

# Pacific Journal of Mathematics

**ON THE REST POINTS OF A NONLINEAR NONEXPANSIVE  
SEMIGROUP**

CHI-LIN YEN

## ON THE REST POINTS OF A NONLINEAR NONEXPANSIVE SEMIGROUP

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Let  $X$  be a reflexive Banach space and  $T$  a nonlinear nonexpansive semigroup on  $X$ . The results which we shall prove are the following:

**THEOREM 1.** Suppose that for any closed convex set  $M$  with the property that  $T(t)M \subseteq M$  for all  $t \geq 0$ ,  $M$  contains a precompact orbit. Then  $T$  has a rest point. Moreover, the set of all rest points of  $T$  is connected.

**THEOREM 2.** Suppose that  $X$  is strictly convex and  $T$  has a bounded orbit. If there is an unbounded increasing sequence  $\{u_i\}$  of positive numbers and point  $x$  such that  $\lim_{i \rightarrow \infty} T(u_i)x$  exists then  $T$  has a rest point. Moreover, if  $\{t_i\}$  is an unbounded increasing sequence of positive numbers such that

$$y = w - \lim_{i \rightarrow \infty} \frac{1}{t_i} \int_0^{t_i} T(t)x \, dt$$

exists, then  $y \in F$ .

Let  $X$  be a Banach space. By a nonlinear nonexpansive strongly continuous semigroup  $T$  on  $X$  (or briefly, a semigroup  $T$  on  $X$ ) we mean that  $T$  is a mapping from  $[0, \infty) \times X$  into  $X$  such that

- (i) for any  $x \in X$ ,  $t_1 \geq 0$ , and  $t_2 \geq 0$ ,  $T(t_1)T(t_2)x = T(t_1 + t_2)x$ ;
- (ii) for any  $x \in X$ ,  $\lim_{t \rightarrow 0^+} T(t)x = T(0)x = x$ ;
- (iii) for any  $x \in X$ ,  $y \in X$ , and  $t \geq 0$ ,  $|T(t)x - T(t)y| \leq |x - y|$ .

Throughout this paper  $T$  will denote a semigroup on  $X$ . We shall give some definitions as follows:

- (1) For  $x \in X$  the orbit of  $x$  is the set  $O_x = \{T(t)x; t \geq 0\}$
- (2)  $F = \{x; T(t)x = x \text{ for all } t \geq 0\}$ , and if  $x \in F$  then  $x$  is called a rest point of  $T$ .
- (3)  $P = \{x; \text{there is } t_0 > 0 \text{ such that } T(t_0)x = x\}$ .
- (4)  $A = \{x; O_x \text{ is precompact}\}$ .
- (5)  $L = \{x; \text{there is a sequence } \{t_i\} \text{ of positive numbers such that } t_i \uparrow \infty \text{ and } \lim_{i \rightarrow \infty} T(t_i)x \text{ exists}\}$ .

Clearly,  $L \supseteq A \supseteq P \supseteq F$ . Moreover, if  $F \neq \phi$  then  $O_x$  is bounded for all  $x \in X$ . The question arises "Is the converse true?" M. Crandall and A. Pazy [2] give an affirmative answer, when  $X$  is a Hilbert space. However, the converse is not true in general (see R. Martin [4]). In this paper some sufficient conditions will be given such that  $F \neq \phi$ .

Our main results are the following:

**THEOREM 1.** *Let  $X$  be a reflexive Banach space. Suppose that for any closed convex set  $M$  with the property that  $T(t)M \subseteq M$  for all  $t \geq 0$ ,  $M \cap A \neq \phi$ . Then  $F \neq \phi$ . Moreover,  $F$  is connected.*

**THEOREM 2.** *Let  $X$  be a strictly convex reflexive Banach space. If  $T$  has a bounded orbit and  $L \neq \phi$ , then  $F \neq \phi$ . Moreover, if  $t_i \uparrow \infty$  and  $y = w - \lim_{i \rightarrow \infty} 1/t_i \int_0^{t_i} T(t)x dt$  for some  $x \in X$ , then  $y \in F$ .*

As an application of Theorem 1 one can verify that if  $X$  is a reflexive Banach space and  $T$  has a bounded orbit, then  $F \neq \phi$  provided that either of the following holds: (i) there is a  $t_0 > 0$  such that  $T(t_0)$  is weakly continuous function on  $X$  or (ii)  $X$  has the property that every  $m$ -dissipative Lipschitz continuous function on  $X$  is demiclosed ( $f$  is demiclosed if  $x_n \rightarrow x_0$  strongly then  $y_0 = fx_0$ ). It is known that if  $X$  is a uniformly convex space, the condition (ii) is fulfilled, (see F. Browder [1]).

