Pacific Journal of Mathematics

ON THE REST POINTS OF A NONLINEAR NONEXPANSIVE SEMIGROUP

CHI-LIN YEN

Vol. 45, No. 2 October 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

ON THE REST POINTS OF A NONLINEAR NONEXPANSIVE SEMIGROUP

CHI-LIN YEN

Let X be a reflexive Banach space and T a nonlinear nonexpansive semigroup on X. The results which we shall prove are the following:

THEOREM 1. **Suppose that for any closed convex set M with the property that $T(t)M \subseteqq M$ for all $t \geqslant 0$, M contains a precompact orbit. Then T has a rest point. Moreover, the set of all rest points of T is connected.**

THEOREM 2. **Suppose that X is strictly convex and T has a bounded orbit. If there is an unbounded increasing sequence $\{u_i\}$ of positive numbers and point x such that $\lim_{i\to\infty} T(u_i)x$ exists then T has a rest point. Moreover, if $\{t_i\}$ is an unbounded increasing sequence of positive numbers such that**

$$y = w - \lim_{i\to\infty} \frac{1}{t_i} \int_0^{t_i} T(t)x\, dt$$

exists, then $y \in F$.

Let X be a Banach space. By a nonlinear nonexpansive strongly continuous semigroup T on X (or briefly, a semigroup T on X) we mean that T is a mapping from $[0, \infty) \times X$ into X such that

(i) for any $x \in X$, $t_1 \geqslant 0$, and $t_2 \geqslant 0$, $T(t_1)T(t_2)x = T(t_1 + t_2)x$;

(ii) for any $x \in X$, $\lim_{t\to 0+} T(t)x = T(0)x = x$;

(iii) for any $x \in X$, $y \in X$, and $t \geqslant 0$, $|T(t)x - T(t)y| \leqslant |x - y|$.

Throughout this paper T will denote a semigroup on X. We shall give some definitions as follows:

(1) For $x \in X$ the orbit of x is the set $O_x = \{T(t)x; t \geqq 0\}$

(2) $F = \{x; T(t)x = x \text{ for all } t \geqslant 0\}$, and if $x \in F$ then x is called a rest point of T.

(3) $P = \{x; \text{there is } t_0 > 0 \text{ such that } T(t_0)x = x\}$.

(4) $A = \{x; O_x \text{ is precompact}\}$.

(5) $L = \{x; \text{there is a sequence } \{t_i\} \text{ of positive numbers such that } t_i \uparrow \infty \text{ and } \lim_{i\to\infty} T(t_i)x \text{ exists}\}$.

Clearly, $L \supseteqq A \supseteqq P \supseteqq F$. Moreover, if $F \neq \phi$ then O_x is bounded for all $x \in X$. The question arises "Is the converse true?" M. Crandall and A. Pazy [2] give an affirmative answer, when X is a Hilbert space. However, the converse is not true in general (see R. Martin [4]). In this paper some sufficient conditions will be given such that $F \neq \phi$.

Our main results are the following:

THEOREM 1. *Let X be a reflexive Banach space. Suppose that for any closed convex set M with the property that $T(t)M \subseteqq M$ for all $t \geqslant 0$, $M \cap A \neq \phi$. Then $F \neq \phi$. Moreover, F is connected.*

THEOREM 2. *Let X be a srictly convex reflexive Banach space. If T has a bounded orbit and $L \neq \phi$, then $F \neq \phi$. Moreover, if $t_i \uparrow \infty$ and $y = w - \lim_{i\to\infty} 1/t_i \int_0^{t_i} T(t)x\,dt$ for some $x \in X$, then $y \in F$.*

As an application of Theorem 1 one can verify that if X is a reflexive Banach space and T has a bounded orbit, then $F \neq \phi$ provided that either of the following holds: (i) there is a $t_0 > 0$ such that $T(t_0)$ is weakly continuous function on X or (ii) X has the property that every m-dissipative Lipschitz continuous function on X is demiclosed (f is demiclosed if $x_n \longrightarrow x_0$ strongly then $y_0 = fx_0$). It is known that if X is a uniformly convex space, the condition (ii) is fulfilled, (see F. Browder [1]).

As an application of Theorem 2 one can verify that if X is a strictly convex, reflexive Banach spach and $A \neq \phi$ then $F \neq \phi$. Furthermore, if $x \in A$ then for some unbounded increasing sequence $\{t_i\}$ of positive numbers $\lim_{t_i\to\infty} 1/t_i \int_0^{t_i} T(u)x\,du$ exists and is an element of F. This result generalizes that of D. Rutedge [5] in which X is a Hilbert space and $P \neq \phi$.

