SPLITTING OF GROUP REPRESENTATIONS

ROBERT PATRICK MARTINEAU
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R. PATRICK MARTINEAU

Let $G$ be a finite group, and $V$, $W$ two modules over the group-ring $KG$, where $K$ is some field. In this note is described a method for proving that every $KG$-extension of $V$ by $W$ is a split extension. The method is applied to the groups $PSL(2, 2^a)$ when $K = GF(2^a)$, giving in this case an alternative proof of a theorem of G. Higman.

1. The method. Fix the finite group $G$ and the field $K$. If $A$ is any left $KG$-module, we let $Cr(G, A)$ denote the $K$-vector space of crossed homomorphisms from $G$ to $A$, that is,

$$Cr(G, A) = \{ f: G \to A \mid f(gh) = gf(h) + f(g), \text{ all } g, h \in G \}.$$

Suppose $G$ is generated by the elements $g_1, \ldots, g_s$ with relations $w_1, \ldots, w_t$. Here $w_1, \ldots, w_t$ are elements of the free group $F$, freely generated by $x_1, \ldots, x_s$, and we say that $g_1, \ldots, g_s$ satisfy the relation $w$ if $\alpha(w) = 1$ where $\alpha$ is the homomorphism from $F$ to $G$ defined by $\alpha(x_i) = g_i$, $i = 1, \ldots, s$.

We shall devise a criterion, in terms of $w_1, \ldots, w_t$, to decide whether or not a map from $G$ to $A$ is a crossed homomorphism. Let $\mathcal{C}$ be the set of maps $f: \{ g_1, \ldots, g_s \} \to A$ which satisfy the following condition: for any $i \in \{ 1, \ldots, s \}$ for which $g_i^{-1} \in \{ g_1, \ldots, g_s \}$, $f(g_i^{-1}) = -g_i^{-1}f(g_i)$.

Now let $w \in F$ and $f \in \mathcal{C}$. We shall define, by induction on the length of $w$, an element $w^*(f)$ of $A$. If $l = 1$, put $w^*(f) = 0$. If $w = x_{e_1}$ for some $e = \pm 1$, then we define $w^*(f) = f(g_i)$ if $g_i \in \{ g_1, \ldots, g_s \}$, and if $g_i \notin \{ g_1, \ldots, g_s \}$, we put $w^*(f) = -g_i^{-1}f(g_i)$. Finally, if $w = v.x_i$ for some $e = \pm 1$, we define $w^*(f) = \alpha(v)f(g_i) + v^*(f)$.

Notice that we do not need $w$ to be in reduced form, since according to the definition,

$$(wx; x_i^{-1})^*(f) = \alpha(w)g_i f(g_i^{-1}) + \alpha(w)f(g_i) + w^*(f)$$

$$= \alpha(w)g_i [ f(g_i^{-1}) + g_i^{-1}f(g_i) ] + w^*(f)$$

$$= w^*(f),$$

and similarly for $wx_i^{-1}x_i$.

[As an example, if $w = x_1 x_2$, then $w^*(f) = g_1g_2f(g_2) + g_2f(g_2) + f(g_2)$.]
Proof. This is true by definition if $v = 1$ or $v = x^i, \varepsilon = \pm 1$. If we have $(wv)^* (f) = \alpha(w) \cdot v^* (f) + w^* (f)$ for two elements $w, v$ of $F$, and $\varepsilon = \pm 1$, then we have

$$(wvx_i^i)^* (f) = \alpha(wv)f(g_i^i) + (wv)^* (f)$$

$$= \alpha(w) \cdot \alpha(v)f(g_i^i) + \alpha(w)v^* (f) + w^* (f)$$

$$= \alpha(w)[\alpha(v)f(g_i^i) + v^* (f)] + w^* (f)$$

$$= \alpha(w)(vx_i^i)^* (f) + w^* (f).$$

Thus the lemma holds by induction on the length of $v$.

**Lemma 2.** If $f \in Cr(G, A)$, then

(i) $f \in \mathcal{C}$

(ii) if $w \in F$ then $w^* (f) = f(\alpha(w))$, and

(iii) for $i = 1, \ldots, t$, $w_i^* (f) = 0$.

**Proof.** If $f \in Cr(G, A)$ then $f(1 \cdot 1) = 1 \cdot f(1) + f(1)$, so $f(1) = 0$. Then $0 = f(1) = f(g_i \cdot g_i^{\varepsilon}) = f(g_i) + f(g_i)$, so that $f \in \mathcal{C}$.

The equation $w^* (f) = f(\alpha(w))$ holds if $w = 1$ or $x_i$, by definition. If $w = x_i^{-1}$, then $w^* (f) = -g_i^{-1}f(g_i) = f(g_i^i)$ since $f \in Cr(G, A)$. If now $w = vx_i^i, \varepsilon = \pm 1$, and $v^* (f) = f(\alpha(v))$, then

$w^* (f) = \alpha(v)f(g_i^i) + v^* (f)$

$$= \alpha(v)f(g_i^i) + f(\alpha(v))$$

$$= f(\alpha(v) \cdot g_i^i) \quad \text{since} \quad f \in Cr(G, A)$$

$$= f(\alpha(w)).$$

Thus (ii) holds by induction on the length of $w$. (iii) now follows immediately, since $\alpha(w_i) = 1$ and $f(1) = 0$.