As an application of Theorem 2 one can verify that if  $X$  is a strictly convex, reflexive Banach space and  $A \neq \phi$  then  $F \neq \phi$ . Furthermore, if  $x \in A$  then for some unbounded increasing sequence  $\{t_i\}$  of positive numbers  $\lim_{i \rightarrow \infty} 1/t_i \int_0^{t_i} T(u)x du$  exists and is an element of  $F$ . This result generalizes that of D. Rutledge [5] in which  $X$  is a Hilbert space and  $P \neq \phi$ .

We need two known lemmas to prove our theorems and we state them below without proof. Lemma 1 was put in the present form by M. Crandall and A. Pazy [2] and Lemma 2 due to R. de Marr [3].

**LEMMA 1.** *Let  $x \in X$  such that  $|T(t)x| \leq M$  for all  $t \geq 0$ . Then  $K = \bigcup_{\tau > 0} \bigcap_{t > \tau} \{y; |y - T(t)x| \leq |x| + M\}$  is a nonempty convex subset of  $X$  such that  $T(t)K \subseteq K$  for all  $t \geq 0$ .*

**LEMMA 2. (R. de Marr).** *Let  $C$  be a compact subset of  $X$  such that  $r = \text{diam } C > 0$ . Then there is an  $x_0 \in \text{clco } C$  and a positive number  $r_1 < r$  such that  $|y - x_0| \leq r_1$  whenever  $y \in C$ .*

We will use the following two lemmas and the above two lemmas to prove Theorem 1.

**LEMMA 3.** *Let  $M$  be a closed subset of  $X$  such that  $T(t)M \subseteq M$  for all  $t \geq 0$ . If  $M \cap A \neq \phi$ , then there is a compact subset  $C$  of  $M$  such that  $T(t)C = C$ .*

*Proof.* Let  $x \in M \cap A$ . Then  $\bar{O}_x$  is a compact subset of  $M$  and  $T(t_1)\bar{O}_x \subseteq T(t_2)\bar{O}_x$  whenever  $t_1 \geq t_2 \geq 0$ . Hence  $C = \bigcap_{t>0} T(t)\bar{O}_x$  is a nonempty compact subset of  $M$ . Furthermore,  $T(t)C = C$  for all  $t \geq 0$ .

LEMMA 4. Let  $x_0, x_1 \in X$  and  $\lambda \in [0, 1]$ . Then

$$M_\lambda = \{y \in X; |x_0 - y| = \lambda |x_1 - x_0|, |x_1 - y| = (1 - \lambda) |x_1 - x_0|\}$$

is a nonempty closed convex bounded subset of  $X$ . Moreover, if  $x_0, x_1 \in F$  then  $T(t)M_\lambda \subseteq M_\lambda$ .

*Proof.*

$$M_\lambda = \{y \in X; |x_0 - y| \leq \lambda |x_0 - x_1|\} \cap \{y \in X; |x_1 - y| \leq (1 - \lambda) |x_0 - x_1|\}$$

contains  $\lambda x_1 + (1 - \lambda)x_0$ . Thus  $M_\lambda$  is a nonempty closed convex bounded subset of  $X$ .

Since  $T(t)x_i = x_i$  for all  $t \geq 0, i = 0, 1$  thus for any  $y \in M_\lambda$ ,

$$|x_0 - T(t)y| = |T(t)x_0 - T(t)y| \leq \lambda |x_0 - x_1|$$

and

$$|x_0 - T(t)y| = |T(t)x_1 - T(t)y| \leq (1 - \lambda) |x_0 - x_1|,$$

that is,  $T(t)y \in M_\lambda$ .

Now we prove Theorem 1.

*Proof of Theorem 1.* By Lemma 1 there is a nonempty closed bounded convex set  $M$  such that  $T(t)M \subseteq M$ . Let  $\{M_\alpha\}$  be a chain of subset of  $M$  such that

(i)  $M_\alpha$  is a nonempty closed bounded convex set satisfying  $T(t)M_\alpha \subseteq M_\alpha$  for all  $\alpha$ .

(ii)  $M_\alpha \subseteq M_\beta$  if  $\alpha \geq \beta$ .

