

# Pacific Journal of Mathematics

**THE COHOMOLOGICAL DESCRIPTION OF A TORUS ACTION**

DAVID GOLBER

## THE COHOMOLOGICAL DESCRIPTION OF A TORUS ACTION

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The theorem proved in this paper is an example of a “regularity” theorem in the study of topological group actions—that is, it shows that a general topological action of a group continues to have certain properties of “linear” actions. Consider an action of a torus  $T$  on a cohomology  $n$ -sphere  $X$ , with fixed point set the cohomology  $r$ -sphere  $F$ . Consider the map  $H^n(X_T; Z) \rightarrow H^n(F_T; Z)$ , and let  $c\eta$  be the image of the generator of  $H^n(X; Z)$ , considered as lying in  $H^{n-r}(BT; Z)$ , where  $c$  is an integer and  $\eta$  has no nontrivial integer divisors. The polynomial part  $\eta$  is well understood. The theorem will evaluate the integer part  $c$  in the following sense: in the linear case,  $c$  can be easily expressed in terms of the dimensions of the fixed point sets of various non-connected subgroups of  $T$ . It is shown that this formula continues to hold in the general topological case, given some weak assumptions. There is also a corresponding result for the case  $F = \emptyset$ .

The main tool will be the fibration  $\pi: X_T \rightarrow B_T \equiv BT$ , where  $X_T$  is as usual  $E_T \times_T X$ . We will use the usual limit arguments to allow ourselves to pretend that  $E_T$  is compact. Cohomology will be sheaf cohomology with compact supports (which will not usually be indicated). The spectral sequence of  $X_T \rightarrow BT$  with coefficients in  $A$  will be denoted  $E_r(X_T; A)$ . The fixed point set of  $T$  acting on  $X$  will be denoted  $F(T, X) \equiv F(T)$ .  $X \sim_Z Y$  ( $X \sim_p Y$ ) will mean that  $X$  is a compact  $Z$ -cohomology ( $Z_p$ -cohomology) manifold with  $Z(Z_p)$  cohomology ring the same as that of  $Y$ .  $\dim_p(X)$  or  $\dim_Z(X)$  will be the usual cohomological dimension of  $X$  over  $Z_p$  or  $Z$ . See [1] or [2] for details. For an abelian group  $A$ , let  $\mathcal{S}A$  be  $A/\text{Torsion}(A)$ .

If a torus  $T$  acts on a space  $X$ , a subtorus  $H$  of  $T$  is said to be *distinguished* if  $F(H) \cong F(K)$  for any subtorus  $K$  which has  $K \cong H$ . In particular, the distinguished corank one subtori of  $T$  are those subtori  $H$  of corank one in  $T$  that have  $F(H) \cong F(T)$ . Recall that given a corank one subtorus of  $T$ , there is a corresponding integer-valued linear functional on the Lie algebra of  $T$ , a corresponding element of  $H^1(T; Z)$  and a corresponding element (not divisible by any integer) in  $H^2(BT; Z)$ .

Now consider a torus  $T$  acting on  $X \sim_Z S^n$ . Let  $F(T) \sim_Z S^r$ , and look at  $F_T \subseteq X_T$ . Consider the cases  $r > 0$ ,  $r = 0$ , and  $r = -1$  ( $F(T) = \emptyset$ ) separately.

In case  $r > 0$ , the map

$$H^n(X; Z) \cong E_\infty^{0,n}(X_T; Z) \longrightarrow E_\infty^{n-r,r}(F_T; Z) \cong H^{n-r}(BT; Z)$$

takes the generator of  $H^n(X; Z)$  to  $c\eta$  where  $c$  is an integer, and  $\eta$  is  $\prod g_i^{(n_i-r)/2}$ . (Here the  $g_i$ 's correspond to the distinguished corank one subtori  $U_i$  of  $T$ , and  $n_i = \dim_Z F(U_i)$ .)

In case  $r = 0$ ,  $F(T) \sim {}_Z S^0$ , we have  $\pi: F_T \cong F \times B_T \rightarrow BT$ , and the inclusion  $F_T \rightarrow X_T$  induces

$$H^n(X; Z) \longrightarrow \tilde{H}^0(F; Z) \otimes H^{n-r}(BT; Z)$$

which takes  $g$  to (generator)  $\otimes c\eta$  as before.

In case  $r = -1$ ,  $F(T) = \emptyset$ , the transgression

$$H^n(X; Z) \longrightarrow H^{n+1}(BT; Z)$$

takes the generator to  $c\eta$ .

