A GENERALIZATION OF KNESER’S CONJECTURE

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Let $M$ be a closed connected 3-manifold such that $\pi_2(M) = 0$. Suppose that $\pi_1(M)$ is a nontrivial free product with amalgamation across the group of a closed connected surface $S$ other than the projective plane or 2-sphere. Then it is shown that there is an embedded surface $S$ in $M$ "realizing" the group structure above.

Our theorem also considers the case when $M$ has boundary and gives an answer to a problem of Neuwirth.

1. Introduction. In 1929 H. Kneser stated the following result in [8].

**Theorem K.** Let $M$ be a closed, connected 3-manifold. Suppose $\pi_1(M)$ is the nontrivial free product of two groups $A_1$ and $A_2$. Then there exists an embedding of the 2-sphere in $M$ which separate $M$ into 3-submanifolds $M_1$ and $M_2$ such that $\pi_1(M_j) = A_j$ for $j = 1, 2$.

Theorem K was confirmed by J. Stallings in his thesis. One would like to generalize Theorem K so that one could realize geometrically more complicated algebraic splittings of $\pi_1(M)$. In [3] we made a generalization of this form which required that $M$ be orientable and closed and that the splitting surface be a closed orientable surface not the 2-sphere.

In the “Splitting Theorem” below we eliminate most of these restraints.

2. Preliminaries. All spaces discussed in this paper are simplicial complexes and all maps are piecewise linear. As usual we will write $G = A \ast^c B$ when the group $G$ is the free product of $A$ and $B$ with amalgamation over $C$. We shall restrict our attention to the case when $A$ and $B$ are proper subgroups of $G$ or equivalently $C$ is a proper subgroup of $A$ and $B$. As usual, we shall denote the boundary, closure, and interior of a subspace $X$ of a space $Y$ by $\text{bd}(X)$, $\text{cl}(X)$, and $\text{int}(X)$ respectively.

Let $X$ be a connected subspace of the space $Y$. Then we shall denote the natural inclusion map from $X$ into $Y$ by $\rho$ and the induced homomorphism from $\pi_i(X)$ into $\pi_i(Y)$ by $\rho_*$. Let a closed connected surface $S$ not the 2-sphere or projective plane be embedded in a space $X$. If $\rho_*: \pi_1(S) \to \pi_1(X)$ is one-to-one, we shall say that $S$ is

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1 Papakyriakopoulos gives an interesting discussion of this conjecture in [9].
incompressible in $X$. Similarly if $S$ is a system of closed surfaces embedded in a space $X$, we shall say that $S$ is incompressible in $X$ if each component of $S$ is incompressible in $X$. We will denote $[0,1]$ by $I$ throughout this paper.

Let $S$ be a closed, connected surface not the 2-sphere or projective plane. Let $M$ be a connected 3-manifold and

$$\pi_1(M) = A_1 \ast_{\pi_1(S)} A_2.$$  

Then we shall say that $\pi_1(M)$ splits across $\pi_1(S)$. We shall say that the splitting of $\pi_1(M)$ above respects the peripheral structure of $M$ if for each component $F$ of the boundary of $M$, a conjugate of $\rho_*\pi_1(F)$ is contained either in $A_1$ or in $A_2$.

Let $S$ be a closed, connected, incompressible surface embedded in $M$. Suppose that $S$ separates $M$ into two 3-submanifolds $M_1$ and $M_2$. Let $\pi_1(S) \cong A_0$ and suppose $\pi_1(M) = A_1 \ast A_2$. Consider the group diagrams given in Figures 1 and 2. The group diagram in Figure 1 is obtained by applying Van Kampen’s theorem to the decomposition of $M$ into $M_1$ and $M_2$. The group diagram in Figure 2 is obtained from the splitting of $\pi_1(M)$ by $A_0$.