Since  $M_\alpha$  is weak-compact, thus  $\bigcap_\alpha M_\alpha \neq \emptyset$ . Further,

$$T(t)(\bigcap_\alpha M_\alpha) \subseteq \bigcap_\alpha M_\alpha.$$

By Zorn's lemma there is a maximal element, say  $M_0$ , in the collection  $\mathcal{S} = \{M_i; M_i \text{ is a nonempty closed bounded convex subset of } M \text{ such that } T(t)M_i \subseteq M_i\}$ . We want to show that  $M_0$  contains exactly one point. Suppose not. By hypothesis,  $M_0 \cap A$  contains at least one point, say  $x$ . By Lemma 3 there is a compact subset  $C$  of  $M_0$  such that  $T(t)C = C$ . By Lemma 2 there is a point  $x_0 \in \text{clco } C \subseteq M_0$  such that  $|y - x_0| \leq r_1 < r = \text{diam } C$  for all  $y \in C$ . Consider the set  $M' = \bigcap_{y \in C} \{z \in M_0; |z - y| \leq r_1\}$ .

We see that  $M'$  is a nonempty closed bounded convex subset of  $M_0$  such that  $T(t)M \subseteq M$ . Since  $r = \text{diam } C$  and  $C$  is compact, thus there are  $x_1, x_2 \in C$  such that  $|x_1 - x_2| = r$ . By the definition of  $M'$  and the fact that  $r_1 < r$ , we have  $x_i \notin M'$  for  $i = 1, 2$ . Thus  $M' \neq M_0$  and the maximality of  $M_0$  is contradicted. Thus  $M_0$  must contain exactly one point which lies in  $F$ . This shows that if  $M$  is a closed convex set satisfying  $T(t)M \subseteq M$  for all  $t \geq 0$  then  $M \cap F \neq \phi$ .

Next we want to show that  $F$  is connected. Suppose not. Then there are two disjoint closed subsets  $A$  and  $B$  of  $X$  such that  $A \cup B \supseteq F$ ,  $A \cap F \neq \phi$  and  $B \cap F \neq \phi$ . Let  $A' = A \cap F$  and  $B' = B \cap F$ . Since  $F$  is closed thus  $A'$  and  $B'$  are closed. For  $x_1 \in A'$ ,  $D(x_1, B') = \inf \{|x_1 - y|; y \in B'\} = k > 0$ . Thus, there is a  $y_1 \in B'$  such that  $|x_1 - y_1| < 5/4 K$ . It follows from Lemma 4 and the above paragraph there is  $z_1 \in M^1 = \{z \in X; |z - x_1| = |z - y_1| = 1/2 |x_1 - y_1|\}$  such that  $z_1 \in F = A' \cup B'$ . Since  $|z_1 - x_1| = 1/2 |x_1 - y_1| < 5/8 K$ ,  $z_1 \in A'$ . Let  $x_2 = z_1$ . Then there is a  $y_1 \in B'$  such that

$$|x_2 - y_2| \leq \text{Min} \left\{ \frac{5}{4} D(x_2, B'), |x_2 - y_1| \right\}.$$

Similarly, there is  $x_3 \in M^2 = \{z \in X; |z - x_2| = |z - y_2| = 1/2 |x_2 - y_2|\}$  such that  $x_3 \in F$ . By the same argument we have  $x_3 \in A'$ . We assume we have chosen  $x_{n+1} \in M^n = \{z \in X; |z - x_n| = |z - y_n| = 1/2 |x_n - y_n|\}$  and  $x_{n+1} \in A'$  and  $y_n \in B'$  such that

$$|y_n - x_n| \leq \text{Min} \left\{ \frac{5}{4} D(x_n, B'), |x_n - y_{n-1}| \right\}$$

for all  $n \leq k-1$  where  $k \geq 3$ . We can choose  $y_k, x_{k+1}$  as follows:

Since  $D(x_k, B') \leq |x_k - y_{k-1}|$ , there is a  $y_k \in B'$  such that

$$|x_k - y_k| \leq \text{Min} \left\{ \frac{5}{4} D(x_k, B'), |x_k - y_{k-1}| \right\}$$

and let  $x_{k+1} \in A'$  such that

$$x_{k+1} \in M^k = \left\{ z \in X; |z - x_k| = |z - y_k| = \frac{1}{2} |x_k - y_k| \right\}.$$

