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**ON THE REST POINTS OF A NONLINEAR NONEXPANSIVE
SEMIGROUP**

CHI-LIN YEN

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Let X be a reflexive Banach space and T a nonlinear nonexpansive semigroup on X . The results which we shall prove are the following:

THEOREM 1. Suppose that for any closed convex set M with the property that $T(t)M \subseteq M$ for all $t \geq 0$, M contains a precompact orbit. Then T has a rest point. Moreover, the set of all rest points of T is connected.

THEOREM 2. Suppose that X is strictly convex and T has a bounded orbit. If there is an unbounded increasing sequence $\{u_i\}$ of positive numbers and point x such that $\lim_{i \rightarrow \infty} T(u_i)x$ exists then T has a rest point. Moreover, if $\{t_i\}$ is an unbounded increasing sequence of positive numbers such that

$$y = w - \lim_{i \rightarrow \infty} \frac{1}{t_i} \int_0^{t_i} T(t)x dt$$

exists, then $y \in F$.

Let X be a Banach space. By a nonlinear nonexpansive strongly continuous semigroup T on X (or briefly, a semigroup T on X) we mean that T is a mapping from $[0, \infty) \times X$ into X such that

- (i) for any $x \in X$, $t_1 \geq 0$, and $t_2 \geq 0$, $T(t_1)T(t_2)x = T(t_1 + t_2)x$;
- (ii) for any $x \in X$, $\lim_{t \rightarrow 0^+} T(t)x = T(0)x = x$;
- (iii) for any $x \in X$, $y \in X$, and $t \geq 0$, $|T(t)x - T(t)y| \leq |x - y|$.

Throughout this paper T will denote a semigroup on X . We shall give some definitions as follows:

- (1) For $x \in X$ the orbit of x is the set $O_x = \{T(t)x; t \geq 0\}$
- (2) $F = \{x; T(t)x = x \text{ for all } t \geq 0\}$, and if $x \in F$ then x is called a rest point of T .
- (3) $P = \{x; \text{there is } t_0 > 0 \text{ such that } T(t_0)x = x\}$.
- (4) $A = \{x; O_x \text{ is precompact}\}$.
- (5) $L = \{x; \text{there is a sequence } \{t_i\} \text{ of positive numbers such that } t_i \uparrow \infty \text{ and } \lim_{i \rightarrow \infty} T(t_i)x \text{ exists}\}$.

Clearly, $L \supseteq A \supseteq P \supseteq F$. Moreover, if $F \neq \phi$ then O_x is bounded for all $x \in X$. The question arises "Is the converse true?" M. Crandall and A. Pazy [2] give an affirmative answer, when X is a Hilbert space. However, the converse is not true in general (see R. Martin [4]). In this paper some sufficient conditions will be given such that $F \neq \phi$.

Our main results are the following:

THEOREM 1. *Let X be a reflexive Banach space. Suppose that for any closed convex set M with the property that $T(t)M \subseteq M$ for all $t \geq 0$, $M \cap A \neq \phi$. Then $F \neq \phi$. Moreover, F is connected.*

THEOREM 2. *Let X be a strictly convex reflexive Banach space. If T has a bounded orbit and $L \neq \phi$, then $F \neq \phi$. Moreover, if $t_i \uparrow \infty$ and $y = w - \lim_{i \rightarrow \infty} 1/t_i \int_0^{t_i} T(t)x dt$ for some $x \in X$, then $y \in F$.*

As an application of Theorem 1 one can verify that if X is a reflexive Banach space and T has a bounded orbit, then $F \neq \phi$ provided that either of the following holds: (i) there is a $t_0 > 0$ such that $T(t_0)$ is weakly continuous function on X or (ii) X has the property that every m -dissipative Lipschitz continuous function on X is demiclosed (f is demiclosed if $x_n \rightarrow x_0$ strongly then $y_0 = fx_0$). It is known that if X is a uniformly convex space, the condition (ii) is fulfilled, (see F. Browder [1]).