The theorem below will identify the integer  $c$ .

Let  $p$  be any prime. (Several of the objects below will depend on  $p$ , although this dependence will not be explicitly indicated.) (The letter  $p$  will also be used as one of indices of a spectral sequence, but hopefully no confusion will result.) For  $i = 1, 2, \dots$  let  $S(i)$  be the subgroup of elements  $t$  of  $T$  such that  $p^i t = 1$ , the identity element of  $T$ . Let  $S(0)$  be the subgroup of  $T$  consisting of 1 only. Clearly  $S(i) \cong (Z_p)^k$ , where  $k$  is the rank of  $T$ . Each  $F(S(i))$  is a  $Z_p$ -cohomology  $n_i$ -sphere for some  $n_i$ .

**THEOREM.** *Suppose that for any prime  $p$  and  $i = 1, 2, \dots$  that  $F(S(i))$  has finitely generated  $Z$ -cohomology. Let  $p^a$  be the largest power of  $p$  that divides  $c$ . Then*

$$\sum_{i=1}^{\infty} [\dim_p F(S(i)) - \dim_Z F] = 2a .$$

*Further,  $F(S(i)) = F$  for  $i > a$ .*

*Proof.* The second claim follows from the first and the fact that  $\dim_p F(S(i)) - \dim_Z F$  is always even. (See [1], Chapter IV.)

We will first do the case  $r > 0$  and then reduce the other two cases to this case.

Consider the spectral sequence of  $F(S(i))_T \rightarrow BT$ . Because  $F(S(i)) \sim {}_p S^{n_i}$  and has finitely generated integral cohomology, it is easy to see from the universal coefficient theorem that  $H^*(F(S(i)); Z)$  has no  $p$ -torsion, that  $H^0(F(S(i)); Z) = Z$ , and that  $\mathcal{S}H^*(F(S(i)); Z) = H^*(S^{n_i}; Z)$ . Because  $r > 0$ , the  $Z_p$  spectral sequence of  $F(S(i))_T \rightarrow B_T$

collapses. It is then easy to verify the following facts about  $E_\infty(F(S(i))_T; Z)$ :

- (i)  $E_\infty(F(S(i))_T; Z)$  has no  $p$ -torsion.
- (ii)  $\mathcal{S} E_\infty^{2,q}(F(S(i))_T; Z) \cong H^p(BT; Z)$  if  $q = 0$  or  $n_i$ , and  $= 0$  otherwise. (As abelian groups with no reference to the multiplicative structure).
- (iii) The bottom row  $E_\infty^{*,0}(F(S(i))_T; Z)$  is isomorphic to  $H^*(BT; Z)$ , as a ring.
- (iv) Let  $h$  be a generator of  $\mathcal{S} E_\infty^{0,n_i}(F(S(i))_T; Z)$ . Multiplication by  $h$  defines a map

$$H^p(BT; Z) \cong E_\infty^{2,0}(F(S(i))_T; Z) \longrightarrow \mathcal{S} E_\infty^{2,n_i}(F(S(i))_T; Z) \cong H^p(BT; Z)$$

This map is monomorphism, and its cokernel is finite with non- $p$  order.

We know  $F(S(i)) \sim_p S^{n_i}$  for  $i = 0, 1, 2, \dots$  where  $n_0 = n$ , and  $F(S(\ell + 1)) = F$  for  $\ell$  large enough, so  $n_{\ell+1} = r$ . We have

$$X = F(S(0)) \supseteq F(S(1)) \supseteq \dots \supseteq F(S(\ell)) \supseteq F(S(\ell + 1)) = F.$$

We can consider the inclusion map  $F_T \rightarrow X_T$  to be the composition

$$F(S(\ell + 1))_T \longrightarrow F(S(\ell))_T \longrightarrow \dots \longrightarrow F(S(1))_T \longrightarrow F(S(0))_T.$$

Let  $h_i$  be a generator of  $\mathcal{S} E_\infty^{0,n_i}(F(S(i))_T; Z)$ . We will show below that the induced map  $\varphi: \mathcal{S} E_\infty(F(S(i))_T; Z) \rightarrow \mathcal{S} E_\infty(F(S(i + 1))_T; Z)$  has  $\varphi(h_i)$  divisible by  $p^{i(n_i - n_{i+1})/2}$  and by no higher power of  $p$ . Using this and the facts (i)-(iv), we can show that  $2a = \sum_{i=0}^{\ell} i(n_i - n_{i+1})$ , which equals  $\sum_{i=1}^{\ell} (n_i - n_{\ell+1})$ , which is our conclusion. Thus we only have to prove our claim about the number of factors of  $p$  dividing  $\varphi(h_i)$ .

