

# Pacific Journal of Mathematics

**SPLITTING OF GROUP REPRESENTATIONS**

ROBERT PATRICK MARTINEAU

# SPLITTING OF GROUP REPRESENTATIONS

R. PATRICK MARTINEAU

Let  $G$  be a finite group, and  $V, W$  two modules over the group-ring  $KG$ , where  $K$  is some field. In this note is described a method for proving that every  $KG$ -extension of  $V$  by  $W$  is a split extension. The method is applied to the groups  $PSL(2, 2^\alpha)$  when  $K = GF(2^\alpha)$ , giving in this case an alternative proof of a theorem of G. Higman.

1. The method. Fix the finite group  $G$  and the field  $K$ . If  $A$  is any left  $KG$ -module, we let  $Cr(G, A)$  denote the  $K$ -vector space of crossed homomorphisms from  $G$  to  $A$ , that is,

$$Cr(G, A) = \{f: G \longrightarrow A \mid f(gh) = gf(h) + f(g), \text{ all } g, h \in G\}.$$

Suppose  $G$  is generated by the elements  $g_1, \dots, g_s$  with relations  $w_1, \dots, w_t$ . Here  $w_1, \dots, w_t$  are elements of the free group  $F$ , freely generated by  $x_1, \dots, x_s$ , and we say that  $g_1, \dots, g_s$  satisfy the relation  $w$  if  $\alpha(w) = 1$  where  $\alpha$  is the homomorphism from  $F$  to  $G$  defined by  $\alpha(x_i) = g_i, i = 1, \dots, s$ .

We shall devise a criterion, in terms of  $w_1, \dots, w_t$ , to decide whether or not a map from  $G$  to  $A$  is a crossed homomorphism. Let  $\mathcal{C}$  be the set of maps  $f: \{g_1, \dots, g_s\} \rightarrow A$  which satisfy the following condition: for any  $i \in \{1, \dots, s\}$  for which  $g_i^{-1} \in \{g_1, \dots, g_s\}$ ,  $f(g_i^{-1}) = -g_i^{-1}f(g_i)$ .

Now let  $w \in F$  and  $f \in \mathcal{C}$ . We shall define, by induction on the length of  $w$ , an element  $w^*(f)$  of  $A$ . If  $w = 1$ , put  $w^*(f) = 0$ . If  $w = x_k^\varepsilon$  for some  $\varepsilon = \pm 1$ , then we define  $w^*(f) = f(g_k^\varepsilon)$  if  $g_k^\varepsilon \in \{g_1, \dots, g_s\}$ , and if  $g_k^\varepsilon \notin \{g_1, \dots, g_s\}$ , we put  $w^*(f) = -g_k^{-1}f(g_k)$ . Finally, if  $w = v.x_k^\varepsilon$  for some  $\varepsilon = \pm 1$ , we define  $w^*(f) = \alpha(v) \cdot f(g_k^\varepsilon) + v^*(f)$ .

Notice that we do not need  $w$  to be in reduced form, since according to the definition,

$$\begin{aligned} (wx_i x_i^{-1})^*(f) &= \alpha(w) \cdot g_i f(g_i^{-1}) + \alpha(w) f(g_i) + w^*(f) \\ &= \alpha(w) g_i [f(g_i^{-1}) + g_i^{-1} f(g_i)] + w^*(f) \\ &= w^*(f), \end{aligned}$$

and similarly for  $w x_i^{-1} x_i$ .

[As an example, if  $w = x_1 x_2^2$ , then  $w^*(f) = g_1 g_2 f(g_2) + g_1 f(g_2) + f(g_1)$ .]

LEMMA 1. If  $v, w \in F$  and  $f \in \mathcal{C}$ , then

$$(wv)^*(f) = \alpha(w) \cdot v^*(f) + w^*(f).$$

*Proof.* This is true by definition if  $v = 1$  or  $v = x_i^\varepsilon$ ,  $\varepsilon = \pm 1$ . If we have  $(wv)^*(f) = \alpha(w) \cdot v^*(f) + w^*(f)$  for two elements  $w, v$  of  $F$ , and  $\varepsilon = \pm 1$ , then we have

$$\begin{aligned} (wvx_i^\varepsilon)^*(f) &= \alpha(wv)f(g_i^\varepsilon) + (wv)^*(f) \\ &= \alpha(w) \cdot \alpha(v)f(g_i^\varepsilon) + \alpha(w)v^*(f) + w^*(f) \\ &= \alpha(w)[\alpha(v)f(g_i^\varepsilon) + v^*(f)] + w^*(f) \\ &= \alpha(w)(vx_i^\varepsilon)^*(f) + w^*(f). \end{aligned}$$

Thus the lemma holds by induction on the length of  $v$ .

