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ON EXTENDING ISOTOPIES

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Let K be a locally compact metric space. An *isotopy* on K is a continuous family of homeomorphisms $h_t: K \rightarrow K$ for $t \in I$ such that $h_0 = \text{id}$. Let $\mathcal{I}(K)$ denote the space of isotopies of K with $C-0$ topology. *Conjecture:* Let X be metric and Y a closed subset of X . Then every map $f: Y \rightarrow \mathcal{I}(K)$ can be continuously extended to X . The conjecture is proved for the following cases: (1) K is a 1-complex, (2) K is compact and X is finite-dimensional, (3) K is compact, Y is compact and finite-dimensional, and X is separable, and (4) Y is of type 1 in a compact space X . Y is of type 1 in X if the closure of the set of points of X which do not have a unique closest point in Y does not intersect Y .

1. Introduction. Let K be a topological space. An *isotopy* on K is a continuous family, for $t \in I$, of homeomorphisms $h_t: K \rightarrow K$ such that $h_0 = \text{id}$. The isotopy is *invertible* if $g_t = (h_t)^{-1}$ is also an isotopy (i.e., is continuous in t). Invertible isotopies can be thought of as level-preserving homeomorphisms of $K \times I$ onto itself which are the identity on $K \times \{0\}$. Throughout the paper, K will be locally compact and metric, so all isotopies on K will automatically be invertible. We will denote the space of isotopies on K with $C-0$ topology by $\mathcal{I}(K)$. We will discuss the following:

Conjecture. Let K be locally compact and metric. Let X be metric and let Y be a closed subset of X . Then every map $\varphi: Y \rightarrow \mathcal{I}(K)$ can be continuously extended to X .

It should be noted that this conjecture is equivalent to saying that $\mathcal{I}(K)$ is an absolute retract for metric spaces. The following theorem states the conjecture for several special cases:

THEOREM. *The conjecture is true in the following cases:*

- (1) K is a one-dimensional simplicial complex (for example, R^1 , S^1 , or I)
- (2) K is compact, X is finite-dimensional
- (3) K is compact, Y is finite-dimensional and compact, and X is separable.
- (4) X is compact and Y is of type 1 in X .

The last case is an easy result, the definition of type being as follows: Let Y be a closed subset of X . Define $H(Y) = \{x \in X \mid x \text{ does not have a unique closest point in } Y\}$. Let $Y_1 = \overline{H(Y)} \cap Y$, $Y_n = Y_{n-1} \cap \overline{H(Y_{n-1})}$ for $n = 2, 3, \dots$.

If $Y_1 = \emptyset$, then we say Y is of *type 1* in X . If $Y_n = \emptyset$ but $Y_{n-1} \neq \emptyset$, we say Y is of *type n* in X . Finally, if all Y_i are nonempty, we say that Y is of *infinite type* in X . There is probably an inductive argument to show that sets of finite type in compact spaces satisfy the conjecture, but sets of infinite type are not uncommon, for instance, the Cantor set in I .

The proof of Part (4) of the Theorem is contained in § 2. The proofs of (1)–(3) are similar to the proof of the Tietze Extension Theorem, an idea suggested to the author by R. D. Anderson, and are contained in later sections.

2. Proof of Part (4). Let $\varphi: Y \rightarrow \mathcal{S}(K)$ be continuous. Let U be an open set containing $H(Y)$ such that $U \cap Y = \emptyset$. Let $f: X \rightarrow I$ be a continuous function such that $f(\bar{U}) = 0$ and $f(Y) = 1$. If $h_t: K \rightarrow K$ is an isotopy, let ${}_s h_t: K \rightarrow K$ for $s \in I$ be the isotopy defined by

$$\begin{aligned} {}_s h_t &= h_t & \text{for } t \leq s \\ {}_s h_t &= h_s & \text{for } t \geq s. \end{aligned}$$

Define $g: X - \overline{H(Y)} \rightarrow Y$ by $g(x)$ is the unique closest point of Y to x . Using compactness of X , it is easy to show that g is continuous.

Finally, define $\Phi: X \rightarrow \mathcal{S}(K)$ by $\Phi(x) = {}_{f(x)}\varphi(g(x))_t$ on $X - \overline{H(Y)}$, and $\Phi(x)$ is the identity isotopy on $\overline{H(Y)}$. Φ is easily seen to be continuous and equals φ on Y .

3. Proof of Part (1). Let $\varphi: Y \rightarrow \mathcal{S}(K)$ be continuous, where K is a 1-complex. Let K' be the set of vertices of K which do not intersect exactly two 1-simplices. Then any isotopy of K is fixed on K' . Also, $K - K'$ is the disjoint union of sets homeomorphic to R^1 or S^1 . Hence we can restrict ourselves to isotopies on these spaces. Furthermore, $\mathcal{S}(R^1)$ is naturally homeomorphic to $\mathcal{S}(I)$. We will prove Part (1) for $K = I$, and it will be clear that the proof generalizes to $K = S^1$ by the covering space of S^1 , and hence to K a 1-complex.