We need two known lemmas to prove our theorems and we state them below without proof. Lemma 1 was put in the present form by M. Crandall and A. Pazy [2] and Lemma 2 due to R. de Marr [3].

LEMMA 1. *Let $x \in X$ such that $|T(t)x| \leqslant M$ for all $t \geqslant 0$. Then $K = \bigcup_{\tau\geqslant 0} \bigcap_{t\geqslant\tau} \{y; |y - T(t)x| \leqslant |x| + M\}$ is a nonempty convex subset of X such that $T(t)K \subseteqq K$ for all $t \geqq 0$.*

LEMMA 2. (R. de Marr). *Let C be a compact subset of X such that $r = \operatorname{diam} C > 0$. Then there is an $x_0 \in \operatorname{clco} C$ and a positive number $r_1 < r$ such that $|y - x_0| \leqslant r_1$ whenever $y \in C$.*

We will use the following two lemmas and the above twe lemmas to prove Theorem 1.

LEMMA 3. *Let M be a closed subset of X such that $T(t)M \subseteqq M$ for all $t \geqslant 0$. If $M \cap A \neq \phi$, then there is a compact subset C of M such that $T(t)C = C$.*

Proof. Let $x \in M \cap A$. Then $\bar{O}_x$ is a compact subset of M and $T(t_1)\bar{O}_x \subseteqq T(t_2)\bar{O}_x$ whenever $t_1 \geqslant t_2 \geqslant 0$. Hence $C = \bigcap_{t>0} T(t)\bar{O}_x$ is a nonempty compact subset of M. Furthermore, $T(t)C = C$ for all $t \geqslant 0$.

LEMMA 4. *Let $x_0, x_1 \in X$ and $\lambda \in [0, 1]$. Then*

$$M_\lambda = \{y \in X;\ |x_0 - y| = \lambda|x_1 - x_0|,\ |x_1 - y| = (1 - \lambda)|x_1 - x_0|\}$$

is a nonempty closed convex bounded subset of X. Moreover, if $x_0, x_1 \in F$ then $T(t)M_\lambda \subseteqq M_\lambda$.

Proof.

$$M_\lambda = \{y \in X;\ |x_0 - y| \leqslant \lambda|x_0 - x_1|\} \cap \{y \in X;\ |x_1 - y| \leqslant (1-\lambda)|x_0 - x_1|\}$$

contains $\lambda x_1 + (1 - \lambda)x_0$. Thus M_λ is a nonempty closed convex bounded subset of X.

Since $T(t)x_i = x_i$ for all $t \geqslant 0$, $i = 0, 1$ thus for any $y \in M_\lambda$,

$$|x_0 - T(t)y| = |T(t)x_0 - T(t)y| \leqq \lambda|x_0 - x_1|$$

and

$$|x_0 - T(t)y| = |T(t)x_1 - T(t)y| \leqslant (1 - \lambda)|x_0 - x_1|,$$

that is, $T(t)y \in M_\lambda$.

Now we prove Theorem 1.

Proof of Theorem 1. By Lemma 1 there is a nonempty closed bounded convex set M such that $T(t)M \subseteqq M$. Let $\{M_\alpha\}$ be a chain of subset of M such that

(i) M_α is a nonempty closed bounded convex set satisfying $T(t)M_\alpha \subseteqq M_\alpha$ for all α.

(ii) $M_\alpha \subseteqq M_\beta$ if $\alpha \geqslant \beta$.

Since M_α is weak-compact, thus $\bigcap_\alpha M_\alpha \neq \phi$. Further,

$$T(t)(\bigcap_\alpha M_\alpha) \subseteqq \bigcap_\alpha M_\alpha .$$

By Zorn's lemma there is a maximal element, say M_0, in the collection $\mathscr{F} = \{M_1;\ M_1$ is a nonempty closed bounded convex subset of M such that $T(t)M_1 \subseteqq M_1\}$. We want to show that M_0 contains exactly one point. Suppose not. By hypothesis, $M_0 \cap A$ contains at least one point, say x. By Lemma 3 there is a compact subset C of M_0 such that $T(t)C = C$. By Lemma 2 there is a point $x_0 \in \text{clco}\, C \subseteqq M_0$ such that $|y - x_0| \leqslant r_1 < r = \text{diam}\, C$ for all $y \in C$. Consider the set $M' = \bigcap_{y \in C} \{z \in M_0;\ |z - y| \leqslant r_1\}$.

