NON-APOSYNDESIS AND NON-HEREDITARY DECOMPOSABILITY

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Let $M$ be a compact metric continuum. If $x \in M$ let $K(x)$ be the set to which an element $y$ of $M$ belongs if and only if $M$ is not aposyndetic at $x$ with respect to $y$. If, for all $x$ in $M$, $x \in \text{Int}(K(x))$, then $M$ is the union of a countable collection of indecomposable subcontinua each of which is the closure of an open set. There exists a compact metric continuum $M$ and a dense subset $J$ of $M$ such that for each $x \in J$, $K(x) = S$, but $M$ contains no indecomposable subcontinua with nonvoid interior. It is the case, however, that if $M$ has a point $x$ such that $\text{Int}(K(x)) = \emptyset$ then $M$ contains an indecomposable subcontinua which intersects $\text{Int}(K(x))$.

Professor F. Burton Jones, in [1], demonstrated that for compact metric continua relationships between nonaposyndesis and indecomposability exist. The examination of such relationships is continued in this paper. The primary concern is to show that certain nonaposyndetic properties, similar to but weaker than those of Theorems 9 and 10 of Jones’ paper, insure the existence of indecomposable subcontinua.

1. Definitions. Let space be metric. For a set $A$, the interior of $A$ and the boundary of $A$ will be denoted by $\text{Int}(A)$ and $\text{Bd}(A)$ respectively. If $A$ and $B$ are sets, then by $A - B$ is meant ${x \in A: x \in B}$. The closure of $A$ will be denoted by $\bar{A}$. Given subsets $A$ and $B$ of a continuum $M$, $M$ is said to be aposyndetic at $A$ with respect to $B$ if there is a subcontinuum $H$ of $M$ such that $A \subset \text{Int}(H) \subset H \subset M - B$. For a subset $A$ of a continuum $M$ let $T_M(A) = \{x \in M: M$ is not aposyndetic at $\{x\}$ with respect to $A\}$ and $K_M(A) = \{x \in M: M$ is not aposyndetic at $A$ with respect to $\{x\}\}$. $T_M(x)$ and $K_M(x)$ will be used for $T_M(\{x\})$ and $K_M(\{x\})$ and the subscript may be omitted where no confusion is likely to result (e.g. $T_M(x) = T(x)$).

2. Essentially indecomposable sets. Theorem 9 of [1] states that a necessary and sufficient condition that the continuum $M$ be indecomposable is that if $x$ and $y$ are points of $M$ then $M$ is nonaposyndetic at $x$ with respect to $y$. We obtain, from this condition, the following definition. A subset $A$ of a continuum $M$ is essentially indecomposable in $M$ (often, in context, essentially indecomposable) if whenever $x$ and $y$ are points of $A$ then $M$ is not aposyndetic at $x$ with respect to $y$. Thus the limit bar in the $\sin \frac{1}{x}$ continuum is
essentially indecomposable in that continuum and any subset of an indecomposable continuum $S$ is essentially indecomposable in $S$.

The following is immediate from the definition.

**Theorem 1.** An essentially indecomposable open subset of a continuum $M$ is an essentially indecomposable subset of each subcontinuum of $M$ which contains it.

**Lemma 1.** If $A$ is an essentially indecomposable open subset of a compact continuum $M$ and $H$ is a subcontinuum of $M$ with $A - H = \emptyset$ then $A$ is contained in a component $C$ of $\overline{M - H}$.

**Proof.** Assume the lemma false. Then $\overline{M - H}$ is the union of disjoint closed sets $B_1$ and $B_2$. Assume, without loss of generality, that $B_1$ contains an open subset $U$ of $A$. Then if $x \in U$ and $y \in B_2 \cap A$, $H \cup B_1$ is a continuum with $x \in \text{Int} (H \cup B_1) \subset H \cup B_1 \subset M - y$. This is a contradiction.

If $F$ is a collection of sets then by $F^*$ is meant the union of the elements of $F$.