![Figure 1](image1)

Then we shall say that the surface $S$ geometrically realizes the algebraic splitting above if there is an isomorphism

$$\varphi : \pi_1(M) \longrightarrow \pi_1(M)$$

such that

1. $\varphi(\pi_1(S)) = A_0$
2. $\varphi(\pi_1(M_j)) = A_j$ for $j = 1, 2$
3. $h_k = \varphi i_k \varphi^{-1}$ for $k = 1, 2, 3, 4$.

3. The splitting theorem.

**Theorem 1.** Let $M$ be a compact, connected 3-manifold such that $\pi_2(M) = 0$. Let $S$ be a closed, connected surface not the 2-sphere or projective plane. Suppose
π₁(M) = A₁ ∗ A₂

and that this splitting preserves the peripheral structure of M. Then there is a geometric splitting realizing the algebraic splitting above.

The proof of Theorem 1 in this paper is similar to the proof of Theorem 1 in [3]. We shall need three lemmas in the proof of Theorem 1 and we shall consider these at this point.

Lemma 1 is the result of a number of well known techniques and is similar in content to Lemma 1.1 in [12]. We shall omit most of the details of the proof.

**Lemma 1.** Let $M$ be a compact, connected 3-manifold such that $\pi_5(M) = 0$. Let $X$ be a connected complex and $S$ a closed incompressible surface embedded in $X$ and having a neighborhood homeomorphic to $S \times I$. We suppose that no component of $S$ is a 2-sphere or projective plane. Let $X_k$, $k = 1, \cdots, n$ be the components of $X - S$. We suppose that $\pi_i(X) = \pi_i(X_k) = 0$ for $i \geq 2$ and $k = 1, \cdots, n$. Let $f: M \rightarrow X$ be a map such that $f_\ast: \pi_i(M) \rightarrow \pi_i(X)$ is one-to-one and $f \text{ bd}(M)$ does not meet $S$. Then there is a homotopy, constant on $\text{bd}(M)$, of $f$ to a map $g$ such that $g^{-1}(S)$ is an incompressible surface in $M$.

**Proof.** One first uses the simplicial approximation theorem to find a map $g_1$ homotopic to $f$ such that $g_1^{-1}(S)$ is a surface in $M$. Next one uses techniques developed by J. Stallings in his thesis to find a map $g$ homotopic to $g_1$ such that $g^{-1}(S)$ is an incompressible surface in $M$. We note that the homotopies used could be held constant on $\text{bd}(M)$ since $f(\text{bd}(M)) \cap S$ was empty. The lemma follows.

A 3-manifold $M$ will be called $p^2$-irreducible if there are no embedded projective planes in $M$ and every 2-sphere in $M$ bounds a 3-ball embedded in $M$.

**Lemma 2.** Let $S_1$ and $S_2$ be disjoint, incompressible, connected surfaces which are embedded in a $P^2$-irreducible 3-manifold $M$. Then if $S_1$ is homotopic to $S_2$ in $M$, $S_1 \cup S_2$ bounds an $S_1 \times I$ embedded in $M$.

**Proof.** It is a consequence of 1.1.5 in [13] that $\pi_j(M) = 0$ for $j \geq 2$ and that the higher homotopy groups of each component of $M - (S_1 \cup S_2)$ are trivial. Let $H: S \times I \rightarrow M$ be a homotopy of $S_1$ to $S_2$. It is a consequence of Lemma 1 that we may assume $H^{-1}(S_1 \cup S_2)$ is incompressible in $S \times I$. Let $S^*$ be a component of $H^{-1}(S_1 \cup S_2)$. Then we claim $S^*$ separates $S \times \{0\}$ from $S \times \{1\}$.
Assume that \( S^* \) does not separate \( S \times I \). Then \([S^*]\) is not homologous to \([S \times \{0\}]\) in \( C_2(S \times I; \mathbb{Z}) \). Since \( H_2(S \times I; \mathbb{Z}) = \mathbb{Z}_2 \), \([S^*]\) bounds a 3-chain in \( C_3(S \times I; \mathbb{Z}) \). Thus \( S^* \) bounds a 3-submanifold \( N \subset S \times I \). Let \( N_1 = \text{cl} (S \times I - N) \). Now by Van Kampen's theorem we have the commutative diagram shown in Figure 3. All homomorphisms in Figure 3 are induced by inclusion.

