>Wild arcs in three-space. I. Families of Fox-Artin arcs</i> .....	585
James Murdoch McPherson, <i>Wild arcs in three-space. III. An invariant of oriented local type for exceptional arcs</i> .....	599
Fred Richman, <i>The constructive theory of countable abelian p-groups</i> .....	621
Edward Barry Saff and J. L. Walsh, <i>On the convergence of rational functions which interpolate in the roots of unity</i> .....	639
Harold Eugene Schlais, <i>Non-aposyndesis and non-hereditary decomposability</i> .....	643
Mark Lawrence Teply, <i>A class of divisible modules</i> .....	653
Edward Joseph Tully, Jr., <i><math>\mathcal{H}</math>-commutative semigroups in which each homomorphism is uniquely determined by its kernel</i> .....	669
Garth William Warner, Jr., <i>Zeta functions on the real general linear group</i> .....	681
Keith Yale, <i>Cocycles with range <math>\{\pm 1\}</math></i> .....	693
Chi-Lin Yen, <i>On the rest points of a nonlinear nonexpansive semigroup</i> .....	699