**Theorem 2.** If $A$ is an essentially indecomposable open subset of a compact continuum $M$ and $F$ is a finite collection of subcontinua of $M$ such that for each $f \in F$, $A - f \neq \emptyset$, then $A$ is contained in one component of $\overline{M - F^*}$.

**Proof.** The theorem follows from Lemma 1 and Theorem 1 using finite induction.

**Theorem 3.** If $A$ is an essentially indecomposable open subset of a continuum $M$, then there is an open subset $U$ of $M$, containing $A$, such that $U$ is maximal with respect to being essentially indecomposable and open. Further if $U \cap V \neq \emptyset$ with $V$ open and essentially indecomposable, then $V \subseteq U$.

**Proof.** Let $B = \{ W \subseteq M: W$ is open, essentially indecomposable, and $W \supseteq A \}$. Let $U = B^*$. Clearly, a subcontinuum $K$ of $M$, containing a point of $U$ in its interior, contains $A$. Thus if $W \in B$, there is a point $x_w$ such that $x_w \in \text{Int} (K)$; so $K \supseteq W$. Hence $U \subseteq K$ and $U$ is essentially indecomposable. If an essentially indecomposable open set $R$ contains $U$ then $R \supseteq A$ and $R \in B$; so $R \subseteq U$.

If $V$ is an essentially indecomposable open set such that $U \cap V \neq \emptyset$, then it follows that $U \cup V$ is essentially indecomposable and open. Thus $U \cup V \in B$ and so $V \subseteq U$. 
A subset $A$ of a continuum $M$ which is maximal with respect to being open and essentially indecomposable in $M$ will be called a \textit{maximal essentially indecomposable open subset of $M$}.

\textbf{Corollary 3.1.} There exist at most countably many maximal essentially indecomposable open subsets of a separable continuum $M$.

\textbf{Theorem 4.} If an essentially indecomposable open subset $A$ of a compact continuum $M$ is such that $M - A$ has at most countably many components, then $A$ is connected.

\textit{Proof.} Assume the theorem false. Then there exists an essentially indecomposable open set $A$ with $\bar{A} = H \cup K$ where $H$ and $K$ are nonempty disjoint closed subsets of $M$ and $M - A$ has at most countably many components. Let $\{C_i : i = 1, 2, \ldots\}$ be a counting of the components of $M - A$. For $i = 1, 2, \ldots$, let $H_i$ be the collection of all components in $\bar{A}$ which contain a point of $C_i$ and let $K_i = \overline{C_i \cup H_i}$.

Since $A \subset \bigcup K_i$ there is a $j$ such that $K_j$ contains an open subset of $A$ and thus $K_j$ contains $A$. Let $A_j = \{h \in H_j : h \subset H\}$. Then $A_j \cup C_j$ is a subcontinuum which contains an open subset of $A$ but does not contain $A$. This is a contradiction.

\textbf{Example 1.} The closure of a maximal essentially indecomposable open set need not be connected. Let $M'$ be a compact plane indecomposable continuum which contains the points $(0, 0)$ and $(1, 0)$. Let $M = \{(x, y, z) : (x, y) \in M' \text{ and } z = 0 \text{ or } x > 1/2 \text{ and } z = x - 1/2\}$. Let $U$ be the open subset of $M$ which is $\{(x, y, z) \in M : x < 1/2\}$. If $x$ and $y$ are points of $U$, it follows that $M$ is not aposyndetic at $x$ with respect to $y$ and $U$ is essentially indecomposable. If $V$ is an open subset of $M$ which properly contains $U$ then one of $A = \{(x, y, z) \in M : x > 1/2, z = 0\}$ or $B = \{(x, y, z) \in M : x > 1/2, z = x - 1/2\}$ contains an open subset of $V$. Assume without loss of generality it is $A$. Then the subcontinuum of $M$ which is $M - A$ contains $U$ but does not contain $V$ and $V$ is essentially decomposable. Thus $U$ is a maximal essentially indecomposable open set. Clearly $\bar{U}$ is not connected.

\textbf{Lemma 2.} In a continuum $M$, if $y \in \text{Int} K(x)$ then $K(y) \subset K(x)$ and, hence, $\text{Int} K(y) \subset \text{Int} K(x)$.

