ON THE CONVERGENCE OF RATIONAL FUNCTIONS WHICH INTERPOLATE IN THE ROOTS OF UNITY

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Results are obtained on the existence and convergence of certain types of rational functions which interpolate in the roots of unity to a function \( f \) which is meromorphic in \( |z| < 1 \) and continuous on \( |z| \leq 1 \). The theorems presented extend results of Fejér and Walsh and Sharma on interpolating polynomials.

In a recent paper [2] the first author investigated the convergence of certain sequences of rational functions which interpolate to a meromorphic function \( f \). The results obtained in [2] apply, for example, when \( f \) is analytic on \( |z| < 1 \), meromorphic in \( |z| < \rho \), \( \rho > 1 \), and the points of interpolation are the roots of unity.

In this paper we study the convergence of rational functions which interpolate in the roots of unity to a function \( f \) which is meromorphic in \( |z| < 1 \) and continuous on \( |z| \leq 1 \). The theorems presented extend those of Fejér [1] and Walsh and Sharma [4] concerning interpolating polynomials. The method of proof of Theorem 1 is basically that of [2].

A rational function \( r_{n\nu}(z) \) is said to be of type \((n, \nu)\) if it is of the form

\[
r_{n\nu}(z) = \frac{p_n(z)}{q_{\nu}(z)}, \quad q_{\nu}(z) \neq 0,
\]

where \( p_n(z) \) and \( q_{\nu}(z) \) are polynomials of degrees at most \( n \) and \( \nu \) respectively.

**Theorem 1.** Let \( f(z) \) be meromorphic with precisely \( \nu \) poles (multiplicity included) in \( D: |z| < 1 \) and otherwise finite and continuous on \( |z| \leq 1 \). Let \( D' \) denote the domain obtained from \( D \) by deleting the \( \nu \) poles of \( f(z) \). Then for all \( n \) sufficiently large there exists a unique rational function \( r_{n\nu}(z) \) of type \((n, \nu)\) which interpolates to \( f(z) \) in the \( n + \nu + 1 \) roots of unity. Each \( r_{n\nu}(z) \) for \( n \) large enough has precisely \( \nu \) finite poles and as \( n \to \infty \) these poles approach respectively the \( \nu \) poles of \( f(z) \) in \( D \). The sequence \( r_{n\nu}(z) \) converges to \( f(z) \) throughout \( D' \), uniformly on any closed subset of \( D' \).

For the case \( \nu = 0 \) the above theorem is due to Fejér [1].

**Proof.** For any function \( g \) defined on \( |z| = 1 \) the unique polynomial of degree at most \( n \) which interpolates to \( g \) in the \( n + 1 \) roots...
of unity shall be denoted by $L_n(g; z)$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the $v$ poles of $f(z)$ in $D$ and set 

$$Q_0(z) = 1, \quad Q_k(z) = \prod_{i=1}^{k} (z - \alpha_i), \quad 1 \leq k \leq n,$$

$$q_n(z) = Q_n(z) + \sum_{k=1}^{v} a_k^{(n)} Q_{n-k}(z).$$

We shall show that for $n$ sufficiently large the coefficients $a_k^{(n)}$ can be chosen so that $Q_v(z)$ divides the interpolating polynomial $L_{n+v}(q_n Q_n f; z)$. For simplicity we assume that the points $\alpha_i$ are distinct, i.e., $f(z)$ has only simple poles in $D$. The case of multiple poles is left to the reader.

Clearly $Q_v(z) | L_{n+v}(q_n Q_n f; z)$ if and only if

$$\sum_{k=1}^{v} c_{j,k}^{(n)} a_k^{(n)} = d_j^{(n)}, \quad j = 1, 2, \ldots, v,$$

where

$$c_{j,k}^{(n)} = L_{n+v}(Q_{k-1} Q_n f; \alpha_j), \quad d_j^{(n)} = -L_{n+v}(Q_v f; \alpha_j).$$

For each $k$ the function $Q_{k-1} Q_n f$ is analytic in $D$ and continuous on $|z| \leq 1$, and so Fejér's theorem implies that

$$\lim_{n \to \infty} c_{j,k}^{(n)} = (Q_{k-1} Q_n f)(\alpha_j), \quad \lim_{n \to \infty} d_j^{(n)} = -(Q_v f)(\alpha_j), \quad 1 \leq j, k \leq v,$$

Since $\alpha_j$ is a simple pole of $f$ we have

$$(Q_{k-1} Q_n f)(\alpha_j) = 0, \quad \text{for } k > j,$$

$$(Q_{k-1} Q_n f)(\alpha_j) \neq 0, \quad \text{for } k = j.$$ 

Hence

$$\lim_{n \to \infty} \det [c_{j,k}^{(n)}] = \prod_{i=1}^{v} (Q_{i-1} Q_n f)(\alpha_i) \neq 0,$$

which implies that for $n$ sufficiently large the linear system (1) can be solved uniquely for the coefficients $a_k^{(n)}$. Furthermore since $d_j^{(n)} \to 0$ as $n \to \infty$, it follows from Cramer's rule that for each $k$, $1 \leq k \leq v$, we have $a_k^{(n)} \to 0$ as $n \to \infty$. Thus

$$\lim_{n \to \infty} q_n(z) = Q_v(z),$$

uniformly on each bounded subset of the plane.

Now set $r_n(z) = L_{n+v}(q_n Q_n f; z)/q_n(z) Q_v(z)$. Then by our choice of the coefficients $a_k^{(n)}$ we have that $r_n(z)$ is a rational function of type $(n, v)$. Also from (2) it follows that for $n$ sufficiently large $q_n(z)$ is different from zero in the $n + v + 1$ roots of unity and so $r_n(z)$ must
interpolate to \( f(z) \) in these points. It is easy to see that \( r_{\nu}(z) \) is uniquely determined by its interpolation property. From Fejér's theorem and (2) we have \( r_{\nu}(z) \to f(z) \) as \( n \to \infty \) uniformly on any closed subset of \( D' \).

Finally note that \( r_{\nu}(z) \) has \( \nu \) formal poles, namely the zeros of \( q_{\nu}(z) \), and as \( n \to \infty \) these poles approach respectively the \( \nu \) poles of \( f(z) \) in \( D \). Since

\[
\lim_{n \to \infty} L_{n+1}(p_{n}, f; z)/Q_{n}(z) = Q_{s}(z)f(z),
\]

uniformly for \( z \) in a neighborhood of each \( \alpha_j \), it follows that for \( n \) sufficiently large no zero of the polynomial \( L_{n+1}(p_{n}, f; z)/Q_{n}(z) \) is a zero of \( q_{\nu}(z) \). Thus the \( \nu \) formal poles of \( r_{\nu}(z) \) are actual poles. This completes the proof of Theorem 1.

Walsh and Sharma [4] have shown that for any function \( g(z) \) analytic in \( |z| < 1 \) and continuous on \( |z| \leq 1 \), the sequence \( L_n(g; z) \) converges to \( g(z) \) on \( |z| = 1 \) in the mean of second order. Applying this result to each of the sequences \( \{L_{n+1}(Q_{n-1}, f; z)\} \), \( 1 \leq k \leq \nu + 1 \), there follows from (2)

**Theorem 2.** The sequence \( r_{\nu}(z) \) of Theorem 1 converges to \( f(z) \) in the mean of second order on \( |z| = 1 \).

Theorems 1 and 2 are another illustration of the close analogy between approximation in the sense of least squares on \( |z| = 1 \) and interpolation in the roots of unity; compare [3, §§7.10, 9.1, 11.6], [4].

**References**


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