Since \( N_1 \supset S \times \{0\} \), \( \rho_4 \) is onto. Since \( \rho_1 \) and \( \rho_2 \) are one-to-one it is a consequence of 2.5 in [1] that \( \rho_1 \) is onto. It is a consequence of the corollary to Theorem A in [5] that \( N \) is a product line bundle.

Thus \( S^* \) separates \( S \times I \). By an argument similar to the one above we can show that the closure of either component of \( S \times I - S^* \) is a product line bundle.

It is now easy to show that \( H^-(S_1 \cup S_2) \) can be assumed to be \( \text{bd} (S \times I) \); one simply considers the restriction of \( H \) to a submanifold of \( S \times I \). Lemma 2 is now a consequence of the corollary to Theorem A in [5].

A result similar to Lemma 3 was suggested to the author by an unknown referee. This suggestion enabled us to greatly simplify the proof to Theorem 1. Lemma 3 is proved using standard arguments in obstruction theory and could be stated in terms of cell complexes and relative homotopy groups. However, it will be immediately obvious in the proof of Theorem 1 that the hypotheses of our Lemma 3 are met.

**Lemma 3.** Let \( M_i \) be a compact, connected, 3-manifold, \( X \) a connected complex, and \( F' \) and \( S \) incompressible connected surfaces in \( M_i \) and \( X \) respectively. We suppose that \( S \) is neither a 2-sphere or protective plane and \( \pi_i(X) = 0 \) for \( i \geq 2 \).

Let \( f: (M_i, F') \rightarrow (X, S) \) be a map of pairs such that for some \( x \in F' \)

\[
 f_* \pi_i(M_i, x) \subset \pi_i(S, f(x))
\]

Then \( f \) is homotopic under a deformation constant on \( F' \) to a map
Proof. We wish to define a map \( H: (N \times I, F \times I) \to (X, S) \) such that

1. \( H(n, 0) = f(n) \) for \( n \in N \)
2. \( H(n, t) = f(n) \) for \( n \in F, t \in I \)
3. \( H(N \times \{1\}) \subset S \).

Of course such an \( H \) will be the desired homotopy.

Let \( N^i \) be the \( i \)-skeleton of some subdivision of the pair \((N, F)\) for \( i = 1, 2, 3 \). We define \( H \) to satisfy (1) and (2) above on \( F \times I \cup N \times \{0\} \).

If \( \alpha \) is any arc embedded in \( N^1 \) which meets \( F \) in its endpoints, the arc \( f(\alpha) \) can be deformed modulo its boundary to lie in \( S \). Thus \( H \) can be extended to \( \alpha \times I \). It follows that \( H \) can be extended to \( N^1 \times I \). If \( D \) is a disk embedded in \( N^2 \) and meeting \( N^1 \) in \( \text{bd}(D) \), we have defined \( H \) on \( \text{bd}(D) \times I \cup D \times \{0\} \). Since \( H(\text{bd}(D) \times \{1\}) \) is nullhomotopic in \( X \), it is nullhomotopic in \( S \) since \( S \) is incompressible. It follows that if \( D \) is not contained in \( F \) we may extend \( H \) to \( D \times \{1\} \). Since \( \pi_d(X) = 0 \), \( H \) can be extended to \( D \times I \). It follows that \( H \) can be extended to \( N^2 \times I \). Similarly we can extend \( H \) to \( N^3 \times I \) since \( \pi_d(S) = 0 \) and \( \pi_d(X) = 0 \). This completes the proof of the lemma.