For a continuum $M$, $\text{Int} I(M)$ denote $\{x \in M : x \in \text{Int} (K_y(x))\}$.

\textbf{Theorem 5.} For a continuum $M$, $I(M)$ is an $F_\sigma$ set.

\textit{Proof.} For each $\varepsilon > 0$ let $I_\varepsilon = \{x \in I(M) : d(x), M - \text{Int} (K(x)) \geq \varepsilon\}$. 
To see that $I_e$ is a closed set let $p$ be a limit point of $I_e$ and suppose 
\{$x_i: i = 1, 2, \cdots$\} is a sequence of points of $I_e$ which converges to $p$.
There exists an integer $N$ such that if $i \geq N$ and $j \geq N$, then 
\[d(x_i, x_j) \leq \varepsilon/2.\]
For such $i$ and $j$ \(\text{Int}(K(x_i)) = \text{Int}(K(x_j))\) by Lemma 2. If $H$ is a subcontinuum of $M$ containing $p$ in its interior, then 
for some $i \geq N$ the point $x_i$ is in $\text{Int}(H)$ and therefore $\text{Int}(K(x_i)) \subset H$. 
But $\text{Int}(K(x_i)) = \text{Int}(K(x_N))$, so $\text{Int}(K(x_N)) \subset K(p)$. Since $d(x_N, p) \leq \varepsilon/2$, 
$p \in \text{Int}(K(x_N))$ and $p \in I(M)$. If $x \in M$ is such that $d(x, p) < \varepsilon$ then 
there is an $e > 0$ such that $d(x, p) + e < \varepsilon$. If $i > N$ is such that 
$d(x_i, p) < e$, then $d(x_i, x) \leq d(x_i, p) + d(x, p) < \varepsilon$ and $x \in \text{Int}(K(x_i))$. As 
above, $\text{Int}(K(x_i)) \subset \text{Int}(K(p))$ and so $d(p, S - \text{Int}K(p)) \geq \varepsilon$. Thus $p \in I_e$.

Since $I(M) = \bigcup_{n=1}^{\infty} I_{1/n}$, $I(M)$ is an $F_\sigma$ set

**Theorem 6.** $I(M)$ intersects an open subset $U$ of a complete continuum $M$ in a second category subset if and only if $U$ contains an essentially indecomposable open set.

**Proof.** If the open subset $U$ of $M$ contains a second category subset which is a subset of $I(M)$, then for some $M = 1, 2, \cdots$, the set $I_{1/n}$ (as defined in the proof of Theorem 5) contains an open subset $V$. If $W$ is an open subset of $V$ of diameter less than $1/2n$, then, for each $x \in W$, $K(x) \supset W$. Thus $W$ is essentially indecomposable.

Conversely, if the complete continuum $M$ contains an essentially indecomposable open set $U$, it then follows that for each $x \in U$, $x \in I(M)$.

**Corollary 6.1.** If, in the compact continuum $M$, $I(M)$ intersects the open subset $U$ of $M$ in a second category set, then $U$ intersects an indecomposable subcontinuum of $M$ which has a nonvoid interior.

**Proof.** From Theorem 6, $U$ contains an essentially indecomposable open subset of $M$. A subcontinuum of $M$ irreducible about $U$ is indecomposable.

**Corollary 6.2.** If $I(M)$ intersects each open subset of the compact continuum $M$ in a second category set then the collection of indecomposable subcontinua of $M$ which have nonvoid interior has a dense union in $M$.

**Example 2.** A compact plane continuum $M$ in which $I(M)$ is dense which contains no indecomposable subcontinua with nonvoid interiors.

