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# THE PERIPHERALITY OF IRREDUCIBLE ELEMENTS OF A LATTICE

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**Irreducible elements, which are not cutpoints, and meet complemented elements are peripheral in certain compact lattices and semilattices.**

The purpose of this note is to extend the results of A. D. Wallace [12]; i.e., to show that meet irreducible elements and meet complemented elements are peripheral in certain topological lattices and semilattices. Meet irreducible elements have played a key role in embedding theorems for topological lattices obtained by K. A. Baker and A. R. Stralka [2] and by the author [10].

**1. Preliminaries.** If $S$ is a semilattice and $x \in S$, then $M(x) = \{y \in S\colon x \leqslant y\}$; $L(x)$ is defined dually; if $x \leqslant y$, then

$$[x, y] = M(x) \cap L(y) .$$

An element $x$ in a semilattice $S$ is *meet irreducible* if $a$, $b \in S$ and $x = a \wedge b$ imply $x = a$ or $x = b$. We denote the set of all meet irreducible elements of $S$ by $MI(S)$. If $S$ has a 0, an element $x \in S$ is *meet complemented* if there is a $y \in S$ such that $y \neq 0$ and $x \wedge y = 0$. The *width* $w(X)$ of a partially ordered set $X$ is the maximum number of elements in a set of incomparable elements.

A topological semilattice $S$ is said to have *small semilattices at* $x$ if $x$ has a basis of neighborhoods which are subsemilattices of $S$; $S$ is a *Lawson semilattice* if it has small semilattices at every point.

Basic definitions, notations, and properties of Alexander cohomology and codimension may be found in [11] and [4]. The point $x$ is *marginal* in a regular space $X$ if and only if for any open set $U$ containing $x$, there exists an open set $V$ containing $x$ and contained in $U$ such that the natural homomorphism $H^*(X) \rightarrow H^*(X \backslash V)$ is an isomorphism [8, Th. 1.3]. A point $x \in X$, a topological space, is *peripheral* if for any open set $U$ containing $x$, there exists an open set $V$ containing $x$ and contained in $U$ such that the homomorphism $i^*\colon H^*(X, X \backslash V) \rightarrow (X, X \backslash U)$ induced by the inclusion mapping $i$ is the trivial or zero homomorphism. A point is *inner* if it is not peripheral.

**2. Peripheral elements.** J. D. Lawson and B. Madison [9, Th. 3.2] have proved that cutpoints of compact, connected spaces are inner points. This result and Theorem 2.1 locate all the meet irreducible elements of a semilattice.

THEOREM 2.1. *Let $S$ be a compact, connected, locally connected Lawson semilattice. If $p \in S$ is meet irreducible and is not a cutpoint of $S$, then $p$ is marginal in $S$.*

*Proof.* Let $p \in U$, an open subset of $S$. Since $p$ is not a cutpoint of $S$, it is known [14, III 4.15] that there exists an open set $V$ containing $p$ such that $V \subset U$ and $S\backslash V$ is connected. If

$$x, y \in S\backslash\{p\},$$

then $x \wedge y \in S\backslash\{p\}$ because $p$ is meet irreducible. Thus $S\backslash\{p\}$ is a locally compact Lawson semilattice. Hence there exists a compact subsemilattice $W \subset S\backslash\{p\}$ such that $S\backslash V \subset W$ [6, Lemma 5.2]. Since $S\backslash V$ is connected, the closed semilattice $B$ generated by $S\backslash V$ is a compact, connected subsemilattice of $W$; $B$ is acyclic [13]; thus $H^*(S) \to H^*(B)$ is an isomorphism. Since $S\backslash V \subset B \subset W \subset S\backslash\{p\}$, we have $p \in S\backslash B \subset V \subset U$. Therefore $p$ is marginal in $S$ and hence peripheral in $S$ [8, Th. 1.8].

COROLLARY 2.2. *Let $L$ be a compact, connected topological lattice of finite codimension. If $p \in L$ is not a cutpoint of $L$ and $p$ is either meet irreducible or join irreducible, then $p$ is marginal in $L$.*

*Proof.* Since $L$ is compact and connected, its breadth is equal to its codimension [7, Cor. 2.4]. Hence $L$ is a Lawson semilattice with respect to either $\vee$ or $\wedge$ [7, Th. 1.1]. Finally, $L$ is locally connected [1, Th. 2]; therefore the conclusion follows from Theorem 2.1.