As an application of Theorem 2 one can verify that if X is a strictly convex, reflexive Banach space and $A \neq \phi$ then $F \neq \phi$. Furthermore, if $x \in A$ then for some unbounded increasing sequence $\{t_i\}$ of positive numbers $\lim_{i \rightarrow \infty} 1/t_i \int_0^{t_i} T(u)x du$ exists and is an element of F . This result generalizes that of D. Rutledge [5] in which X is a Hilbert space and $P \neq \phi$.

We need two known lemmas to prove our theorems and we state them below without proof. Lemma 1 was put in the present form by M. Crandall and A. Pazy [2] and Lemma 2 due to R. de Marr [3].

LEMMA 1. *Let $x \in X$ such that $|T(t)x| \leq M$ for all $t \geq 0$. Then $K = \bigcup_{\tau > 0} \bigcap_{t \geq \tau} \{y; |y - T(t)x| \leq |x| + M\}$ is a nonempty convex subset of X such that $T(t)K \subseteq K$ for all $t \geq 0$.*

LEMMA 2. (R. de Marr). *Let C be a compact subset of X such that $r = \text{diam } C > 0$. Then there is an $x_0 \in \text{clco } C$ and a positive number $r_1 < r$ such that $|y - x_0| \leq r_1$ whenever $y \in C$.*

We will use the following two lemmas and the above two lemmas to prove Theorem 1.

LEMMA 3. *Let M be a closed subset of X such that $T(t)M \subseteq M$ for all $t \geq 0$. If $M \cap A \neq \phi$, then there is a compact subset C of M such that $T(t)C = C$.*

Proof. Let $x \in M \cap A$. Then \bar{O}_x is a compact subset of M and $T(t_1)\bar{O}_x \subseteq T(t_2)\bar{O}_x$ whenever $t_1 \geq t_2 \geq 0$. Hence $C = \bigcap_{t>0} T(t)\bar{O}_x$ is a nonempty compact subset of M . Furthermore, $T(t)C = C$ for all $t \geq 0$.

LEMMA 4. Let $x_0, x_1 \in X$ and $\lambda \in [0, 1]$. Then

$$M_\lambda = \{y \in X; |x_0 - y| = \lambda|x_1 - x_0|, |x_1 - y| = (1 - \lambda)|x_1 - x_0|\}$$

is a nonempty closed convex bounded subset of X . Moreover, if $x_0, x_1 \in F$ then $T(t)M_\lambda \subseteq M_\lambda$.

Proof.

$$M_\lambda = \{y \in X; |x_0 - y| \leq \lambda|x_0 - x_1|\} \cap \{y \in X; |x_1 - y| \leq (1 - \lambda)|x_0 - x_1|\}$$

contains $\lambda x_1 + (1 - \lambda)x_0$. Thus M_λ is a nonempty closed convex bounded subset of X .

Since $T(t)x_i = x_i$ for all $t \geq 0, i = 0, 1$ thus for any $y \in M_\lambda$,

$$|x_0 - T(t)y| = |T(t)x_0 - T(t)y| \leq \lambda|x_0 - x_1|$$

and

$$|x_0 - T(t)y| = |T(t)x_1 - T(t)y| \leq (1 - \lambda)|x_0 - x_1|,$$

that is, $T(t)y \in M_\lambda$.

Now we prove Theorem 1.

Proof of Theorem 1. By Lemma 1 there is a nonempty closed bounded convex set M such that $T(t)M \subseteq M$. Let $\{M_\alpha\}$ be a chain of subset of M such that

(i) M_α is a nonempty closed bounded convex set satisfying $T(t)M_\alpha \subseteq M_\alpha$ for all α .

(ii) $M_\alpha \subseteq M_\beta$ if $\alpha \geq \beta$.

