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ON THE CONVERGENCE OF RATIONAL FUNCTIONS WHICH INTERPOLATE IN THE ROOTS OF UNITY

EDWARD BARRY SAFF AND J. L. WALSH

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ON THE CONVERGENCE OF RATIONAL FUNCTIONS WHICH INTERPOLATE IN THE ROOTS OF UNITY

E. B. SAFF AND J. L. WALSH

Results are obtained on the existence and convergence of certain types of rational functions which interpolate in the roots of unity to a function f which is meromorphic in $|z| < 1$ and continuous on $|z| \leqq 1$. The theorems presented extend results of Fejér and Walsh and Sharma on interpolating polynomials.

In a recent paper [2] the first author investigated the convergence of certain sequences of rational functions which interpolate to a meromorphic function f. The results obtained in [2] apply, for example, when f is analytic on $|z| \leqq 1$, meromorphic in $|z| < \rho$, $\rho > 1$, and the points of interpolation are the roots of unity.

In this paper we study the convergence of rational functions which interpolate in the roots of unity to a function f which is meromorphic in $|z| < 1$ and continuous on $|z| \leqq 1$. The theorems presented extend those of Fejér [1] and Walsh and Sharma [4] concerning interpolating polynomials. The method of proof of Theorem 1 is basically that of [2].

A rational function $r_{n\nu}(z)$ is said to be *of type* (n, ν) if it is of the form

$$r_{n\nu}(z) = p_n(z)/q_\nu(z) , \qquad q_\nu(z) \not\equiv 0 ,$$

where $p_n(z)$ and $q_\nu(z)$ are polynomials of degrees at most n and ν respectively.

THEOREM 1. *Let $f(z)$ be meromorphic with precisely ν poles (multiplicity included) in D: $|z| < 1$ and otherwise finite and continuous on $|z| \leqq 1$. Let D' denote the domain obtained from D by deleting the ν poles of $f(z)$. Then for all n sufficiently large there exists a unique rational function $r_{n\nu}(z)$ of type (n, ν) which interpolates to $f(z)$ in the $n + \nu + 1$ roots of unity. Each $r_{n\nu}(z)$ for n large enough has precisely ν finite poles and as $n \to \infty$ these poles approach respectively the ν poles of $f(z)$ in D. The sequence $r_{n\nu}(z)$ converges to $f(z)$ throughout D', uniformly on any closed subset of D'.*

For the case $\nu = 0$ the above theorem is due to Fejér [1].

Proof. For any function g defined on $|z| = 1$ the unique polynomial of degree at most n which interpolates to g in the $n + 1$ roots of unity shall be denoted by $L_n(g; z)$.

Let $\alpha_1, \alpha_2, \cdots, \alpha_\nu$ be the ν poles of $f(z)$ in D and set

$$Q_0(z) = 1 , \quad Q_k(z) = \prod_{i=1}^{k} (z - \alpha_i) , \quad 1 \leqq k \leqq \nu ,$$

$$q_n(z) = Q_\nu(z) + \sum_{k=1}^{\nu} a_k^{(n)} Q_{k-1}(z) .$$

We shall show that for n sufficiently large the coefficients $a_k^{(n)}$ can be chosen so that $Q_\nu(z)$ divides the interpolating polynomial $L_{n+\nu}(q_n Q_\nu f; z)$. For simplicity we assume that the points α_j are distinct, i.e., $f(z)$ has only simple poles in D. The case of multiple poles is left to the reader.

Clearly $Q_\nu(z) \mid L_{n+\nu}(q_n Q_\nu f; z)$ if and only if

$$\sum_{k=1}^{\nu} c_{jk}^{(n)} a_k^{(n)} = d_j^{(n)} , \qquad j = 1, 2, \cdots, \nu , \tag{1}$$

where

$$c_{jk}^{(n)} = L_{n+\nu}(Q_{k-1} Q_\nu f; \alpha_j) , \qquad d_j^{(n)} = -L_{n+\nu}(Q_\nu^2 f; \alpha_j) .$$

For each k the function $Q_{k-1} Q_\nu f$ is analytic in D and continuous on $|z| \leqq 1$, and so Fejér's theorem implies that

$$\lim_{n\to\infty} c_{jk}^{(n)} = (Q_{k-1} Q_\nu f)(\alpha_j) , \quad \lim_{n\to\infty} d_j^{(n)} = -(Q_\nu^2 f)(\alpha_j) , \quad 1 \leqq j, k \leqq \nu ,$$

Since α_j is a simple pole of f we have

$$\begin{aligned} (Q_{k-1} Q_\nu f)(\alpha_j) &= 0 , && \text{for } k > j , \\ (Q_{k-1} Q_\nu f)(\alpha_j) &\neq 0 , && \text{for } k = j . \end{aligned}$$

Hence

$$\lim_{n\to\infty} \det [c_{jk}^{(n)}] = \prod_{l=1}^{\nu} (Q_{l-1} Q_\nu f)(\alpha_l) \neq 0 ,$$

which implies that for n sufficiently large the linear system (1) can be solved uniquely for the coefficients $a_j^{(n)}$. Furthermore since $d_j^{(n)} \to 0$ as $n \to \infty$, it follows from Cramer's rule that for each k, $1 \leqq k \leqq \nu$, we have $a_k^{(n)} \to 0$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} q_n(z) = Q_\nu(z) , \tag{2}$$

uniformly on each bounded subset of the plane.

Now set $r_{n\nu}(z) \equiv L_{n+\nu}(q_n Q_\nu f; z)/q_n(z) Q_\nu(z)$. Then by our choice of the coefficients $a_k^{(n)}$ we have that $r_{n\nu}(z)$ is a rational function of type (n, ν). Also from (2) it follows that for n sufficiently large $q_n(z)$ is different from zero in the $n + \nu + 1$ roots of unity and so $r_{n\nu}(z)$ must interpolate to $f(z)$ in these points. It is easy to see that $r_{n\nu}(z)$ is uniquely determined by its interpolation property. From Fejér's theorem and (2) we have $r_{n\nu}(z) \to f(z)$ as $n \to \infty$ uniformly on any closed subset of D'.

Finally note that $r_{n\nu}(z)$ has ν formal poles, namely the zeros of $q_n(z)$, and as $n \to \infty$ these poles approach respectively the ν poles of $f(z)$ in D. Since

$$\lim_{n\to\infty} L_{n+\nu}(q_n Q_\nu f; z)/Q_\nu(z) = Q_\nu(z)f(z) ,$$

uniformly for z in a neighborhood of each α_j, it follows that for n sufficiently large no zero of the polynomial $L_{n+\nu}(q_n Q_\nu f; z)/Q_\nu(z)$ is a zero of $q_n(z)$. Thus the ν formal poles of $r_{n\nu}(z)$ are *actual* poles. This completes the proof of Theorem 1.

Walsh and Sharma [4] have shown that for any function $g(z)$ analytic in $|z| < 1$ and continuous on $|z| \leqq 1$, the sequence $L_n(g; z)$ converges to $g(z)$ on $|z| = 1$ in the mean of second order. Applying this result to each of the sequences $\{L_{n+\nu}(Q_{k-1}Q_\nu f; z)\}$, $1 \leqq k \leqq \nu + 1$, there follows from (2)

THEOREM 2. *The sequence $r_{n\nu}(z)$ of Theorem* 1 *converges to $f(z)$ in the mean of second order on $|z| = 1$.*

Theorems 1 and 2 are another illustration of the close analogy between approximation in the sense of least squares on $|z| = 1$ and interpolation in the roots of unity; compare [3, §§ 7.10, 9.1, 11.6], [4].

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