Note that

$$\begin{aligned} |x_{n+1} - y_{n+1}| &\leq |x_{n+1} - y_n| = \frac{1}{2} |x_n - y_n| \leq \cdots \leq \left(\frac{1}{2}\right)^n |x_1 - y_1| \\ &< \left(\frac{1}{2}\right)^n \left(\frac{5}{4} K\right) \end{aligned}$$

and

$$|x_{n+1} - x_n| = |x_{n+1} - y_n| < \left(\frac{1}{2}\right)^n \left(\frac{5}{4}K\right).$$

Thus,  $\{x_n\}$  is a Cauchy sequence and so  $\{x_n\}$  converges to some point, say  $x_0$  in  $A'$ . Also  $D(x_{n+1}, B') \leq |x_{n+1} - y_{n+1}| < (1/2)^n ((5/4)K) \rightarrow 0$ , so  $D(x_0, B') = 0$ . Since  $B'$  is closed  $x_0 \in B'$ . This is a contradiction to  $\phi = A \cap B \ni x_0$ . Therefore,  $F$  is connected.

In order to prove Theorem 2 we need the following lemmas.

LEMMA 5. *If  $x_0 \in X$  such that  $x_0 = \lim_{i \rightarrow \infty} T(t_i)x$  for some  $x \in X$  and some unbounded increasing sequence  $\{t_i\}$  of positive numbers, then there is an unbounded increasing sequence  $\{s_i\}$  of positive numbers, such that*

$$\lim_{i \rightarrow \infty} T(s_i)x_0 = x_0.$$

*Indication of proof.* By an inductive process, for each  $i$ , choose  $n_{i+1}$  such that  $t_{n_{i+1}} - t_{i+1} \geq 1 + t_{n_i} - t_i$ ,  $i = 1, 2, 3, \dots$  and  $n_1 = 1$ . Let  $s_i = t_{n_i} - t_i$ . Then,

$$\begin{aligned} |T(s_i)x_0 - x_0| &\leq |T(s_i)T(t_i)x - x_0| + 2|T(t_i)x - x_0| \\ &= |T(t_{n_i})x - x_0| + 2|T(t_i)x - x_0| \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \end{aligned}$$

That is,  $\lim_{i \rightarrow \infty} T(s_i)x_0 = x_0$ .

LEMMA 6. *Let  $X$  be a strictly convex Banach space. If*

$$\lim_{i \rightarrow \infty} T(s_i)x_0 = x_0$$

*for some increasing unbounded sequence  $\{s_i\}$  of positive numbers, then for any  $n$ , any  $\lambda_1, \dots, \lambda_n$  such that  $\lambda_i \geq 0$ ,  $\sum_{i=1}^n \lambda_i = 1$  and any  $x_1, \dots, x_n$  in  $0_{x_0}$ ,*

$$(1) \quad T(t) \left( \sum_{i=1}^n \lambda_i x_i \right) = \sum_{i=1}^n \lambda_i T(t)x_i \quad \text{for all } t \geq 0.$$

*Indication of proof.* Clearly, (1) is true for the case  $n = 1$ . Using inductive argument we may assume that (1) holds for all  $n \leq k$  where  $k \geq 1$ . We shall show that (1) holds for the case  $n = k + 1$ , that is, for any  $\lambda_i, \lambda_i \neq 1$ ,  $\sum_{i=1}^{k+1} \lambda_i = 1$ , and any  $x_1, \dots, x_{k+1}$  in  $0_{x_0}$ ,

$$T(t) \left( \sum_{i=1}^{k+1} \lambda_i x_i \right) = \sum_{i=1}^{k+1} \lambda_i T(t)x_i.$$

Let  $y = \sum_{i=1}^{k+1} \lambda_i x_i$ ,  $z = (1 - \lambda_1)^{-1} \sum_{i=1}^{k+1} \lambda_i x_i$ . Then  $y = \lambda_1 x_1 + (1 - \lambda_1)z$ , and

$$(2) \quad |T(t)y - T(t)x_1| \leq |y - x_1|, \quad |T(t)y - T(t)z| \leq |y - z|$$

for all  $t \geq 0$ .

$$(3) \quad |T(t)x_1 - T(t)z| \leq |T(t)y - T(t)x_1| + |T(t)y - T(t)z| \\ \leq |y - x_1| + |y - z| = |x_1 - z|.$$