Let $D$ be the closed square disk in $E^2$ whose opposite vertices are $(-1, -1)$ and $(1, 1)$. Let $D_i^1$ and $D_i^2$ be subsets of $D$ homeomorphic to $D - \{(1, 0)\}$ which spiral out to $\text{Bd}(D)$, as indicated in Figure 1, so that $\text{Bd}(D)$ is the limiting set of each of the spirals.
Define the spaces $M_n$ inductively as follows. Let $M_1 = D_1^1 \cup D_1^2$. Let $f_1^1$ and $f_1^2$ be homeomorphisms of $D - \{(1, 0)\}$ onto $D_1^1$ and $D_1^2$ respectively. Let $M_2 = \overline{f_1^1(M_1 - \{(1, 0)\})} \cup \overline{f_1^2(M_1 - \{(1, 0)\})}$. For $n > 2$, assume $M_{n-1}$ has been defined and let $D_{n-1}^1, D_{n-1}^2, \ldots, D_{n-1}^{2^{(n-1)}}$ be a counting of the images in $M_{n-1}$ of $D_1^1$ and $D_1^2$ respectively. Let $f_{n-1}^1, f_{n-1}^2, \ldots, f_{n-1}^{2^{(n-1)}}$ be homeomorphisms of $D - \{(1, 0)\}$ onto $D_{n-1}^1, D_{n-1}^2, \ldots, D_{n-1}^{2^{(n-1)}}$, such that for each $x \in D - \{(1, 0)\}$

\[ d(f_{n-1}^j(x), D - f_{n}^j[D - \{(1, 0)\}]) < \frac{1}{n-1}. \]

Let $M_n = \bigcup f_{n-1}^j(S_i - \{(1, 0)\}), j = 1, 2, \ldots, 2^{(n-1)}$, and then define $M$ to be $\cap M_n, n = 1, 2, \ldots$.

If $n$ is a positive integer, $i \leq 2^n$ and $p$ and $q$ are points in $\text{Bd} (D_n^i)$, then it is clear that $M$ is not aposyndetic at $p$ with respect to $q$. Since, if $K \supseteq n$ and $j \leq 2^n$ is such that $D_{k}^j \subset D_n^i$, each point of $\text{Bd} (D_n^i)$ is a boundary point of $D_{k}^j$, we have for each $x \in \text{Bd} (D_n^i)$, $K(x) \supset \{\text{Bd} (D_n^i): D_{k}^j \subset D_n^i\}$. It follows from (1) that, for such $x$, $K(x)$ is dense in $D_n^i \cap M$ and, since $K(x)$ is closed, it thus contains $\overline{D_n^i} \cap M$. If for some $n$ and $i$, $x \in \text{Bd} (D_n^i)$ then $x \in \text{Int} (K(x))$. Since, as above, $\{x: x \in \text{Bd} (D_n^i), n = 1, 2, \ldots, i \leq 2^n\}$ is dense, $I(M)$ is dense.

If $K$ is a subcontinuum of $M$ with interior, then there are integers $N, i, j$, with $i \neq j$, such that $\text{Int} (K) \cap D_N^i \neq \emptyset$ and $\text{Int} (K) \cap D_N^j \neq \emptyset$. The subcontinua $\overline{K \cap D_N^i}$ and $K \cap (M - D_N^i)$ of $K$ decompose $K$.

**Lemma 3.** Let $A$ be a maximal essentially indecomposable open
subset of a compact continuum \( M \) and \( x \in I(M) \). Then if \( x \in \overline{A} \), \( M \) is aposyndetic at \( x \) with respect to \( A \).

**Proof.** If \( x \in \overline{A} \) and \( U \) is an open subset of \( M - A \) such that \( U \subseteq \text{Int}(K(x)) \), then \( A \cup U \) is not essentially indecomposable. It follows from Lemma 1 that there is a subcontinuum \( H \) of \( M \) such that \( H \supset A \) and \( (A \cup U) - H \neq \emptyset \) — i.e., \( M \) is aposyndetic at \( A \) with respect to \( U - H \). Since \( U \) is any open subset of \( \text{Int}(K(x)) - A \),

\[
B = \{ y \in \text{Int}(K(x)) : T(y) \cap A = \emptyset \}
\]

is dense in \( \text{Int}(K(x)) - A \). Let \( K = \{ T(y) : y \in B \}^* \). Since \( T(y) \) is connected [1, Theorem 3] and \( x \in T(y) \) for each \( y \in B \), \( K \) is a continuum. Since \( x \in \overline{A} \), \( x \in \text{Int}(K) \) and, since \( \overline{K} \cap A = \emptyset \), we have the lemma.