Let ${}_i f_t \in \mathcal{S}(I)$ for $i = 1, 2, \dots, n$, and let $s_i \in I$ be such that $\sum_{i=1}^n s_i = 1$. Define the s_i -average of the isotopies ${}_i f_t$ to be the isotopy $f_t(x) = \sum_{i=1}^n s_i {}_i f_t(x)$ for $x \in I$, $t \in I$. It is easy to see that $f_t(x)$ is an isotopy, and furthermore, that

$$(*) \quad \max_{i \leq n} d(f_t, {}_i f_t) \leq \max_{i, j \leq n} d({}_i f_t, {}_j f_t).$$

Here d denotes the sup metric on $\mathcal{S}(K)$ inherited from K since K is compact. It will also be used to denote the metric on X .

We will inductively define continuous functions $\varphi_n: X \rightarrow \mathcal{I}(K)$ satisfying:

- (a) for $y \in Y$, $d(\varphi_n(Y), \varphi(Y)) < 1/2^n$
- (b) for $i, j \geq n$, $x \in X$ such that $d(x, Y) \geq 1/n$, then $\varphi_j(x) = \varphi_i(x)$
- (c) $d(\varphi_n, \varphi_{n-1}) < 1/2^{n-1}$.

If we can do the above, then $\varphi = \lim_{n \rightarrow \infty} \varphi_n$ is the required extension. It is equal to φ on Y by (a), it converges on $X - Y$ by (b), and it is continuous on Y by (c).

Suppose we have inductively defined the φ_i satisfying the above for $i \leq n - 1$. We will define φ_n as follows.

Let $\Phi_n: Y \rightarrow \mathcal{I}(K)$ be defined by $\Phi_n(y) = \varphi(y) \circ (\varphi_{n-1}(y))^{-1}$. (It is understood that the composition of isotopies is at each level the composition of the homeomorphisms at the same level.) Then $d(\Phi_n(y), \text{id}) = d(\varphi(y), \varphi_{n-1}(y)) < 1/2^{n-1}$ by (a).

Let $g: Y \rightarrow (0, 1]$ be a continuous function such that if given $y_1, y_2 \in Y$ and $x \in X$ such that $d(y_1, x) < g(y_1)$ and $d(y_2, x) < g(y_2)$, then $d(\Phi_n(y_1), \Phi_n(y_2)) < 1/2^n$. Such a g is easily constructed using continuity of Φ_n .

Let $N_\delta(A)$ denote the open δ -neighborhood of the set A . Let $\mathcal{C} = \{N_{g(y)}(y) \cap N_{1/m}(Y) \mid y \in Y\} \cup \{X - Y\}$. \mathcal{C} is an open cover of X . Since X is paracompact, let \mathcal{C}' be a locally finite refinement of \mathcal{C} . For each $u \in \mathcal{C}'$, associate an element $f_u \in \mathcal{I}(K)$ as follows: If $u \subset X - Y$, then $f_u = \text{id}$. If $u \subset N_{g(y)}(y)$ for some $y \in Y$, then $f_u = \Phi_n(y)$. Let $\{S_u: X \rightarrow I\}$ be a partition of unity of \mathcal{C}' .

We will define $\varphi_n: X \rightarrow \mathcal{I}(K)$ as follows. Pick $x \in X$. Let u_1, \dots, u_m be the elements of \mathcal{C}' such that $S_{u_i}(x) \neq 0$ for $i = 1, \dots, m$.

Define $\Phi'_n(x)$ to be the s_{u_i} -average of the isotopies f_u and let $\varphi_n(x) = \Phi'_n(x) \circ \varphi_{n-1}(x)$. By construction of the cover \mathcal{C}' , if $y_1, y_2 \in Y$ and $x \in X$ and $\{x, y_1\} \subset U_1 \in \mathcal{C}'$ and $\{x, y_2\} \subset U_2 \in \mathcal{C}'$, then $d(\Phi_n(y_1), \Phi_n(y_2)) < 1/2^n$, hence by (*), $d(\Phi'_n(y), \Phi_n(y)) < 1/2^n$ for $y \in Y$. In addition, since $d(\Phi_n(y), \text{id}) < 1/2^{n-1}$, it follows that $d(\Phi'_n(x), \text{id}) < 1/2^{n-1}$ for $x \in X$. Hence for $y \in Y$, $d(\varphi_n(y), \varphi(y)) = d(\Phi'_n(y), \Phi_n(y)) < 1/2^n$ and $d(\varphi_n, \varphi_{n-1}) = d(\Phi'_n(x), \text{id}) < 1/2^{n-1}$. That part (b) of the inductive hypothesis also holds is trivial.