We remark, though we shall not need this, that a converse of this result is also true, namely:

**Lemma 3.** If $w_i, \ldots, w_t$ are defining relations for $G$, and if $f \in \mathcal{C}$ satisfies $w_i^* (f) = 0$ for $i = 1, \ldots, t$, then $f$ can be extended (uniquely) to an element of $Cr(G, A)$.

**Proof.** First of all we show that if $u \in \ker \alpha$, then $u^* (f) = 0$. Now $\ker \alpha = \langle w_i, \ldots, w_t \rangle_F$, that is, the subgroup of $F$ generated by all elements of the form $v^{-1}w_iv, v \in F$. By definition, $1^* (f) = 0$, so by Lemma 1, $\alpha(v^{-1}) \cdot v^* (f) + (v^{-1})^* (f) = 0$. Again by Lemma 1,

$$(v^{-1}w_i)^* (f) = \alpha(v^{-1}w_i) \cdot v^* (f) + (v^{-1}w_i)^* (f)$$

$$= \alpha(v^{-1}) \cdot \alpha(w_i) \cdot v^* (f) + \alpha(v^{-1})w_i^* (f) + (v^{-1})^* (f).$$

Since $\alpha(w_i) = 1$ and $w_i^* (f) = 0$, we have $(v^{-1}w_i)^* (f) = 0$. Finally by Lemma 1, if $w^* (f) = 0$ and $v^* (f) = 0$ then $(wv)^* (f) = 0$. Thus $u^* (f) = 0$ for all $u \in \ker \alpha$. 


Now if \( g \) is any element of \( G \), then \( g = \alpha(w) \) for some \( w \in F \). Define \( f(g) = w^*(f) \). Then this definition depends only on \( g \), for if \( g = \alpha(v) \) also, then \( wv^{-1} \in \ker \alpha \), say \( wv^{-1} = u \). But now \( w = uv \), so by Lemma 1, \( w^*(f) = \alpha(u) \cdot v^*(f) + u^*(f) = v^*(f) \) since \( \alpha(u) = 1 \) and \( u^*(f) = 0 \).

Now if \( g, h \in G \), say \( g = \alpha(w), h = \alpha(v) \), then \( f(gh) = (wv)^*(f) = \alpha(w)v^*(f) + w^*(f) \) by Lemma 1 so \( f(gh) = f(h) + f(g) \), as required.

The uniqueness of \( f \) is immediate from the fact that \( f \) is already defined on a set of generators of \( G \).

Lemmas 2(iii) and 3 tell us how to find \( \dim_k(Cr(G, A)) \): we look in \( A \) for elements \( a_1, \ldots, a_s \) satisfying the relations \( w^*(f) = 0 \) which are necessary if \( f \) is to be an element of \( Cr(G, A) \) with \( f(g_i) = a_i, i = 1, \ldots, s \). The point of doing this is explained in the next result.

Let \( V, W \) be two left \( KG \)-modules. The dual module \( W^* \) is given the structure of a left \( KG \)-module by defining \( (gw^*)(w) = w^*(gw) \) for \( g \in G, w^* \in W^* \) and \( w \in W \). Then \( V \otimes_k W^* = A \) is a left \( KG \)-module if we define \( g(v \otimes w^*) = gv \otimes gw^* \). Let \( C_A(G) \) denote \( \{ a \mid a \in A \text{ and } ga = a \text{ for all } g \in G \} \).

**Lemma 4.** If \( \dim_k(Cr(G, A)) \geq \dim_k(A) - \dim_k(C_A(G)) \), then every \( KG \)-extension of \( V \) by \( W \) is a split extension.

**Proof.** By Theorem 10, page 235, of [2], there is a one-to-one correspondence between classes of equivalent \( KG \)-extensions of \( V \) by \( W \), and elements of \( H^1(G, A) \), and by [2], page 231, \( H^1(G, A) \) is the quotient space \( Cr(G, A)/P \), where \( P \) is the subspace of principal crossed homomorphisms, that is, \( P = \{ f: G \rightarrow A \mid \text{for some } a \in A, f(g) = ga - a \text{ for all } g \in G \} \).

To prove Lemma 4, therefore, if suffices to show that \( \dim P \geq \dim_k(Cr(G, A)) \), and so by the hypothesis, we need only prove \( \dim P \geq \dim_k(A) - \dim_k(C_A(G)) \).