**Lemma 4.** If, for a compact continuum \( M \), \( I(M) = M \) and \( A \) is a maximal essentially indecomposable open set, then \( \overline{A} \) is connected.

**Proof.** Assume, on the contrary, that \( A \) is a maximal essentially indecomposable open set and yet \( \overline{A} = A_1 \cup A_2 \) where \( A_1 \) and \( A_2 \) are separated subsets of \( M \). Let \( U \) be an open subset of \( M - A_2 \) containing \( A \), with \( \text{Bd}(U) \cap \overline{A} = \emptyset \). By Lemma 3, \( M \) is aposyndetic at each \( x \in \text{Bd}(U) \) with respect to \( A \). Hence there is a finite collection \( \{H_1, H_2, \ldots, H_n\} \) of subcontinua whose interiors cover \( \text{Bd}(U) \) such that for each \( i = 1, 2, \ldots, n \), \( H_i \cap A = \emptyset \). By Theorem 2, there is a subcontinuum \( C \) of \( M \) with \( C \subseteq M - \bigcup H_i \) such that \( A \subseteq C \). But then \( C \subseteq M - \text{Bd}(U) \) and so \( \text{Bd}(U) \) does not separate \( M \) between \( A_1 \) and \( A_2 \). This is a contradiction.

**Theorem 7.** If \( M \) is a compact continuum with \( I(M) = M \), then \( M \) is the union of a countable collection of indecomposable continua each of which is the closure of a member of a closure preserving collection of maximal essentially indecomposable open subsets of \( M \).

**Proof.** It follows from Theorem 6 and Corollary 3.1 that there is a countable collection of maximal essentially indecomposable open subsets of \( M \) whose union is dense in \( M \). Let \( \{ U_n : n = 1, 2, 3, \ldots \} \) be a counting of this collection. Since for each \( i \), \( U_i \) is connected (Lemma 4) it follows that \( U_i \) is an indecomposable continuum.

Let \( U = \{ U_{n_i} : i = 1, 2, \ldots \} \) be a subcollection of \( \{ U_n : n = 1, 2, \ldots \} \) and \( x \) be an element of \( M \) such that \( x \in \overline{U_{n_i}} \) for any \( i \). Then, by Lemma 3, for each \( i \), \( M \) is aposyndetic at \( x \) with respect to \( U_{n_i} \). Thus \( \text{Int}(K(x)) \cap \overline{U_{n_i}} = \emptyset \) for each \( i \) and since \( x \in \text{Int}(K(x)) \), \( x \in \overline{U_{n_i}} \). Therefore \( \{ U_n : n = 1, 2, \ldots \} \) is closure preserving and \( M = \bigcup U_i \).
**Example 3.** There exists a compact metric continuum $M$ such that $I(M) = M$ and $M$ is not the union of the closures of any finite collection of maximal essentially indecomposable open sets.

For each $i = 1, 2, \ldots$ let $M_i$ be a planar indecomposable subcontinuum of $E^2$ of diameter less than $1/i$ such if $i \neq j$ then $M_i \cap M_j = (0, 0, 0)$. Let $M = \bigcup M_i$.

Clearly $M = I(M)$. The maximal essentially indecomposable open subsets of $M$ are $\text{Int}(M_j)$ relative to $M$. No finite collection of the closures of $\text{Int}(M_j)$ has union $M$.

3. **Non-aposyndesis and the existence of indecomposable subcontinua.** An indecomposable subcontinuum $M$ is $K(x)$ for each $x \in M$ [1, Theorem 9], i.e., $M$ is $I(M)$. In Example 2, $I(M)$ is dense. Note also that if one takes a nested sequence $E$ of the $D'_n$'s used in the construction of Example 2, then the subspace $\tilde{M}_i$ of $M$ (where $M_i$ is $\bigcup \{D'_n \cap M : D'_n \in E\}$) is an indecomposable continuum. This is seen by observing that if $x$ and $y$ are points of $\tilde{M}_i$ then $\tilde{M}_i$ is not aposyndetic at $x$ with respect to $y$ and applying Theorem 9 of [1] again. Since in each of the above examples there exists an indecomposable subcontinuum and $I(M)$ is infinite the following two considerations seem natural:

(a) If $I(M) \neq \emptyset$ then must it necessarily be infinite; and
(b) If $I(M) \neq \emptyset$ then must $M$ contain an indecomposable subcontinuum?