Peripheral elements need not belong to $(MI(L) \cup JI(L))^*$. Examples may be found in $I^3$, the unit cube.

THEOREM 2.3. *Let $S$ be a compact, connected, locally connected topological semilattice with 1. If $p \in S$ and $M(p)^\circ = \varnothing$, then $M(p)$ is contained in the set of peripheral elements of $L$.*

*Proof.* We define $F: S \times S \to S$ by $F(x, y) = x \wedge y$ for all $x, y \in S$. Then $F$ is continuous and $F(1, x) = x$ for all $x \in S$. If $s \in M(p)$ and $s$ is an inner point of $S$, then there exists an open set $U$ containing 1 such that for each $u \in U$ there is a $v \in S$ with $u \wedge v = s$ [8, Th. 3.4]. This implies $U \subset M(s) \subset M(p)$ so that $M(p)^\circ \neq \varnothing$ contrary to hypothesis. Thus $s$ is peripheral in $S$; since $s$ was an arbitrary element of $M(p)$, $M(p)$ consists entirely of peripheral elements of $S$.

As noted above a compact, connected topological lattice $L$ is locally connected and has a 1; thus if $M(p)^\circ = \varnothing$ in such a lattice, then $M(p)$ consists of peripheral elements of $L$.

The set of peripheral elements of a topological space need not be closed [8, p. 261]. However, we have the following.

COROLLARY 2.4. *Let $L$ be a compact, connected topological lattice of finite codimension. If $A = \{x \in L\colon M(x)^\circ = \varnothing\}$, then each element of $A^*$ is peripheral in $L$.*

*Proof.* Let $x \in A^*$ and suppose that $x$ is an inner point of $L$. Let $\{x_\alpha\}$ be a net in $A$ which converges to $x$. Then $\{x_\alpha \vee x\}$ also converges to $x$, and since $M(x_\alpha \vee x) \subset M(x_\alpha)$, we must have

$$M(x_\alpha \vee x)^\circ = \varnothing .$$

Thus $\{x_\alpha \vee x\} \subset A$. By Theorem 2.3 and our assumption that $x$ is inner, $M(x)^\circ \neq \varnothing$. Since the codimension of $L$ is finite we may choose an inner point $y$ of $L$ such that $y \in M(x)^\circ$ [8, Th. 2.6]; thus $y$ is also an inner point of $M(x)$ [8, Th. 1.4]. Since $x$ is the zero of $M(x)$ and $y$ is inner in $M(x)$, it follows from the proof of Thorem 2.3 that there must be an open subset $U$ of $M(x)$ which contains $x$ and such that $u \in U$ implies $u \leqslant y$. The net $\{x_\alpha \vee x\}$ converges to $x$ and $\{x_\alpha \vee x\} \subset M(x)$; therefore there exists an $\alpha$ such that $x_\alpha \vee x \in U$. Hence $x_\alpha \vee x \leqslant y$; therefore $y \in M(x_\alpha \vee x)$ which implies

$$M(y) \subset M(x_\alpha \vee x) .$$

But $M(y)^\circ \subset M(y) \subset M(x_\alpha \vee x)$ and $M(y)^\circ \neq \varnothing$ since $y$ is an inner point of $L$. Thus $M(x_\alpha \vee x)^\circ \neq \varnothing$ contrary to $x_\alpha \vee x \in A$. This contradiction completes the proof.

The set $A^*$ of Corollary 2.4 has some interesting properties not necessarily held by either the set of all peripheral elements or by the set of all meet irreducible elements of a lattice.

PROPOSITION 2.5. *Let $L$ and $A^*$ be as in Corollary 2.4.*

(1) *$x \leqslant y$ and $x \in A^*$ imply $y \in A^*$.*

(2) *$A^*$ is connected.*

(3) *If the breadth of $L$, $b(L)$, is two then $A^*$ consists of meet irreducible elements of $L$.*

(4) *If $b(L) = 2$ and $w(MI(L)) = n$, then $A^*$ is the union of at most $n$ compact, connected chains.*

(5) *If $A^*$ is a sublattice of $L$, then $A^* = M(\bigwedge A^*)$ and $b(A^*) = cd(A^*) < b(L) = cd(L)$.*

*Proof.* (1) Clearly $A$ is an increasing set; hence $A^*$ is also.

(2) For each $x \in A^*$, $M(x)$ is a connected subset of $A^*$ which contains 1. Thus $A^*$ is connected.