We see that M' is a nonempty closed bounded convex subset of M_0 such that $T(t)M \subseteqq M$. Since $r = \operatorname{diam} C$ and C is compact, thus there are $x_1, x_2 \in C$ such that $|x_1 - x_2| = r$. By the definition of M' and the fact that $r_1 < r$, we have $x_i \notin M'$ for $i = 1, 2$. Thus $M' \neq M_0$ and the maximality of M_0 is contradicted. Thus M_0 must contain exactly one point which lies in F. This shows that if M is a closed convex set satisfying $T(t)M \subseteqq M$ for all $t \geqslant 0$ then $M \cap F \neq \phi$.

Next we want to show that F is connected. Suppose not. Then there are two disjoint closed subsets A and B of X such that $A \cup B \supseteqq F$, $A \cap F \neq \phi$ and $B \cap F \neq \phi$. Let $A' = A \cap F$ and $B' = B \cap F$. Since F is closed thus A' and B' are closed. For $x_1 \in A'$, $D(x_1, B') = \inf\{|x_1 - y|; y \in B'\} = k > 0$. Thus, there is a $y_1 \in B'$ such that $|x_1 - y_1| < 5/4\, K$. It follows from Lemma 4 and the above paragraph there is $z_1 \in M^1 = \{z \in X; |z - x_1| = |z - y_1| = 1/2\,|x_1 - y_1|\}$ such that $z_1 \in F = A' \cup B'$. Since $|z_1 - x_1| = 1/2\,|x_1 - y_1| < 5/8\, K$, $z_1 \in A'$. Let $x_2 = z_1$. Then there is a $y_1 \in B'$ such that

$$|x_2 - y_2| \leqslant \operatorname{Min}\left\{\frac{5}{4} D(x_2, B'), |x_2 - y_1|\right\}.$$

Similarly, there is $x_3 \in M^2 = \{z \in X; |z - x_2| = |z - y_2| = 1/2\,|x_2 - y_2|\}$ such that $x_3 \in F$. By the same argument we have $x_3 \in A'$. We assume we have chosen $x_{n+1} \in M^n = \{z \in X; |z - x_n| = |z - y_n| = 1/2\,|x_n - y_n|\}$ and $x_{n+1} \in A'$ and $y_n \in B'$ such that

$$|y_n - x_n| \leqslant \operatorname{Min}\left\{\frac{5}{4} D(x_n, B'), |x_n - y_{n-1}|\right\}$$

for all $n \leqslant k - 1$ where $k \geqslant 3$. We can choose y_k, x_{k+1} as follows:

Since $D(x_k, B') \leqslant |x_k - y_{k-1}|$, there is a $y_k \in B'$ such that

$$|x_k - y_k| \leqslant \operatorname{Min}\left\{\frac{5}{4} D(x_k, B'), |x_k - y_{k-1}|\right\}$$

and let $x_{k+1} \in A'$ such that

$$x_{k+1} \in M^k = \left\{z \in X; |z - x_k| = |z - y_k| = \frac{1}{2}|x_k - y_k|\right\}.$$

Note that

$$|x_{n+1} - y_{n+1}| \leqslant |x_{n+1} - y_n| = \frac{1}{2}|x_n - y_n| \leqslant \cdots \leqslant \left(\frac{1}{2}\right)^n |x_1 - y_1| < \left(\frac{1}{2}\right)^n \left(\frac{5}{4} K\right)$$

and

$$|x_{n+1} - x_n| = |x_{n+1} - y_n| < \left(\frac{1}{2}\right)^n \left(\frac{5}{4}K\right).$$

Thus, $\{x_n\}$ is a Cauchy sequence and so $\{x_n\}$ converges to some point, say x_0 in A'. Also $D(x_{n+1}, B') \leqq |x_{n+1} - y_{n+1}| < (1/2)^n ((5/4)K) \rightarrow 0$, so $D(x_0, B') = 0$. Since B' is closed $x_0 \in B'$. This is a contradiction to $\phi = A \cap B \ni x_0$. Therefore, F is connected.

In order to prove Theorem 2 we need the following lemmas.