Consideration (a) is answered negatively by Example 4 while Theorem 9 shows that the answer to consideration (b) is yes.

**Example 4.** There exists a compact plane continuum $M$ which has exactly one point $p$ such that $p \in \text{Int}(K(p))$ (i.e., $I(M) = \{p\}$). For this point $K(p) = M$. 

![Figure 2](image-url)
Let $C$ be the unit square in the plane, $p \in C$, and, as indicated in Figure 2, $D_1, D_2, \ldots$ be a sequence of closed topological disks in the plane having $C$ as its limit set, such that $D_i \cap D_j = \{p\}$ if $i \neq j$. Let $M_i = \{D_i: i = 1, 2, \ldots\} \cup C$. Assume $M_n$ to be defined and let $\{D_i^n: i = 1, 2, \ldots\}$ be a counting of the maximal topological disks in $M_n$. Let $f_i^n$ be a homeomorphism of $M_i$ into $D_i^n$ such that $f_i^n(p) = p$, $f_i^n(C) = \text{Bd}(D_i^n)$ and, for $x \in M_i$,

\[ (2) \quad d(f_i^n(x), E_z - f_i^n(x)) < \frac{1}{n}. \]

Let $M_{n+1} = \bigcup \{f_i^n(M_i): i = 1, 2, 3, \ldots\}$ and $M = \bigcup \{M_n: n = 1, 2, \ldots\}$. If $C_i^n$ is the simple closed curve which is the boundary of $D_i^n$, then it is clear that for any $i$ and $n$, $M$ is not aposyndetic at $p$ with respect to any point of $C_i^n$. Thus

\[ K(p) \supset \bigcup \{C_i^n: n = 1, 2, \ldots, i = 1, 2, \ldots\} . \]

It follows from condition (2) that $K(p) = M$. Clearly, if an element $x$ of $M$ is different from $p$ and $y \in K(x)$, there exists a sequence $\{y_i: i = 1, 2, 3, \ldots\}$ converging to $y$ such that, for each $i$, $y_i \notin K(x)$. Thus $K(x)$ is nowhere dense for $x \neq p$ and $p$ is the only point of $M$ with $\text{Int}(K(p)) \neq \emptyset$.

**Theorem 8.** A necessary and sufficient condition that the compact continuum $M$ not be hereditarily decomposable is that, for some subcontinuum $K$ of $M$ and point $p$ of $K$, there exists a sequence $U_1, U_2, \ldots$, of pairwise disjoint open subsets of $M$ which converges to $p$ such that, for each $i$, $U_i \cap K \neq \emptyset$, $p \notin U_i$ and, if $i < j$, the $p$-component of $K - U_i$ does not intersect $\bar{U}_j$.

**Proof.** The sufficiency is established first. Assume $K$, $p$, and $U_1, U_2, \ldots$ are as in the statement of the theorem. Let $H$ be a subcontinuum of $K$ irreducible with respect to intersecting each $\bar{U}_j$. Assume $H$ is decomposable. Then $H = A \cup B$ where $A$ and $B$ are proper subcontinua of $H$. Assume without loss of generality $A \cap \bar{U}_j \neq \emptyset$ for infinitely many $j$. Then $p \in A$. It follows, that for each $i$ there is a $j > i$ such that $\bar{U}_j \cap A \neq \emptyset$. Hence for each $i$, $A \cap U_i \neq \emptyset$. But $A$ is a proper subcontinuum of $H$. This contradicts the choice of $H$.