(3) Let $x \in A^*$. Since $M(x) \subset A^*$ and $b(L) = 2$, $b(M(x)) = cd(M(x)) = 1$ [8, Th. 3.2]. Thus $x = a \wedge b$ implies $x = a$ or $x = b$.

(4) Since $A^* \subset MI(L)$, $w(A^*) \leqslant n$. Hence by Dilworth's theorem [5, Th. 1.1], $A^*$ is the union of $n$ or fewer chains. These chains may be chosen to be compact and connected.

(5) If $A^*$ is a sublattice of $L$, then $z = \bigwedge A^*$ belongs to $A^*$ and $A^* = M(z)$. As noted above $b(A^*) = cd(A^*) < cd(L) = b(L)$.

If $b(L) > 2$, then $M(x) \subset A^*$ need not imply $M(x)$ is a chain; examples may be found in $I^3$, the unit cube.

Let $L = I^2 \backslash \{(x, y)\colon 0 \leqslant x < 1/4, 3/4 < y \leqslant 1\}$. Then $L$ is a compact, connected, distributive topological lattice of breadth two and $A^*$ is a proper subset of $MI(L)$.

EXAMPLE 2.6. Let

$$L = \{(x_i)\colon 0 \leqslant x_i \leqslant 1\} \cup \{(x_i)\colon -1 \leqslant x_i \leqslant 0\} \subset \prod_{i=1}^{\infty} R_i\ ,$$

$R_i$ the set of real numbers for $i = 1, 2, \cdots$. With the order and topology inherited from $\prod_{i=1}^{\infty} R_i$, i.e., $(x_i) \leqslant (y_i)$ if and only if $x_i \leqslant y_i$ for $i = 1, 2, \cdots$, $L$ is a compact, connected topological lattice. Since $p = (p_i)$ with $p_i = 0$ for $i = 1, 2, \cdots$ is a cutpoint of $L$, $p$ is an inner point of $L$. Any $(x_i) \in L$ with $0 < x_i \leqslant 1$ for infinitely many $i$ has the property that $M((x_i))^\circ$ is empty. Thus

$$p \in \{(x_i)\colon M((x_i))^\circ = \varnothing\}^*\ .$$

THEOREM 2.7. *Let $L$ be a compact, connected topological lattice. If $a, b \in L$ and $a$ is a meet complement for $b$, then $[0, a]$ and $[0, b]$ are contained in the set of peripheral elements of $L$.*

*Proof.* We define $F\colon L \times L \to L$ by $F(x, y) = x \vee y$ for all $x, y \in L$. Then $F$ is continuous and $F(0, y) = y$ for all $y \in L$. Let $x \in (0, a] = [0, a]\backslash\{0\}$; then $x \wedge b \leqslant a \wedge b = 0$ which implies $x \wedge b = 0$. If $x$ is not peripheral in $L$, then there exists an open set $U$ containing 0 such that for each $s \in U$ there is a $t \in L$ for which $s \vee t = x$ [8, Th. 3.4]. Since $[0, b] = b \wedge L$, it is connected; thus $U \cap (0, b] \neq \varnothing$. Let $s \in U \cap (0, b]$ and let $t \in L$ be such that $s \vee t = x$. Then $s \leqslant x$ and $s \leqslant b$; thus $s \leqslant x \wedge b = 0$ which implies $s = 0$ contrary to

$s \in (0, b]$. Hence $x$ is peripheral in $L$. That 0 is peripheral is a consequence of Theorem 2.3. Thus each element of $[0, a]$ is peripheral in $L$. The proof for $[0, b]$ is similar.

The following corollaries are immediate.

COROLLARY 2.8. *Let $L$ be a compact, connected topological lattice. If $a, b \in L$ are not related, then $[a, a \vee b]$, $[b, a \vee b]$, $[a \wedge b, a]$, and $[a \wedge b, b]$ are contained in the set of peripheral elements of*

$$[a \wedge b, a \vee b] .$$

COROLLARY 2.9. *Let $L$ be a compact, connected topological lattice. If for $p \in L$ there is a $q \in L$ such that $q$ is not related to $p$ and either $p \in M(p \wedge q)^\circ$, or $p \in L(p \vee q)^\circ$, then $p$ is peripheral in $L$.*

3. Questions. 3.1. It is known [8, Ex. 1.9] that peripheral elements of topological spaces need not be marginal. Is this true for compact, connected lattices?

3.2. If $B$ is the set of all peripheral (marginal) elements of a compact lattice $L$, is $B$ closed in $L$?

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