Since  $|T(t_i)x_1 - T(t_i)z| \downarrow |x_1 - z|$  as  $i \rightarrow \infty$ , thus we have

$$|T(t)y - T(t)x_1| + |T(t)y - T(t)z| = |T(t)x_1 - T(t)z|.$$

By the strict convexity of  $X$ , (2) and (3) we have that

$$T(t)y = \lambda_1 T(t)x_1 + (1 - \lambda_1)T(t)z.$$

By the inductive hypothesis,

$$T(t)y = \sum_{i=1}^{k+1} \lambda_i T(t)x_i.$$

LEMMA 7. Let  $x_0, X$  be as in Lemma 6. If there is an unbounded increasing sequence  $\{u_i\}$  of positive numbers such that

$$y = w - \lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0 dt, \text{ then } y \in F.$$

*Proof.* Let

$$y_i = \frac{1}{u_i} \int_0^{u_i} T(t)x_0 dt.$$

For  $\varepsilon > 0, r > 0$  fixed, there is an  $N > 0$  such that if  $M \geq |T(t)x_0|$  for all  $t \geq 0$ ,

$$\frac{rM}{u_i} < \frac{\varepsilon}{3} \text{ whenever } i \geq N.$$

It follows from Lemma 6 that

$$T(r)y_i = \frac{1}{u_i} \int_r^{u_i+r} T(t)x_0 dt = y_i + \frac{1}{u_i} \left( \int_{u_i}^{u_i+r} - \int_0^r \right) T(t)x_0 dt.$$

Thus  $|T(r)y_i - y_i| < 2\varepsilon/3$  for all  $i \geq N$ . Since  $y = w - \lim_{i \rightarrow \infty} y_i$ , there exists a  $k > 0, \lambda_1, \lambda_2, \dots, \lambda_k \geq 0$  such that  $\sum_{i=1}^k \lambda_i = 1$  and  $|y - \sum_{i=1}^k \lambda_i y_{i+N-1}| < \varepsilon/6$ . Hence,

$$|T(r)y - y| \leq \left| T(r)y - \sum_{i=1}^k \lambda_i T(r)y_{i+N-1} \right| \\ + \left| \sum_{i=1}^k \lambda_i (T(r)y_{i+N-1}) \right| + \left| y - \sum_{i=1}^k \lambda_i y_{i+N-1} \right| \\ < 2\varepsilon/6 + 2\varepsilon/3 = \varepsilon.$$

Since  $\varepsilon$  and  $r$  are arbitrary positive numbers, thus  $y \in F$ .

**LEMMA 8.** *Let  $\{t_i\}$  be an unbounded increasing sequence of positive numbers and  $x$  in  $X$ . If  $T$  has a bounded orbit and*

$$x_0 = \lim_{i \rightarrow \infty} T(t_i)x ,$$

then

$$\lim_{u \rightarrow \infty} \frac{1}{u} \int_0^u (T(t)x - T(t)x_0)dt = 0 .$$

*Proof.* For  $\varepsilon > 0$  be given there is an positive integer  $n$  such that

$$| T(t_i)x - x_0 | < \varepsilon \qquad \text{for all } i \geq n .$$

Let  $u$  be any positive number great than  $t_n$ . Then

$$\begin{aligned} \left| \frac{1}{u} \int_0^u (T(t)x - T(t)x_0)dt \right| &\leq \frac{1}{u} \int_0^{u-t_n} | T(t)T(t_n)x - T(t)x_0 | dt \\ &\quad + \frac{1}{u} \int_0^{t_n} | T(t)x | dt + \frac{1}{u} \int_{u-t_n}^u | T(t)x_0 | dt \\ &< \frac{u - t_n}{u} \varepsilon \\ &\quad + \frac{1}{u} \left( \int_0^{t_n} | T(t)x | dt + \int_{u-t_n}^u | T(t)x_0 | dt \right) . \end{aligned}$$

Since orbits are bounded the last term in above inequality will tend to 0 as  $u \rightarrow \infty$ . Hence, we prove the assertion.

*Proof of Theorem 2.* By Lemma 5, Lemma 7 and reflexivity of  $X$ , there is an increasing unbounded sequence  $\{u_i\}$  of positive numbers such that

$$w - \lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0 dt$$

exists and is in  $F$ , where  $x_0 = \lim_{i \rightarrow \infty} T(t_i)x$ . Also, it follows from Lemma 8

$$\lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} (T(t)x - T(t)x_0) dt = 0 .$$

Thus,

$$w - \lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x dt = w - \lim_{i \rightarrow \infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0 dt \quad \text{is in } F .$$



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