Consider the following diagram:

$$\begin{array}{ccc}
 \mathcal{S} E_\infty(F(S(i))_T; Z) & \xrightarrow{\varphi} & \mathcal{S} E_\infty(F(S(i + 1))_T; Z) \\
 \uparrow \alpha & & \uparrow \beta \\
 \mathcal{S} E_\infty(F(S(i))_{T/S(i)}; Z) & \xrightarrow{\psi} & \mathcal{S} E_\infty(F(S(i + 1))_{T/S(i)}; Z) \\
 \downarrow \gamma & & \downarrow \delta \\
 E_\infty(F(S(i))_{T/S(i)}; Z_p) & \longrightarrow & E_\infty(F(S(i + 1))_{T/S(i)}; Z_p) \\
 \downarrow \epsilon & & \downarrow \zeta \\
 E_\infty(F(S(i))_{S(i+1)/S(i)}; Z_p) & \xrightarrow{\tau} & E_\infty(F(S(i + 1))_{S(i+1)/S(i)}; Z_p)
 \end{array}$$

Let  $k_i$  be the generator of  $\mathcal{S} E_\infty^{0,n_i}(F(S(i))_{T/S(i)}; Z)$ . It is easy to see that  $\alpha(k_i)$  is a non- $p$  multiple of  $h_i$ . Now the map  $\beta$  on the

$E^{2,0}$  terms is  $Z^k \cong H^2(BT/S(i); Z) \rightarrow H^2(BT; Z) \cong Z^k$ , which is multiplication by  $p^i$ . Since  $\psi(k_i)$  lies in filtration degree  $n_i - n_{i+1}$ , we can see using (iv) above that  $\beta(\psi(k_i))$  contains precisely  $i(n_i - n_{i+1})/2$  more factors of  $p$  than  $\psi(k_i)$  does. Thus it is sufficient to show that  $\psi(k_i)$  is not divisible by  $p$ .

The map  $\delta$  is reduction mod  $p$ , so it suffices to show that  $\delta\psi(k_i) \neq 0$ . Therefore it suffices to show that  $\tau\varepsilon\gamma(k_i) \neq 0$ . But  $S(i+1)/S(i) \cong (Z_p)^k$  acts on  $F(S(i)) \sim_p S^{n_i}$ , with fixed point set  $F(S(i+1)) \sim_p S^{n_{i+1}}$ , and  $\varepsilon\gamma(k_i)$  is the generator of  $E_{\infty}^{0, n_i}(F(S(i))_{S(i+1)/S(i)}; Z_p)$ . In these circumstances, we must have  $\tau\varepsilon\gamma(k_i)$  nonzero (see [1], Chapter XIII, and [3]) which finishes the proof in the case  $r > 0$ .

The cases  $r = 0$  and  $r = -1$  are handled by replacing  $X$  by  $SX$  and  $S^2X$  respectively, where  $SX$  denotes the nonreduced suspension. The action of  $T$  on  $SX$  (or  $S^2X$ ) then has a nonempty connected fixed point set, so the problem is reduced to the previous case. It is not hard to see that if  $X \sim_z S^n$ , then  $SX \sim_z S^{n+1}$ . Thus we only need to show: (1) Suppose  $T$  acts on  $X \sim_z S^n$  with  $F(T) = \emptyset$ . Consider the actions of  $T$  on  $X$  ( $F(T, X) = \emptyset, r = -1$ ) and on  $SX$  ( $F(T, SX) \sim_z S^0, r = 0$ ). One gets an integer  $c$  from each action. We need to show that the two  $c$ 's are the same, at least up to sign. And (2) Suppose  $T$  acts on  $X \sim_z S^n$  with  $F(T) \sim_z S^0$ . Consider the actions of  $T$  on  $X$  ( $F(T, X) \sim_z S^0, r = 0$ ) and on  $SX$  ( $F(T; SX) \sim_z S^1, r = 1$ ). Again, one gets two  $c$ 's which we need to prove are the same up to sign.

The second case, going from  $r = 0$  to  $r = 1$ , is easy; one merely uses the naturality of the suspension map.