To establish the necessity, let $K$ be an indecomposable subcontinuum of $M$ and $p \in K$. Let $U_i$ be an open set intersecting $K$ such that $p \notin \bar{U}_i$. Since $K$ is indecomposable, $p$ is not an interior point, relative to $K$, of $P_i$, the $p$-component of $K - U_i$. Thus there is an open set $U_2$ of diameter less than $1/2$, which intersects $K$, such that
If the set $U_n$ has been defined, let $P_n$ be the $p$-component of $K - U_n$. Since $P_n = \bigcup \{P_i: i = 1, 2, \cdots, n\}$ contains no interior relative to $K$, $p$ is not an interior point, relative to $K$, of $P_n$. Let $U_{n+1}$ be an open set of diameter less than $1/(n+1)$ which intersects $K$, such that $\overline{U_{n+1}} \cap P_n = \emptyset$, $d(p, U_{n+1}) < 1/(n+1)$ and $\overline{U_{n+1}} \cap \bigcup \{\overline{U_i}: i = 1, 2, \cdots, n\} = \emptyset$. Clearly $K$, $p$ and $U_1, U_2, \cdots$ are as required.

**Theorem 9.** If $M$ is a compact continuum and for some $x \in M$, $\text{Int}(K(x)) \neq \emptyset$, then $M$ is not hereditarily decomposable.

**Proof.** Let $U_0$ be a nonempty open subset of $\text{Int}(K(x))$ with $\overline{U_0} \subset \text{Int}(K(x))$ such that $x \notin \overline{U_0}$. Let $B$ be the set of points in $\overline{U_0}$ which are points of $C_x$, the $x$-component of $M - U_0$, and $U_1$ be an open subset of $U_0$ with $\overline{U_1} \subset U_0$. Let $\varepsilon_1$ be a positive real number less than $1/2$ such that $N_{\varepsilon_1}(B) \cap U_1 = \emptyset$ and $N_{\varepsilon_1}(B) \subset \text{Int}(K(x))$. Since $M$ is not aposyndetic at $x$ with respect to any point of $U_1$, it can be seen as follows, that $C_i$, the $x$-component of $M - U_i$, does not contain $N_{\varepsilon_1}(B)$. Assume the statement false. Since $C_x$ does not contain $U_1$, $x \in M - C_x$. Clearly, $x$ is not in any component of $M - C_i$ which intersects $U_i$. Thus for each $p \in U_i$, there exists mutually separated sets $A_p$ and $B_p$ such that $A_p \cup B_p = M - C_i$, $x \in A_p$ and $p \in B_p$. For $p \in U_i$, $C_i \cup A_p$ is a continuum. But $x \in \text{Int}(C_i \cup A_p)$ and $C_i \cup A_p \subset M - \{p\}$. This is a contradiction. Let $U_2$ be a nonempty open subset of $N_{\varepsilon_1}(B)$ such that $\overline{U_2} \subset M - C_i$ and $\overline{U_2} \subset N_{\varepsilon_1}(B) - B$. Let $C_2$ be the $x$-component of $M - U_2$. Note, as follows, that $\text{Int}(C_2)$ does not contain $B$. Assume that $B \subset \text{Int}(C_2)$. Since $U_2 \subset K(x)$, $x \in M - C_2$. Further, $C$, the $x$-component of $M - C_2$, does not intersect $U_2$ since $C \subset C_i$ and $U_2 \cap C = \emptyset$. Thus there is a separation of $M - C_2$ between $x$ and a point of $U_2$. As above, this is a contradiction.

If the set $U_n$ has been defined, let $C_n$ be the $x$-component of $M - U_n$ and $\varepsilon_n$ be a positive real number less than $1/2^n$ such that $N_{\varepsilon_n}(B) \cap \bigcup_{i=1}^{n-1} U_i = \emptyset$ and $N_{\varepsilon_n}(B) \subset \text{Int}(K(x))$. It follows, as above, that $C_n$ does not contain $N_{\varepsilon_n}(B)$. Let $U_{n+1}$ be an open subset of $N_{\varepsilon_n}(B)$ such that $\overline{U_{n+1}} \cap C_n = \emptyset$ and $\overline{U_{n+1}} \subset N_{\varepsilon_n}(B) - B$.