Proof of Theorem 1. It follows from generalization 1 in [9] that we can replace finitely many prime homotopy 3-cells in \( M \) and obtain an irreducible 3-manifold. Since an incompressible surface can be made to miss a finite collection of disjoint 2-spheres, we may assume that \( M \) is irreducible. It is a consequence of Theorem 1 in [2] that \( M \) does not admit any embeddings of the projective plane since \( \pi_3(M) = 0 \) and \( \pi_1(M) \) is not finite.

Let \( (M_{a_j}, q_j) \) be the covering space of \( M \) associated with \( A_j \subset \pi_1(M, x) \) for \( j = 1, 2 \). Let \( g: (S, y) \to (M, x) \) be a map such that

\[
g_*\pi_1(S, y) = A_1 \cap A_2.
\]

Let \( g_j: S \to M_{a_j} \) be a map covering \( g \), i.e., \( q_j g_j = g: S \to M \). Let \( X_j \) be the mapping cylinder of \( g_j \) over \( M_{a_j} \), i.e., \( X_j \) is the union of \( M_{a_j} \) with \( S \times I \) with identification \( g_j(n) = (n, 0) \) for \( n \) in \( S \) and \( j = 1, 2 \). Let \( X = X_1 \cup X_2 \) identifying \( (n, 1) \) in \( X_1 \) with \( (n, 1) \) in \( X_2 \) for \( n \) in \( S \). As was shown in [2] \( \pi_i(X) = \pi_i(X_j) = 0 \) for \( j = 1, 2 \) and \( i \geq 2 \). Also \( \pi_1(X) \approx \pi_1(M) \).

Let \( G: X \to M \) be defined by

1. \( G|_{M_{a_j}} = q_j \) for \( j = 1, 2 \)
(2) \( G(n, t) = g(n) \) for \((n, t) \in S \times I \). Then \( G_*: \pi_1(X) \to \pi_1(M) \) is an isomorphism. We denote \( S \times \{1\} \subset X \) by \( S \).

We wish to construct a map \( \hat{G}: M \to X \) such that

(1) \( \hat{G}_*: \pi_1(M) \to \pi_1(X) \) is \( G_*^{-1} \)

(2) \( \hat{G}(\text{bd}(M)) \) does not meet \( S \subset X \).

Let \( \text{bd}(M) = \bigcup_{k=1}^m F_k \) where \( F_k \) is a closed connected surface. By assumption \( \rho_k: \pi_1(F_k) \to \pi_1(M) \) is conjugate to a subgroup either of \( A_1 \) or \( A_2 \). We assume \( A_1 \). It follows that there is a map \( \hat{\rho}_k: F_k \to M_{A_1} \) covering \( \rho_k \). We define \( \hat{G}(F_k) = \hat{\rho}_k \rho_k^{-1} \).

Let \( \{\alpha_k: k = 1, \ldots, n\} \) be a collection of simple arcs in \( X \) such that

(1) \( G(\alpha(\alpha_k)) \cap G(\alpha_{k'}) = \emptyset \) for \( k \neq k' \)

(2) \( \alpha_k \) runs from \( y \) in \( S \) to a point in \( \hat{\rho}(F_k) \)

(3) \( G(\alpha_k) \) is a simple arc.

We extend \( \hat{G} \) to \( G(\alpha_k) \) to be any homeomorphism onto \( \alpha_k \). Note that \( \hat{G} \) has been defined on \( Y = \text{bd}(M) \cup \bigcup_{k=1}^m G(\alpha_k) \) such that for each loop \( l \subset Y \) based at \( x \) \( (G\hat{G})_*[l] = [l] \) in \( \pi_1(M, x) \). Thus we can extend \( G \) by using well known techniques so that \( \hat{G}_* = G_*^{-1} \) since \( G_*^{-1} \) is an isomorphism and \( \pi_i(X) = 0 \) for \( i \geq 2 \).