**LEMMA 2.** *If  $f \in Cr(G, A)$ , then*

- (i)  $f \in \mathcal{C}$
- (ii) if  $w \in F$  then  $w^*(f) = f(\alpha(w))$ , and
- (iii) for  $i = 1, \dots, t$ ,  $w_i^*(f) = 0$ .

*Proof.* If  $f \in Cr(G, A)$  then  $f(1 \cdot 1) = 1 \cdot f(1) + f(1)$ , so  $f(1) = 0$ . Then  $0 = f(1) = f(g_i \cdot g_i^{-1}) = g_i f(g_i^{-1}) + f(g_i)$ , so that  $f \in \mathcal{C}$ .

The equation  $w^*(f) = f(\alpha(w))$  holds if  $w = 1$  or  $x_i$ , by definition. If  $w = x_i^{-1}$ , then  $w^*(f) = -g_i^{-1}f(g_i) = f(g_i^{-1})$  since  $f \in Cr(G, A)$ . If now  $w = vx_i^\varepsilon$ ,  $\varepsilon = \pm 1$ , and  $v^*(f) = f(\alpha(v))$ , then

$$\begin{aligned} w^*(f) &= \alpha(v)f(g_i^\varepsilon) + v^*(f) \\ &= \alpha(v)f(g_i^\varepsilon) + f(\alpha(v)) \\ &= f(\alpha(v) \cdot g_i^\varepsilon) \quad \text{since } f \in Cr(G, A) \\ &= f(\alpha(w)). \end{aligned}$$

Thus (ii) holds by induction on the length of  $w$ . (iii) now follows immediately, since  $\alpha(w_i) = 1$  and  $f(1) = 0$ .

We remark, though we shall not need this, that a converse of this result is also true, namely:

**LEMMA 3.** *If  $w_1, \dots, w_t$  are defining relations for  $G$ , and if  $f \in \mathcal{C}$  satisfies  $w_i^*(f) = 0$  for  $i = 1, \dots, t$ , then  $f$  can be extended (uniquely) to an element of  $Cr(G, A)$ .*

*Proof.* First of all we show that if  $u \in \ker \alpha$ , then  $u^*(f) = 0$ . Now  $\ker \alpha = \langle w_1, \dots, w_t \rangle^F$ , that is, the subgroup of  $F$  generated by all elements of the form  $v^{-1}w_i v$ ,  $v \in F$ . By definition,  $1^*(f) = 0$ , so by Lemma 1,  $\alpha(v^{-1}) \cdot v^*(f) + (v^{-1})^*(f) = 0$ . Again by Lemma 1,

$$\begin{aligned} (v^{-1}w_i v)^*(f) &= \alpha(v^{-1}w_i) \cdot v^*(f) + (v^{-1}w_i)^*(f) \\ &= \alpha(v^{-1}) \cdot \alpha(w_i) \cdot v^*(f) + \alpha(v^{-1})w_i^*(f) + (v^{-1})^*(f). \end{aligned}$$

Since  $\alpha(w_i) = 1$  and  $w_i^*(f) = 0$ , we have  $(v^{-1}w_i v)^*(f) = 0$ . Finally by Lemma 1, if  $w^*(f) = 0$  and  $v^*(f) = 0$  then  $(wv)^*(f) = 0$ . Thus  $u^*(f) = 0$  for all  $u \in \ker \alpha$ .

Now if  $g$  is any element of  $G$ , then  $g = \alpha(w)$  for some  $w \in F$ . Define  $f(g) = w^*(f)$ . Then this definition depends only on  $g$ , for if  $g = \alpha(v)$  also, then  $wv^{-1} \in \ker \alpha$ , say  $wv^{-1} = u$ . But now  $w = uv$ , so by Lemma 1,  $w^*(f) = \alpha(u) \cdot v^*(f) + u^*(f) = v^*(f)$  since  $\alpha(u) = 1$  and  $u^*(f) = 0$ .