Assume without loss of generality that $U_1, U_2, \cdots$ converges to a point $b$ of $B$. Suppose $\overline{U_i} \cap C_x \neq \emptyset$ for some $i$. Since $\overline{U_i} \cap C_x = \emptyset$, there is a smallest $i > 1$ such that $\overline{U_i} \cap C_x \neq \emptyset$. But then $\overline{U_i} \cap C_{i-1} \neq \emptyset$. This contradicts our choice of $U_i$. It now follows that $M$, $b$, and $U_1, U_2, \cdots$ satisfy the condition of Theorem 8.

**Theorem 10.** If an element $x$ of a compact continuum $M$ has $\text{Int}(K(x)) \neq \emptyset$ and $K$ is the collection of indecomposable subcontinua.
of $M$, then $K^*$ is dense in $\text{Int}(K(x))$.

Proof. The choice for $U_o$ in the proof of Theorem 9 can be made in such a way that $U_i$ is any open subset of $\text{Int}(K(x))$ such that $x \in \bar{U}_i$ and $\bar{U}_i \subset \text{Int}(K(x))$. From the proof of Theorem 8, there exists an indecomposable subcontinuum at $M$ which intersects $U_i$. Since every open subset of $\text{Int}(K(x))$ contains such an open set, $K^*$ is dense in $\text{Int}(K(x))$.

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Kenneth Paul Baclawski and Kenneth Kapp, *Induced topologies for quasigroups and loops* ................................................................. 393
D. G. Bourgin, *Fixed point and min – max theorems* ................................. 403
J. L. Brenner, *Zolotarev’s theorem on the Legendre symbol* .......................... 413
Jospeh Atkins Childress, Jr., *Restricting isotopies of spheres* ......................... 415
John Edward Coury, *Some results on lacunary Walsh series* ....................... 419
James B. Derr and N. P. Mukherjee, *Generalized Sylow tower groups. II* ........ 427
Paul Frazier Duvall, Jr., Peter Fletcher and Robert Allen McCoy, *Isotopy Galois spaces* ................................................................. 435
Mary Rodriguez Embry, *Strictly cyclic operator algebras on a Banach space* . . . 443
Abi (Abiadbollah) Fattahi, *On generalizations of Sylow tower groups* ............. 453
Burton I. Fein and Murray M. Schacher, *Maximal subfields of tensor products* . 479
Ervin Fried and J. Sichler, *Homomorphisms of commutative rings with unit element* ................................................................. 485
Kenneth R. Goodearl, *Essential products of nonsingular rings* ....................... 493
George Grätzer, Bjarni Jónsson and H. Lakser, *The amalgamation property in equational classes of modular lattices* ........................................... 507
H. Groemer, *On some mean values associated with a randomly selected simplex in a convex set* ................................................................. 525
Marcel Herzog, *Central 2-Sylow intersections* ................................................ 535
Joel Saul Hillel, *On the number of type-k translation-invariant groups* .......... 539
Ronald Brian Kirk, *A note on the Mackey topology for (Cb(X)* , Cb(X))* .......... 543
J. W. Lea, *The peripherality of irreducible elements of lattice* .......................... 555
John Stewart Locker, *Self-adjointness for multi-point differential operators* ....... 561
Robert Patrick Martineau, *Splitting of group representations* ....................... 571
Robert Massagli, *On a new radical in a topological ring* ............................... 577
Fred Richman, *The constructive theory of countable abelian p-groups* .......... 621
Edward Barry Saff and J. L. Walsh, *On the convergence of rational functions which interpolate in the roots of unity* ................................. 639
Harold Eugene Schlais, *Non-aposyndesis and non-hereditary decomposability* .............................................................................. 643
Mark Lawrence Teply, *A class of divisible modules* ......................................... 653
Edward Joseph Tully, Jr., *H-commutative semigroups in which each homomorphism is uniquely determined by its kernel* ............................... 669
Garth William Warner, Jr., *Zeta functions on the real general linear group* ...... 681
Keith Yale, *Cocyles with range \{±1\} .......................................................... 693