Since M_α is weak-compact, thus $\bigcap_\alpha M_\alpha \neq \emptyset$. Further,

$$T(t)\left(\bigcap_\alpha M_\alpha\right) \subseteq \bigcap_\alpha M_\alpha.$$

By Zorn's lemma there is a maximal element, say M_0 , in the collection $\mathcal{S} = \{M_i; M_i \text{ is a nonempty closed bounded convex subset of } M \text{ such that } T(t)M_i \subseteq M_i\}$. We want to show that M_0 contains exactly one point. Suppose not. By hypothesis, $M_0 \cap A$ contains at least one point, say x . By Lemma 3 there is a compact subset C of M_0 such that $T(t)C = C$. By Lemma 2 there is a point $x_0 \in \text{clco } C \subseteq M_0$ such that $|y - x_0| \leq r_1 < r = \text{diam } C$ for all $y \in C$. Consider the set $M' = \bigcap_{y \in C} \{z \in M_0; |z - y| \leq r_1\}$.

We see that M' is a nonempty closed bounded convex subset of M_0 such that $T(t)M \subseteq M$. Since $r = \text{diam } C$ and C is compact, thus there are $x_1, x_2 \in C$ such that $|x_1 - x_2| = r$. By the definition of M' and the fact that $r_1 < r$, we have $x_i \notin M'$ for $i = 1, 2$. Thus $M' \neq M_0$ and the maximality of M_0 is contradicted. Thus M_0 must contain exactly one point which lies in F . This shows that if M is a closed convex set satisfying $T(t)M \subseteq M$ for all $t \geq 0$ then $M \cap F \neq \phi$.

Next we want to show that F is connected. Suppose not. Then there are two disjoint closed subsets A and B of X such that $A \cup B \supseteq F$, $A \cap F \neq \phi$ and $B \cap F \neq \phi$. Let $A' = A \cap F$ and $B' = B \cap F$. Since F is closed thus A' and B' are closed. For $x_1 \in A'$, $D(x_1, B') = \inf \{|x_1 - y|; y \in B'\} = k > 0$. Thus, there is a $y_1 \in B'$ such that $|x_1 - y_1| < 5/4 K$. It follows from Lemma 4 and the above paragraph there is $z_1 \in M^1 = \{z \in X; |z - x_1| = |z - y_1| = 1/2 |x_1 - y_1|\}$ such that $z_1 \in F = A' \cup B'$. Since $|z_1 - x_1| = 1/2 |x_1 - y_1| < 5/8 K$, $z_1 \in A'$. Let $x_2 = z_1$. Then there is a $y_1 \in B'$ such that

$$|x_2 - y_2| \leq \text{Min} \left\{ \frac{5}{4} D(x_2, B'), |x_2 - y_1| \right\}.$$

Similarly, there is $x_3 \in M^2 = \{z \in X; |z - x_2| = |z - y_2| = 1/2 |x_2 - y_2|\}$ such that $x_3 \in F$. By the same argument we have $x_3 \in A'$. We assume we have chosen $x_{n+1} \in M^n = \{z \in X; |z - x_n| = |z - y_n| = 1/2 |x_n - y_n|\}$ and $x_{n+1} \in A'$ and $y_n \in B'$ such that

$$|y_n - x_n| \leq \text{Min} \left\{ \frac{5}{4} D(x_n, B'), |x_n - y_{n-1}| \right\}$$

for all $n \leq k - 1$ where $k \geq 3$. We can choose y_k, x_{k+1} as follows:

Since $D(x_k, B') \leq |x_k - y_{k-1}|$, there is a $y_k \in B'$ such that

$$|x_k - y_k| \leq \text{Min} \left\{ \frac{5}{4} D(x_k, B'), |x_k - y_{k-1}| \right\}$$

and let $x_{k+1} \in A'$ such that

$$x_{k+1} \in M^k = \left\{ z \in X; |z - x_k| = |z - y_k| = \frac{1}{2} |x_k - y_k| \right\}.$$

Note that

$$\begin{aligned} |x_{n+1} - y_{n+1}| &\leq |x_{n+1} - y_n| = \frac{1}{2} |x_n - y_n| \leq \cdots \leq \left(\frac{1}{2}\right)^n |x_1 - y_1| \\ &< \left(\frac{1}{2}\right)^n \left(\frac{5}{4} K\right) \end{aligned}$$

and

$$|x_{n+1} - x_n| = |x_{n+1} - y_n| < \left(\frac{1}{2}\right)^n \left(\frac{5}{4}K\right).$$