LEMMA 5. *If $x_0 \in X$ such that $x_0 = \lim_{i\to\infty} T(t_i)x$ for some $x \in X$ and some unbounded increasing sequence $\{t_i\}$ of positive numbers, then there is an unbounded increasing sequence $\{s_i\}$ of positive numbers, such that*

$$\lim_{1\to\infty} T(s_i)x_0 = x_0 .$$

Indication of proof. By an inductive process, for each i, choose n_{i+1} such that $t_{n_{i+1}} - t_{i+1} \geqslant 1 + t_{n_i} - t_i$, $i = 1, 2, 3, \cdots$ and $n_1 = 1$. Let $s_i = t_{n_i} - t_i$. Then,

$$\begin{aligned} |T(s_i)x_0 - x_0| &\leqq |T(s_i)T(t_i)x - x_0| + 2|T(t_i)x - x_0| \\ &= |T(t_{n_i})x - x_0| + 2|T(t_i)x - x_0| \longrightarrow 0 \quad \text{as} \quad i \longrightarrow \infty . \end{aligned}$$

That is, $\lim_{i\to\infty} T(s_i)x_0 = x_0$.

LEMMA 6. *Let X be a strictly convex Banach space. If*

$$\lim_{i\to\infty} T(s_i)x_0 = x_0$$

for some increasing unbounded sequence $\{s_i\}$ of positive numbers, then for any n, any $\lambda_1, \cdots, \lambda_n$ such that $\lambda_i \geqslant 0$, $\sum_{i=1}^n \lambda_i = 1$ and any $x_1, \cdots, x_n$ in 0_{x_0},

$$T(t)\left(\sum_{i=1}^n \lambda_i x_i\right) = \sum_{i=1}^n \lambda_i T(t)x_i \quad \text{for all } t \geqslant 0 . \tag{1}$$

Indication of proof. Clearly, (1) is true for the case $n = 1$. Using inductive argument we may assume that (1) holds for all $n \leqslant k$ where $k \geqslant 1$. We shall show that (1) holds for the case $n = k + 1$, that is, for any λ_i, $\lambda_i \neq 1$, $\sum_{i=1}^{k+1} \lambda_i = 1$, and any $x_1, \cdots, x_{k+1}$ in 0_{x_0},

$$T(t)\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = \sum_{i=1}^{k+1} \lambda_i T(t)x_i .$$

Let $y = \sum_{i=1}^{k+1} \lambda_i x_i$, $z = (1 - \lambda_1)^{-1} \sum_{i=1}^{k+1} \lambda_i x_i$. Then $y = \lambda_1 x_1 + (1 - \lambda_1)z$, and

$$(2) \qquad |T(t)y - T(t)x_1| \leqslant |y - x_1|,\ |T(t)y - T(t)z| \leqslant |y - z| \quad \text{for all } t \geqslant 0 .$$

$$(3) \qquad \begin{aligned} |T(t)x_1 - T(t)z| &\leqslant |T(t)y - T(t)x_1| + |T(t)y - T(t)z| \\ &\leqslant |y - x_1| + |y - z| = |x_1 - z| . \end{aligned}$$

Since $|T(t_i)x_1 - T(t_i)z| \downarrow |x_1 - z|$ as $i \to \infty$, thus we have

$$|T(t)y - T(t)x_1| + |T(t)y - T(t)z| = |T(t)x_1 - T(t)z| .$$

By the strict convexity of X, (2) and (3) we have that

$$T(t)y = \lambda_1 T(t)x_1 + (1 - \lambda_1) T(t)z .$$

By the inductive hypothesis,

$$T(t)y = \sum_{i=1}^{k+1} \lambda_i T(t)x_i .$$

LEMMA 7. *Let x_0, X be as in Lemma 6. If there is an unbouded increasing sequence $\{u_i\}$ of positive numbers such that*

$$y = w - \lim_{i\to\infty} \frac{1}{u_i} \int_0^{u_i} T(t)x_0\, dt, \text{ then } y \in F .$$

Proof. Let

$$y_i = \frac{1}{u_i} \int_0^{u_i} T(t)x_0\, dt .$$

For $\varepsilon > 0$, $r > 0$ fixed, there is an $N > 0$ such that if $M \geqslant |T(t)x_0|$ for all $t \geqslant 0$,

$$\frac{rM}{u_i} < \frac{\varepsilon}{3} \text{ whenever } i \geqslant N .$$

It follows from Lemma 6 that

$$T(r)y_i = \frac{1}{u_i} \int_r^{u_i+r} T(t)x_0\, dt = y_i + \frac{1}{u_i} \left(\int_{u_i}^{u_i+r} - \int_0^r \right) T(t)x_0\, dt .$$