4. Composition sequences. Let K be a compact metric space. A *composition sequence* of isotopies of K is a finite collection $\{i f_i, s_i\}_{i \leq n}$ of isotopies $i f_i: K \rightarrow K$ and numbers $s_i \in I$ such that $\sum s_i = 1$. The *composition* of $\{i f_i, s_i\}_{i \leq n}$ is an isotopy $f_i: K \rightarrow K$ defined as follows:

Let $p_m = \sum_{i=1}^m s_i$ for $m \leq n$.

If $p_m \leq t \leq p_{m+1}$ and $x \in K$, then

$$f_t(x) = {}_{m+1}f_t(x) \circ {}_{m+1}f_{p_m}^{-1}(x) \circ {}_m f_{p_m}(x) \circ \dots \circ {}_2 f_{p_2}(x) \circ {}_2 f_{p_1}^{-1}(x) \circ {}_1 f_{p_1}(x).$$

We will now need several technical lemmas about composition sequences

LEMMA 1. *Let $\{f_i, s_i\}_{i \leq n}$ be a composition sequence satisfying $d(f_i, \text{id}) < \varepsilon$. Then the composition f_i satisfies $d(f_i, \text{id}) < 2n\varepsilon$.*

Proof. Trivial.

LEMMA 2. *Let $f_i \in \mathcal{S}(K)$ and let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for $0 \leq s \leq t \leq 1$, and $x, y \in K$ such that $d(x, y) < \delta$, then $d(f_s^{-1}(x), f_s^{-1}(y)) < \varepsilon$, $d(f_t(x), f_t(y)) < \varepsilon$ and $d(f_t \circ f_s^{-1}(x), f_t \circ f_s^{-1}(y)) < \varepsilon$.*

Proof. Easy to verify using compactness of K and I .

LEMMA 3. *Given $\varepsilon > 0$ and $f_i \in \mathcal{S}(K)$. Then there exists a $\delta > 0$ such that if $g_i \in \mathcal{S}(K)$ satisfies $d(f_i, g_i) < \delta$, then $d(f_i^{-1}, g_i^{-1}) < \varepsilon$.*

Proof. Again easy, using compactness of K and I .

LEMMA 4. *Given $f_i \in \mathcal{S}(K)$, there exists a function $l: (0, 1] \rightarrow (0, 1]$ such that for any $\varepsilon \in (0, 1]$, $g_i \in \mathcal{S}(K)$, $x, y \in K$ such that $d(f_i, g_i) < l(\varepsilon)$ and $d(x, y) < l(\varepsilon)$ and $0 \leq s \leq t \leq 1$, then*

$$d(f_t \circ f_s^{-1}(x), g_t \circ g_s^{-1}(y)) < \varepsilon .$$

Proof. $d(f_t \circ f_s^{-1}(x), g_t \circ g_s^{-1}(y)) \leq d(f_t \circ f_s^{-1}(x), f_t \circ f_s^{-1}(y)) + d(f_t \circ f_s^{-1}(y), f_t \circ g_s^{-1}(y)) + d(f_t \circ g_s^{-1}(y), g_t \circ g_s^{-1}(y))$. The first term is minimized by Lemma 2, the second by Lemma 3 and uniform continuity of f_i , and the third is minimized by $l(\varepsilon)$.

LEMMA 5. *Let $f_i \in \mathcal{S}(K)$ and let $\varepsilon > 0$ and n be an integer. Then there exists a $\delta > 0$ such that if $\{g_i, s_i\}_{i \leq n}$ is a composition sequence such that $d(g_i, f_i) < \delta$ for all $i \leq n$, then the composition g_i satisfies $d(g_i, f_i) < \varepsilon$.*

Proof. Let $\delta = l^n(\varepsilon)$ (the composition of n l s) where l is the function of Lemma 4.

5. **Proofs of Parts (2) and (3).** We will now prove Part (2) in a fashion analogous to that of Part (1). Let $\varphi: Y \rightarrow \mathcal{S}(K)$ be continuous, where K is compact and $\dim(x) = n - 1$. We will inductively define functions $\varphi_m: X \rightarrow \mathcal{S}(K)$ satisfying:

(a) for $y \in Y$, $d(\varphi_m(y), \varphi(y)) < 1/2_m$

(b) for $i, j \geq m$ and $x \in X$ such that $d(x, Y) \geq 1/m$, $\varphi_i(x) = \varphi_j(x)$

(c) $d(\varphi_m, \varphi_{m-1}) \leq 2n/2^{m-1}$.

Again, define $\Phi_n(y) = \varphi(y) \circ (\varphi_{n-1}(y))^{-1}$.