Now if  $g, h \in G$ , say  $g = \alpha(w)$ ,  $h = \alpha(v)$ , then  $f(gh) = (wv)^*(f) = \alpha(w)v^*(f) + w^*(f)$  by Lemma 1 so  $f(gh) = gf(h) + f(g)$ , as required.

The uniqueness of  $f$  is immediate from the fact that  $f$  is already defined on a set of generators of  $G$ .

Lemmas 2(iii) and 3 tell us how to find  $\dim_K(Cr(G, A))$ : we look in  $A$  for elements  $a_1, \dots, a_s$  satisfying the relations  $w_i^*(f) = 0$  which are necessary if  $f$  is to be an element of  $Cr(G, A)$  with  $f(g_i) = a_i$ ,  $i = 1, \dots, s$ . The point of doing this is explained in the next result.

Let  $V, W$  be two left  $KG$ -modules. The dual module  $W^*$  is given the structure of a left  $KG$ -module by defining  $(gw^*)(w) = w^*(g^{-1}w)$  for  $g \in G$ ,  $w^* \in W^*$  and  $w \in W$ . Then  $V \otimes_K W^* = A$  is a left  $KG$ -module if we define  $g(v \otimes w^*) = gv \otimes gw^*$ . Let  $C_A(G)$  denote  $\{a \mid a \in A \text{ and } ga = a \text{ for all } g \in G\}$ .

**LEMMA 4.** *If  $\dim_K(Cr(G, A)) \leq \dim_K(A) - \dim_K(C_A(G))$ , then every  $KG$ -extension of  $V$  by  $W$  is a split extension.*

*Proof.* By Theorem 10, page 235, of [2], there is a one-to-one correspondence between classes of equivalent  $KG$ -extensions of  $V$  by  $W$ , and elements of  $H^1(G, A)$ , and by [2], page 231,  $H^1(G, A)$  is the quotient space  $Cr(G, A)/P$ , where  $P$  is the subspace of principal crossed homomorphisms, that is,  $P = \{f: G \rightarrow A \mid \text{for some } a \in A, f(g) = ga - a \text{ for all } g \in G\}$ .

To prove Lemma 4, therefore, it suffices to show that  $\dim P \geq \dim(Cr(G, A))$ , and so by the hypothesis, we need only prove  $\dim P \geq \dim A - \dim(C_A(G))$ .

Let  $\{a_{r+1}, \dots, a_n\}$  be a basis for  $C_A(G)$ , and extend it to a basis  $\{a_1, \dots, a_r, a_{r+1}, \dots, a_n\}$  for  $A$ . For  $i = 1, \dots, r$  define  $f_i(g) = ga_i - a_i$  for all  $g \in G$ , so that  $f_i \in P$ . If we have  $\sum_{i=1}^r \alpha_i f_i = 0$  with  $\alpha_i \in K$ ,  $i = 1, \dots, r$ , then for all  $g \in G$ ,  $\sum_{i=1}^r \alpha_i (ga_i - a_i) = 0$ , so that for all  $g \in G$ ,  $\sum_{i=1}^r \alpha_i a_i = g(\sum_{i=1}^r \alpha_i a_i)$ .

Thus  $\sum_{i=1}^r \alpha_i a_i \in C_A(G)$ , so  $\alpha_i = 0$  for  $i = 1, \dots, r$ . Hence  $f_1, \dots, f_r$  are linearly independent, and the Lemma is proved.