Thus $|T(r)y_i - y_i| < 2\varepsilon/3$ for all $i \geqslant N$. Since $y = w - \lim_{i\to\infty} y_i$, there exists a $k > 0$, $\lambda_1, \lambda_2, \cdots, \lambda_k \geqslant 0$ such that $\sum_{i=1}^k \lambda_i = 1$ and $|y - \sum_{i=1}^k \lambda_i y_{i+N-1}| < \varepsilon/6$. Hence,

$$\begin{aligned} |T(r)y - y| &\leqslant \left| T(r)y - \sum_{i=1}^k \lambda_i T(r) y_{i+N-1} \right| \\ &\quad + \left| \sum_{i=1}^k \lambda_i (T(r) y_{i+N-1}) \right| + \left| y - \sum_{i=1}^k \lambda_i y_{i+N-1} \right| \\ &< 2\varepsilon/6 + 2\varepsilon/3 = \varepsilon . \end{aligned}$$

Since ε and r are arbitrary positive numbers, thus $y \in F$.

LEMMA 8. *Let $\{t_i\}$ be an unbounded increasing sequence of positive numbers and x in X. If T has a bounded orbit and*

$$x_0 = \lim_{i\to\infty} T(t_i)x ,$$

then

$$\lim_{u\to\infty} \frac{1}{u}\int_0^n (T(t)x - T(t)x_0)dt = 0 .$$

Proof. For $\varepsilon > 0$ be given there is an positive integer n such that

$$|T(t_i)x - x_0| < \varepsilon \qquad \text{for all } i \geqslant n .$$

Let u be any positive number great than t_n. Then

$$\begin{aligned}\left|\frac{1}{u}\int_0^u (T(t)x - T(t)x_0)dt\right| &\leqslant \frac{1}{u}\int_0^{u-t_n} |T(t)T(t_n)x - T(t)x_0|\,dt \\ &\quad + \frac{1}{u}\int_0^{t_n} |T(t)x|\,dt + \frac{1}{u}\int_{u-t_n}^{u} |T(t)x_0|\,dt \\ &< \frac{u-t_n}{u}\in \\ &\quad + \frac{1}{u}\left(\int_0^{t_n} |T(t)x|\,dt + \int_{u-t_n}^{u} |T(t)x_0|\,dt\right) .\end{aligned}$$

Since orbits are bounded the last term in above inequality will tend to 0 as $u \to \infty$. Hence, we prove the assertion.

Proof of Theorem 2. By Lemma 5, Lemma 7 and reflexivity of X, there is an increasing unbounded sequence $\{u_i\}$ of positive numbers such that

$$w - \lim_{i\to\infty} \frac{1}{u_i}\int_0^{u_i} T(t)x_0\,dt$$

exists and is in F, where $x_0 = \lim_{i\to\infty} T(t_i)x$. Also, it follows from Lemma 8

$$\lim_{i\to\infty} \frac{1}{u_i}\int_0^{u_i} (T(t)x - T(t)x_0)\,dt = 0 .$$

Thus,

$$w - \lim_{i\to\infty} \frac{1}{u_i}\int_0^{u_i} T(t)x\,dt = w - \lim_{i\to\infty} \frac{1}{u_i}\int_0^{u_i} T(t)x_0\,dt \qquad \text{is in } F .$$

References

1. F. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc., **74** (1968), 660-665.
2. M. Crandall and A. Pazy, *Semigroups of nonlinear contractions and dissipative sets*, J. Functional Analysis, **3** (1969), 376-418.
3. R. de Marr, *Common fixed points of commuting commuting contraction mappings*, Proc. J. Math., **13** (1963), 1139-1141.
4. R. Martin, *Lyapunov function and autonomous differential equation in Banach space*, to appear in Math. Systems Theory.
5. D. Rutledge, *A generator for a semigroup of nonlinear transformations*, Proc. Amer. Math. Soc., **20** (1969), 491-498.

Received November 24, 1971. The results are partially based upon the author's Ph. D. thesis, written under the direction of Professor G. F. Webb at Vanderbilt University. This work was supported by N. S. C. Republic of China.

ACADEMIA SINICA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 45, No. 2 October, 1973