

# Pacific Journal of Mathematics

**ON THE CONVERGENCE OF RATIONAL FUNCTIONS WHICH  
INTERPOLATE IN THE ROOTS OF UNITY**

EDWARD BARRY SAFF AND J. L. WALSH

## ON THE CONVERGENCE OF RATIONAL FUNCTIONS WHICH INTERPOLATE IN THE ROOTS OF UNITY

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**Results are obtained on the existence and convergence of certain types of rational functions which interpolate in the roots of unity to a function  $f$  which is meromorphic in  $|z| < 1$  and continuous on  $|z| \leq 1$ . The theorems presented extend results of Fejér and Walsh and Sharma on interpolating polynomials.**

In a recent paper [2] the first author investigated the convergence of certain sequences of rational functions which interpolate to a meromorphic function  $f$ . The results obtained in [2] apply, for example, when  $f$  is analytic on  $|z| \leq 1$ , meromorphic in  $|z| < \rho$ ,  $\rho > 1$ , and the points of interpolation are the roots of unity.

In this paper we study the convergence of rational functions which interpolate in the roots of unity to a function  $f$  which is meromorphic in  $|z| < 1$  and continuous on  $|z| \leq 1$ . The theorems presented extend those of Fejér [1] and Walsh and Sharma [4] concerning interpolating polynomials. The method of proof of Theorem 1 is basically that of [2].

A rational function  $r_{n\nu}(z)$  is said to be of type  $(n, \nu)$  if it is of the form

$$r_{n\nu}(z) = p_n(z)/q_\nu(z), \quad q_\nu(z) \not\equiv 0,$$

where  $p_n(z)$  and  $q_\nu(z)$  are polynomials of degrees at most  $n$  and  $\nu$  respectively.

**THEOREM 1.** *Let  $f(z)$  be meromorphic with precisely  $\nu$  poles (multiplicity included) in  $D: |z| < 1$  and otherwise finite and continuous on  $|z| \leq 1$ . Let  $D'$  denote the domain obtained from  $D$  by deleting the  $\nu$  poles of  $f(z)$ . Then for all  $n$  sufficiently large there exists a unique rational function  $r_{n\nu}(z)$  of type  $(n, \nu)$  which interpolates to  $f(z)$  in the  $n + \nu + 1$  roots of unity. Each  $r_{n\nu}(z)$  for  $n$  large enough has precisely  $\nu$  finite poles and as  $n \rightarrow \infty$  these poles approach respectively the  $\nu$  poles of  $f(z)$  in  $D$ . The sequence  $r_{n\nu}(z)$  converges to  $f(z)$  throughout  $D'$ , uniformly on any closed subset of  $D'$ .*

For the case  $\nu = 0$  the above theorem is due to Fejér [1].

*Proof.* For any function  $g$  defined on  $|z| = 1$  the unique polynomial of degree at most  $n$  which interpolates to  $g$  in the  $n + 1$  roots

of unity shall be denoted by  $L_n(g; z)$ .

Let  $\alpha_1, \alpha_2, \dots, \alpha_\nu$  be the  $\nu$  poles of  $f(z)$  in  $D$  and set

$$Q_0(z) = 1, \quad Q_k(z) = \prod_{i=1}^k (z - \alpha_i), \quad 1 \leq k \leq \nu,$$

$$q_n(z) = Q_\nu(z) + \sum_{k=1}^{\nu} a_k^{(n)} Q_{k-1}(z).$$

We shall show that for  $n$  sufficiently large the coefficients  $a_k^{(n)}$  can be chosen so that  $Q_\nu(z)$  divides the interpolating polynomial  $L_{n+\nu}(q_n Q_\nu f; z)$ . For simplicity we assume that the points  $\alpha_j$  are distinct, i.e.,  $f(z)$  has only simple poles in  $D$ . The case of multiple poles is left to the reader.