2.  $SL(2, 2^n)$ . As an application we take  $G = SL(2, 2^n)$  and  $K = GF(2^n)$ . Let  $V = V_0$  be the 'natural' 2-dimensional representation of  $G$  over  $K$ . Then  $G$  is generated by elements  $g_1, g_2, g_3$  whose action on  $V_0$  can be represented by matrices  $\begin{pmatrix} 01 \\ 10 \end{pmatrix}, \begin{pmatrix} 10 \\ 11 \end{pmatrix}, \begin{pmatrix} \theta 0 \\ 0 \theta^{-1} \end{pmatrix}$ , where  $\theta$  is

a primitive  $(2^n - 1)$ st root of 1. A short calculation shows that  $g_1, g_2$  and  $g_3$  satisfy the relations

$$(*) \quad \begin{cases} w_1 = (x_1 x_2)^3 & w_2 = (x_1 x_3)^2 \\ w_3 = x_1^2, & w_4 = x_2^2, & w_5 = x_3^k, \quad \text{where } k = 2^n - 1. \end{cases}$$

We take  $W = (V_i)^*$ , where  $V_i$  is the (2-dimensional) representation of  $G$  over  $K$  obtained by applying the field automorphism  $\beta \rightarrow \beta^{2^i}$  to the entries of the matrices above. (In fact, all 2-dimensional irreducible representations of  $G$  over  $K$  are of this form—see [1], Theorem 8.2). Thus  $W^*$  has a basis with respect to which the matrices of  $g_1, g_2, g_3$  are respectively  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} \psi^i & 0 \\ 0 & \psi^{i-1} \end{pmatrix}$ , where  $\psi = \theta^{2^i}$ .

Let  $A = V \otimes_{\kappa} W^*$ , take  $f \in Cr(G, A)$  and suppose  $f(g_i) = a_i$ ,  $i = 1, 2, 3$ . Then from (\*) and Lemma 2(iii) we have

$$(1) \quad 0 = w_1^*(f) = (g_1 g_2 g_1 g_2 + g_1 g_2 + 1)a_1 + (g_1 g_2 g_1 g_2 g_1 + g_1 g_2 g_1 + g_1)a_2$$

$$(2) \quad 0 = w_2^*(f) = (g_1 g_3 + 1)a_1 + (g_1 g_3 g_1 + g_1)a_3$$

$$(3) \quad 0 = w_3^*(f) = (g_1 + 1)a_1$$

$$(4) \quad 0 = w_4^*(f) = (g_2 + 1)a_2$$

$$(5) \quad 0 = w_5^*(f) = (g_3^{k-1} + g_3^{k-2} + \dots + g_3 + 1)a_3.$$

If we use the relations (\*), and equations (3) and (4), equation (1) can be re-written as

$$(1') \quad (g_2 + g_1 g_2 + 1)a_1 + (1 + g_2 g_1 + g_1)a_2 = 0.$$

If we multiply equation (2) by  $g_1$  and note that  $g_1^2 = 1$  and  $g_1 a_1 = a_1$  (equation (3)), then we obtain

$$(2') \quad (g_3 + 1)a_1 + (g_3 g_1 + 1)a_3 = 0.$$

Let  $\bar{g}_1, \bar{g}_2, \bar{g}_3$  be matrices representing  $g_1, g_2, g_3$  respectively in  $A$ . Then it is straightforward to calculate that the rank of the matrix

$$M = \begin{pmatrix} \bar{g}_2 + \bar{g}_1 \bar{g}_2 + 1 & 1 + \bar{g}_2 \bar{g}_1 + \bar{g}_1 & 0 \\ \bar{g}_3 + 1 & 0 & \bar{g}_3 \bar{g}_1 + 1 \\ \bar{g}_1 + 1 & 0 & 0 \\ 0 & \bar{g}_2 + 1 & 0 \\ 0 & 0 & \bar{h} \end{pmatrix}$$

where  $\bar{h} = \sum_{t=0}^{k-1} \bar{g}_3^t$ , is 8 if  $i \neq 0$  and 9 if  $i = 0$ .

Secondly, it is easy to show that  $C_A(G) = 0$  if  $i \neq 0$ , and that  $\dim_{\kappa}(C_A(G)) = 1$  if  $i = 0$ . Thus in either case,  $\dim_{\kappa}(Cr(G, A)) \leq 3.4 - \text{rank}(M) \leq \dim_{\kappa} A - \dim_{\kappa}(C_A(G))$ . Hence by Lemma 4, for any  $i$ , any  $KG$ -extensions of  $V$  by  $W$  is a split extension.

## REFERENCES

1. G. Higman, *Odd Characterisations of Finite Simple Groups*, Lecture notes, University of Michigan, 1968.
2. D. G. Northcott, *An Introduction to Homological Algebra*, Cambridge University Press, 1962.

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