Let $g: Y \rightarrow (0, 1]$ be a function such that if x_1, \dots, x_n are points of X such that $d(x_i, y) < g(y)$ for some $y \in Y$ and all $i \leq n$, then any composition f_t of the isotopies $\Phi_n(x_i)$ satisfies $d(f_t, \Phi_n(y)) < 1/2^n$. Such a function is constructed using Lemma 4 and the continuity of Φ_n .

We now take a cover \mathcal{C} of X as sets of the form $\{N_{g(y)} \cap N_{1/m}(Y) \mid y \in Y\} \cup \{X - Y\}$ as in § 3. Take a refinement or order $n - 1$ (see Theorem V 1, p. 48 of [2]), call it \mathcal{C}' . Order the elements of \mathcal{C}' , take a portion of unity, and define $\Phi'_n: X \rightarrow \mathcal{I}(K)$ by using compositions instead of s_i -averages. Letting $\varphi_n(x) = \Phi'_n(x) \circ \varphi_{n-1}(x)$ as before, the inductive hypotheses are again met. Note that (c), $d(\varphi_m, \varphi_{m-1}) \leq 2n/2^{m-1}$ follows from Lemma 1.

Part (3) follows as a corollary of (2). Since X is separable, it may be imbedded in an endslice of the Hilbert cube Q . Call the imbedding i . Since Y is compact, $i(Y)$ is closed, and hence a Z -set in the cube. If Y has dimension n , it may also be imbedded in I^{2n+1} , call the imbedding j . Then $i(Y)$ and $j(Y)$ are Z -sets in the cube. By [1] there exists a homeomorphism $h: Q \rightarrow I^{2n+1} \times Q$ such that $h \mid i(Y)$ is onto $j(Y) \times \{0\}$.

By Part (2), we can extend the map $\varphi \circ (h \circ i)^{-1}$ on $(h \circ i)(Y)$ to $I^{2n+1} \times \{0\}$. There is then a natural extension to $I^{2n+1} \times Q$ and hence to $(h \circ i)(x)$.

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Allan Francis Abrahamse, <i>Uniform integrability of derivatives on σ-lattices</i>	1
Ronald Alter and K. K. Kubota, <i>The diophantine equation $x^2 + D = p^n$</i>	11
Grahame Bennett, <i>Some inclusion theorems for sequence spaces</i>	17
William Cutler, <i>On extending isotopies</i>	31
Robert Jay Daverman, <i>Factored codimension one cells in Euclidean n-space</i>	37
Patrick Barry Eberlein and Barrett O'Neill, <i>Visibility manifolds</i>	45
M. Edelstein, <i>Concerning dentability</i>	111
Edward Graham Evans, Jr., <i>Krull-Schmidt and cancellation over local rings</i>	115
C. D. Feustel, <i>A generalization of Kneser's conjecture</i>	123
Avner Friedman, <i>Uniqueness for the Cauchy problem for degenerate parabolic equations</i>	131
David Golber, <i>The cohomological description of a torus action</i>	149
Alain Goulet de Rugy, <i>Un théorème du genre "Andô-Edwards" pour les Fréchet ordonnés normaux</i>	155
Louise Hay, <i>The class of recursively enumerable subsets of a recursively enumerable set</i>	167
John Paul Helm, Albert Ronald da Silva Meyer and Paul Ruel Young, <i>On orders of translations and enumerations</i>	185
Julien O. Hennefeld, <i>A decomposition for $B(X)^*$ and unique Hahn-Banach extensions</i>	197
Gordon G. Johnson, <i>Moment sequences in Hilbert space</i>	201
Thomas Rollin Kramer, <i>A note on countably subparacompact spaces</i>	209
Yves A. Lequain, <i>Differential simplicity and extensions of a derivation</i>	215
Peter Lorimer, <i>A property of the groups $\text{Aut PU}(3, q^2)$</i>	225
Yasou Matsugu, <i>The Levi problem for a product manifold</i>	231
John M.F. O'Connell, <i>Real parts of uniform algebras</i>	235
William Lindall Paschke, <i>A factorable Banach algebra without bounded approximate unit</i>	249
Ronald Joel Rudman, <i>On the fundamental unit of a purely cubic field</i>	253
Tsuan Wu Ting, <i>Torsional rigidities in the elastic-plastic torsion of simply connected cylindrical bars</i>	257
Philip C. Tonne, <i>Matrix representations for linear transformations on analytic sequences</i>	269
Jung-Hsien Tsai, <i>On E-compact spaces and generalizations of perfect mappings</i>	275
Alfons Van Daele, <i>The upper envelope of invariant functionals majorized by an invariant weight</i>	283
Giulio Varsi, <i>The multidimensional content of the frustum of the simplex</i>	303