Clearly  $Q_\nu(z) | L_{n+\nu}(q_n Q_\nu f; z)$  if and only if

$$(1) \quad \sum_{k=1}^{\nu} c_{jk}^{(n)} a_k^{(n)} = d_j^{(n)}, \quad j = 1, 2, \dots, \nu,$$

where

$$c_{jk}^{(n)} = L_{n+\nu}(Q_{k-1} Q_\nu f; \alpha_j), \quad d_j^{(n)} = -L_{n+\nu}(Q_\nu^2 f; \alpha_j).$$

For each  $k$  the function  $Q_{k-1} Q_\nu f$  is analytic in  $D$  and continuous on  $|z| \leq 1$ , and so Fejér's theorem implies that

$$\lim_{n \rightarrow \infty} c_{jk}^{(n)} = (Q_{k-1} Q_\nu f)(\alpha_j), \quad \lim_{n \rightarrow \infty} d_j^{(n)} = -(Q_\nu^2 f)(\alpha_j), \quad 1 \leq j, k \leq \nu,$$

Since  $\alpha_j$  is a simple pole of  $f$  we have

$$\begin{aligned} (Q_{k-1} Q_\nu f)(\alpha_j) &= 0, & \text{for } k > j, \\ (Q_{k-1} Q_\nu f)(\alpha_j) &\neq 0, & \text{for } k = j. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \det [c_{jk}^{(n)}] = \prod_{l=1}^{\nu} (Q_{l-1} Q_\nu f)(\alpha_l) \neq 0,$$

which implies that for  $n$  sufficiently large the linear system (1) can be solved uniquely for the coefficients  $a_j^{(n)}$ . Furthermore since  $d_j^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$ , it follows from Cramer's rule that for each  $k$ ,  $1 \leq k \leq \nu$ , we have  $a_k^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus

$$(2) \quad \lim_{n \rightarrow \infty} q_n(z) = Q_\nu(z),$$

uniformly on each bounded subset of the plane.

Now set  $r_{n\nu}(z) \equiv L_{n+\nu}(q_n Q_\nu f; z) / q_n(z) Q_\nu(z)$ . Then by our choice of the coefficients  $a_k^{(n)}$  we have that  $r_{n\nu}(z)$  is a rational function of type  $(n, \nu)$ . Also from (2) it follows that for  $n$  sufficiently large  $q_n(z)$  is different from zero in the  $n + \nu + 1$  roots of unity and so  $r_{n\nu}(z)$  must

interpolate to  $f(z)$  in these points. It is easy to see that  $r_{n\nu}(z)$  is uniquely determined by its interpolation property. From Fejér's theorem and (2) we have  $r_{n\nu}(z) \rightarrow f(z)$  as  $n \rightarrow \infty$  uniformly on any closed subset of  $D'$ .

Finally note that  $r_{n\nu}(z)$  has  $\nu$  formal poles, namely the zeros of  $q_n(z)$ , and as  $n \rightarrow \infty$  these poles approach respectively the  $\nu$  poles of  $f(z)$  in  $D$ . Since

$$\lim_{n \rightarrow \infty} L_{n+\nu}(q_n Q_\nu f; z)/Q_\nu(z) = Q_\nu(z)f(z),$$

uniformly for  $z$  in a neighborhood of each  $\alpha_j$ , it follows that for  $n$  sufficiently large no zero of the polynomial  $L_{n+\nu}(q_n Q_\nu f; z)/Q_\nu(z)$  is a zero of  $q_n(z)$ . Thus the  $\nu$  formal poles of  $r_{n\nu}(z)$  are *actual* poles. This completes the proof of Theorem 1.

Walsh and Sharma [4] have shown that for any function  $g(z)$  analytic in  $|z| < 1$  and continuous on  $|z| \leq 1$ , the sequence  $L_n(g; z)$  converges to  $g(z)$  on  $|z| = 1$  in the mean of second order. Applying this result to each of the sequences  $\{L_{n+\nu}(Q_{k-1} Q_\nu f; z)\}$ ,  $1 \leq k \leq \nu + 1$ , there follows from (2)

**THEOREM 2.** *The sequence  $r_{n\nu}(z)$  of Theorem 1 converges to  $f(z)$  in the mean of second order on  $|z| = 1$ .*

Theorems 1 and 2 are another illustration of the close analogy between approximation in the sense of least squares on  $|z| = 1$  and interpolation in the roots of unity; compare [3, §§ 7.10, 9.1, 11.6], [4].

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