Let \( \{ a_{r+1}, \ldots, a_s \} \) be a basis for \( C_A(G) \), and extend it to a basis \( \{ a_1, \ldots, a_r, a_{r+1}, \ldots, a_s \} \) for \( A \). For \( i = 1, \ldots, r \) define \( f_i(g) = ga_i - a_i \) for all \( g \in G \), so that \( f_i \in P \). If we have \( \sum_{i=1}^{r} \alpha_i f_i = 0 \) with \( \alpha_i \in K \), \( i = 1, \ldots, r \), then for all \( g \in G, \sum_{i=1}^{r} \alpha_i (ga_i - a_i) = 0 \), so that for all \( g \in G, \sum_{i=1}^{r} \alpha_i a_i = g(\sum_{i=1}^{r} \alpha_i a_i) \).

Thus \( \sum_{i=1}^{r} \alpha_i a_i \in C_A(G) \), so \( \alpha_i = 0 \) for \( i = 1, \ldots, r \). Hence \( f_1, \ldots, f_r \) are linearly independent, and the Lemma is proved.

2. **SL(2, 2^s).** As an application we take \( G = SL(2, 2^s) \) and \( K = GF(2^s) \). Let \( V = V_0 \) be the 'natural' 2-dimensional representation of \( G \) over \( K \). Then \( G \) is generated by elements \( g_1, g_2, g_3 \) whose action on \( V_0 \) can be represented by matrices \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} \), where \( \theta \) is
a primitive \((2^n - 1)\)st root of 1. A short calculation shows that 
\(g_1, g_2, \) and \(g_3\) satisfy the relations

\[
\begin{align*}
(w_1 &= (x_1 x_2)^k, \quad w_2 = (x_1 x_2)^k, \quad w_3 = x_3^k, \quad w_4 = x_3^k, \quad w_5 = x_5^k, \quad \text{where } k = 2^n - 1. \\
\end{align*}
\]

We take \(W = (V_i)^*, \) where \(V_i\) is the (2-dimensional) representation of \(G\) over \(K\) obtained by applying the field automorphism \(\beta \rightarrow \beta^{2i}\) to the entries of the matrices above. (In fact, all 2-dimensional irreducible representations of \(G\) over \(K\) are of this form—see [1], Theorem 8.2). Thus \(W^*\) has a basis with respect to which the matrices of \(g_1, g_2, g_3\) are respectively \((01, 10)\), \((10, 11)\) and \((\psi 0, 0 0^{-1})\), where \(\psi = \theta^{2i}\).

Let \(A = V \otimes_K W^*\), take \(f \in Cr(G, A)\) and suppose \(f(g_i) = a_i, \ i = 1, 2, 3.\) Then from (*) and Lemma 2(iii) we have

\[
\begin{align*}
(1) \quad 0 &= w_i(f) = (g_2 g_1 g_3 + g_3 g_2 + 1)a_i + (g_1 g_3 g_2 + g_3 g_1 g_2 + g_1 g_2 + g_3 + g_1) a_2 \\
(2) \quad 0 &= w_i(f) = (g_1 a_i + (g_3 g_2 + g_2 + g_3 + 1)a_2 \\
(3) \quad 0 &= w_i(f) = (g_i + 1) a_i \\
(4) \quad 0 &= w_i(f) = (g_3 + 1) a_2 \\
(5) \quad 0 &= w_i(f) = (g_1^{-1} + g_1^{-2} + \cdots + g_3 + 1)a_i.
\end{align*}
\]

If we use the relations (*), and equations (3) and (4), equation (1) can be re-written as

\[
(1') \quad (g_2 g_1 g_3 + g_3 g_2 + 1)a_1 + (1 + g_2 g_1 + g_3) a_2 = 0.
\]

If we multiply equation (2) by \(g_1\) and note that \(g_1^i = 1\) and \(g_1 a_1 = a_i\) (equation (3)), then we obtain

\[
(2') \quad (g_3 + 1) a_1 + (g_1 g_3 + 1) a_3 = 0.
\]

Let \(\bar{g}_1, \bar{g}_2, \bar{g}_3\) be matrices representing \(g_1, g_2, g_3\) respectively in \(A.\) Then it is straightforward to calculate that the rank of the matrix

\[
M = \begin{pmatrix}
 \bar{g}_1 + \bar{g}_2 + 1 & 1 + \bar{g}_2 \bar{g}_1 + \bar{g}_1 & 0 \\
 \bar{g}_3 + 1 & 0 & \bar{g}_3 \bar{g}_1 + 1 \\
 \bar{g}_1 + 1 & 0 & 0 \\
 0 & \bar{g}_2 + 1 & 0 \\
 0 & 0 & \bar{h}
\end{pmatrix}
\]

where \(\bar{h} = \sum_{i=0}^{t-1} \bar{g}_i,\) is 8 if \(i \neq 0\) and 9 if \(i = 0.\)

Secondly, it is easy to show that \(C_i(G) = 0\) if \(i \neq 0,\) and that \(\dim_K (C_i(G)) = 1\) if \(i = 0.\) Thus in either case, \(\dim_K (Cr(G, A)) \leq 3.4 - \text{rank}(M) \leq \dim_K A - \dim_K (C_i(G)).\) Hence by Lemma 4, for any \(i,\) any \(KG\)-extentions of \(V\) by \(W\) is a split extension.
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