Thus, $\{x_n\}$ is a Cauchy sequence and so $\{x_n\}$ converges to some point, say x_0 in A' . Also $D(x_{n+1}, B') \leq |x_{n+1} - y_{n+1}| < (1/2)^n ((5/4)K) \rightarrow 0$, so $D(x_0, B') = 0$. Since B' is closed $x_0 \in B'$. This is a contradiction to $\phi = A \cap B \ni x_0$. Therefore, F is connected.

In order to prove Theorem 2 we need the following lemmas.

LEMMA 5. *If $x_0 \in X$ such that $x_0 = \lim_{i \rightarrow \infty} T(t_i)x$ for some $x \in X$ and some unbounded increasing sequence $\{t_i\}$ of positive numbers, then there is an unbounded increasing sequence $\{s_i\}$ of positive numbers, such that*

$$\lim_{i \rightarrow \infty} T(s_i)x_0 = x_0.$$

Indication of proof. By an inductive process, for each i , choose n_{i+1} such that $t_{n_{i+1}} - t_{i+1} \geq 1 + t_{n_i} - t_i$, $i = 1, 2, 3, \dots$ and $n_1 = 1$. Let $s_i = t_{n_i} - t_i$. Then,

$$\begin{aligned} |T(s_i)x_0 - x_0| &\leq |T(s_i)T(t_i)x - x_0| + 2|T(t_i)x - x_0| \\ &= |T(t_{n_i})x - x_0| + 2|T(t_i)x - x_0| \longrightarrow 0 \text{ as } i \longrightarrow \infty. \end{aligned}$$

That is, $\lim_{i \rightarrow \infty} T(s_i)x_0 = x_0$.

LEMMA 6. *Let X be a strictly convex Banach space. If*

$$\lim_{i \rightarrow \infty} T(s_i)x_0 = x_0$$

for some increasing unbounded sequence $\{s_i\}$ of positive numbers, then for any n , any $\lambda_1, \dots, \lambda_n$ such that $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ and any x_1, \dots, x_n in 0_{x_0} ,

$$(1) \quad T(t) \left(\sum_{i=1}^n \lambda_i x_i \right) = \sum_{i=1}^n \lambda_i T(t)x_i \text{ for all } t \geq 0.$$

Indication of proof. Clearly, (1) is true for the case $n = 1$. Using inductive argument we may assume that (1) holds for all $n \leq k$ where $k \geq 1$. We shall show that (1) holds for the case $n = k + 1$, that is, for any $\lambda_i, \lambda_i \neq 1$, $\sum_{i=1}^{k+1} \lambda_i = 1$, and any x_1, \dots, x_{k+1} in 0_{x_0} ,

$$T(t) \left(\sum_{i=1}^{k+1} \lambda_i x_i \right) = \sum_{i=1}^{k+1} \lambda_i T(t)x_i.$$

Let $y = \sum_{i=1}^{k+1} \lambda_i x_i$, $z = (1 - \lambda_1)^{-1} \sum_{i=1}^{k+1} \lambda_i x_i$. Then $y = \lambda_1 x_1 + (1 - \lambda_1)z$, and

$$(2) \quad |T(t)y - T(t)x_1| \leq |y - x_1|, \quad |T(t)y - T(t)z| \leq |y - z|$$

for all $t \geq 0$.

$$(3) \quad |T(t)x_1 - T(t)z| \leq |T(t)y - T(t)x_1| + |T(t)y - T(t)z|$$

$$\leq |y - x_1| + |y - z| = |x_1 - z|.$$

Since $|T(t_i)x_1 - T(t_i)z| \downarrow |x_1 - z|$ as $i \rightarrow \infty$, thus we have

$$|T(t)y - T(t)x_1| + |T(t)y - T(t)z| = |T(t)x_1 - T(t)z|.$$

By the strict convexity of X , (2) and (3) we have that

$$T(t)y = \lambda_1 T(t)x_1 + (1 - \lambda_1)T(t)z.$$

By the inductive hypothesis,

$$T(t)y = \sum_{i=1}^{k+1} \lambda_i T(t)x_i.$$

LEMMA 7. Let x_0, X be as in Lemma 6. If there is an unbounded increasing sequence $\{u_i\}$ of positive numbers such that

$$y = w - \lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0 dt, \text{ then } y \in F.$$