We have now established the existence of the desired map \( \hat{G} \). It is a consequence of Lemma 1 that we may assume \( \hat{G}_*^{-1}(S) \) is an incompressible surface in \( M \). Denote the components of \( \hat{G}_*^{-1}(S) \) by \( S_i \) for \( i = 1, \ldots, m \). Since \( S_i \) and \( S \) are incompressible in \( M \) and \( X \) respectively,

\[
(\hat{G} | S_i)_*: \pi_i(S_i) \longrightarrow \pi_i(S) \subset \pi_i(X)
\]

is one-to-one. It is a consequence of Theorem 1 in [5] that \( \hat{G} | S_i \) is homotopic to a covering map. Thus one can assume that \( \hat{G} | S_i \) is a local homeomorphism. (One can change \( \hat{G} \) on a small neighborhood of \( S_i \) to achieve this result.)

We may choose any point \( z \) in \( M \) and have that

\[
\hat{G}_*: \pi_1(M, z) \longrightarrow \pi_1(X, \hat{G}(z))
\]

is an isomorphism. Suppose \( \hat{G} | S_i \) is not a homeomorphism and \( z \) in \( S_i \). Let \( \Phi \) be the isomorphism of \( \pi_1(M, z) \) onto \( \pi_1(X, \hat{G}(z)) \) induced by \( \hat{G} \). Then \( \Phi(\rho_*\pi_1(S_i, z)) \subseteq \rho_*\pi_1(S, \hat{G}(z)) \). Since \( \Phi^{-1}(\rho_*\pi_1(S, \hat{G}(z)) \) is a subgroup of \( \pi_1(M, z) \) properly containing \( \rho_*\pi_1(S_i, z) \) we would have by Theorem 6 in [7] that \( S_i \) bounds a twisted line bundle \( N \subset M \). One can easily show using the techniques in [7], as has been done in [4], that \( p_*\pi_1(N, z) \) may be taken to be \( \Phi^{-1}(\rho_*\pi_1(S, \hat{G}(z)) \). It is now a consequence of Lemma 3 that we may assume that \( \hat{G} | S_i \) is a homeomorphism for \( i = 1, \ldots, m \).
As was shown in the proof of Theorem 1 in [3] every pair of components \( S_1 \) and \( S_2 \) in \( \hat{G}^{-1}(S) \) are homotopic. Thus by Lemma 2, \( S_1 \cup S_2 \) bounds an \( S \times I \) embedded in \( M \). As was shown in the proof of Theorem 1 in [3] we may assume that there is a single component \( S_i \) in \( \hat{G}^{-1}(S) \) and \( S_i \) is homeomorphic to \( S \). The desired algebraic result now follows as in the proof of Theorem 1 in [3] and Theorem 1 is now established.

4. An application. In [10] Neuwirth asks, "Every knot group contains the group \((a, b; [a, b])\). This subgroup may be obtained from the natural inclusion of the fundamental group of a nonsingular torus in the knot group. Suppose a knot contains the group of a closed surface of genus \( g \). Does there exist a nonsingular closed surface of genus \( g \) whose fundamental group is injected monomorphically into the knot group by the natural inclusion?".

W. Heil has shown in [6] that, if the subgroup in question is normal, such a surface does not exist.

**THEOREM 2.** A knot complement admits an incompressible embedding of a closed surface of genus \( g > 1 \) if and only if its fundamental group splits across the group of the surface in question and said splitting preserves the peripheral structure of the fundamental group of the knot complement.

**Proof.** Since any closed surface embedded in \( S^3 \) separates \( S^3 \), one half of the theorem follows from Van Kampen's theorem. The other half follows from Theorem 1.

**REFERENCES**

4. ----------, *S-maximal subgroups of \( \pi_1(M^3) \)*, Canad. J. Math., XXIV (1972), 439-449.

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