Proof. Let

$$y_i = \frac{1}{u_i} \int_0^{u_i} T(t)x_0 dt.$$

For $\varepsilon > 0$, $r > 0$ fixed, there is an $N > 0$ such that if $M \geq |T(t)x_0|$ for all $t \geq 0$,

$$\frac{rM}{u_i} < \frac{\varepsilon}{3} \text{ whenever } i \geq N.$$

It follows from Lemma 6 that

$$T(r)y_i = \frac{1}{u_i} \int_r^{u_i+r} T(t)x_0 dt = y_i + \frac{1}{u_i} \left(\int_{u_i}^{u_i+r} - \int_0^r \right) T(t)x_0 dt.$$

Thus $|T(r)y_i - y_i| < 2\varepsilon/3$ for all $i \geq N$. Since $y = w - \lim_{i \rightarrow \infty} y_i$, there exists a $k > 0$, $\lambda_1, \lambda_2, \dots, \lambda_k \geq 0$ such that $\sum_{i=1}^k \lambda_i = 1$ and $|y - \sum_{i=1}^k \lambda_i y_{i+N-1}| < \varepsilon/6$. Hence,

$$|T(r)y - y| \leq \left| T(r)y - \sum_{i=1}^k \lambda_i T(r)y_{i+N-1} \right|$$

$$+ \left| \sum_{i=1}^k \lambda_i (T(r)y_{i+N-1}) \right| + \left| y - \sum_{i=1}^k \lambda_i y_{i+N-1} \right|$$

$$< 2\varepsilon/6 + 2\varepsilon/3 = \varepsilon.$$

Since ε and r are arbitrary positive numbers, thus $y \in F$.

LEMMA 8. Let $\{t_i\}$ be an unbounded increasing sequence of positive numbers and x in X . If T has a bounded orbit and

$$x_0 = \lim_{i \rightarrow \infty} T(t_i)x,$$

then

$$\lim_{u \rightarrow \infty} \frac{1}{u} \int_0^u (T(t)x - T(t)x_0)dt = 0.$$

Proof. For $\varepsilon > 0$ be given there is an positive integer n such that

$$|T(t_i)x - x_0| < \varepsilon \quad \text{for all } i \geq n.$$

Let u be any positive number great than t_n . Then

$$\begin{aligned} \left| \frac{1}{u} \int_0^u (T(t)x - T(t)x_0)dt \right| &\leq \frac{1}{u} \int_0^{u-t_n} |T(t)T(t_n)x - T(t)x_0| dt \\ &\quad + \frac{1}{u} \int_0^{t_n} |T(t)x| dt + \frac{1}{u} \int_{u-t_n}^u |T(t)x_0| dt \\ &< \frac{u - t_n}{u} \varepsilon \\ &\quad + \frac{1}{u} \left(\int_0^{t_n} |T(t)x| dt + \int_{u-t_n}^u |T(t)x_0| dt \right). \end{aligned}$$

Since orbits are bounded the last term in above inequality will tend to 0 as $u \rightarrow \infty$. Hence, we prove the assertion.

Proof of Theorem 2. By Lemma 5, Lemma 7 and reflexivity of X , there is an increasing unbounded sequence $\{u_i\}$ of positive numbers such that

$$w - \lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0 dt$$

exists and is in F , where $x_0 = \lim_{i \rightarrow \infty} T(t_i)x$. Also, it follows from Lemma 8

$$\lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} (T(t)x - T(t)x_0) dt = 0.$$

Thus,

$$w - \lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x dt = w - \lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0 dt \quad \text{is in } F.$$

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