In the first case, going from  $r = -1$  to  $r = 0$ , we have an action of  $T$  on  $X \sim_z S^n$  with  $F = \emptyset$ , so  $n$  is odd. In the spectral sequence of  $p: X_T \rightarrow B_T$ , the generator of  $H^n(X)$  transgresses to  $c\eta \in H^{n+1}(BT)$ ,  $c\eta \neq 0$ . On the other hand, the spectral sequence of  $q: (SX)_T \rightarrow BT$  collapses. Then it is easy to check that  $H^n(X_T) = 0$ ,  $H^{n+1}(X_T) = H^{n+1}(BT)/\langle c\eta \rangle$ ,  $H^{n+2}((SX)_T) = 0$ , and there is a split short exact sequence

$$0 \longrightarrow H^{n+1}(BT) \xrightarrow{q^*} H^{n+1}((SX)_T) \xrightarrow{j^*} H^{n+1}(SX) \longrightarrow 0,$$

where  $j: SX \rightarrow (SX)_T$  is the inclusion of the fiber. One can consider  $(SX)_T$  to be the space  $(X_T \times I)/\sim$ , where  $(a, 1) \sim (b, 1)$  and  $(a, 0) \sim (b, 0)$  iff  $p(a) = p(b)$ . The map  $q: (SX)_T \rightarrow BT$  is given by  $[a, t] \rightarrow p(a)$ . The sets  $E_0 = \{[a, 0] \in (SX)_T\}$  and  $E_1 = \{[a, 1] \in (SX)_T\}$  are each mapped homeomorphically onto  $BT$  by  $q$ . Let  $i_0$  and  $i_1: BT \rightarrow (SX)_T$  be the corresponding two sections of  $q$ . Showing that the action on  $SX$  gives rise to the same  $c$  as that on  $X$  reduces to showing that in

$$H^{n+1}(SX) \xleftarrow{j^*} H^{n+1}((SX)_T) \xrightarrow{i_0^* - i_1^*} H^{n+1}(BT),$$

$(i_0^* - i_1^*) \circ (j^*)^{-1}$  takes the generator of  $H^{n+1}(SX)$  to  $\pm c\eta$ .

Let  $U = \{[a, t] \in (SX)_T \mid t > 1/4\}$ ; let  $V = \{[a, t] \in (SX)_T \mid t < 3/4\}$ .  $U$  and  $V$  have  $E_0$  and  $E_1$  as strong deformation retracts,  $U \cup V = (SX)_T$ , and  $U \cap V$  has  $\{[a, 1/2] \in (SX)_T\} \cong X_T$  as a strong deformation retract. Consider the Meyer-Vietoris sequence of pair  $(U, V)$ :

$$\begin{array}{ccccccccc} H^n(U \cap V) & \longrightarrow & H^{n+1}(U \cup V) & \longrightarrow & H^{n+1}(U) \oplus H^{n+1}(V) & \longrightarrow & H^{n+1}(U \cap V) & \longrightarrow & H^{n+2}(U \cup V) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ H^n(X_T) & & H^{n+1}((SX)_T) & & H^{n+1}(BT) \oplus H^{n+1}(BT) & & H^{n+1}(X_T) & & H^{n+2}((SX)_T) \\ \parallel & & & & & & \parallel & & \parallel \\ 0 & & & & & & \frac{H^{n+1}(BT)}{\langle c\eta \rangle} & & 0 \end{array}$$

so we have the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{n+1}(BT) & \xlongequal{\quad} & H^{n+1}(BT) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow q^* & & \downarrow \Delta & & \downarrow & & \\ 0 & \longrightarrow & H^{n+1}((SX)_T) & \longrightarrow & H^{n+1}(BT) \oplus H^{n+1}(BT) & \longrightarrow & \frac{H^{n+1}(BT)}{\langle c\eta \rangle} & \longrightarrow & 0 \\ & & \downarrow j^* & \searrow i_0^* - i_1^* & \downarrow D & & \parallel & & \\ 0 & \longrightarrow & H^{n+1}(SX) & \longrightarrow & H^{n+1}(BT) & \longrightarrow & \frac{H^{n+1}(BT)}{\langle c\eta \rangle} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where  $\Delta$  is the diagonal map and  $D$  is given by  $D(a, b) = a - b$ . The top two rows and all three columns are exact, so that bottom row is also exact. The horizontal map on the bottom left is  $(i_0^* - i_1^*) \circ (j^*)^{-1}$ , so we can see that this map takes the generator of  $H^{n+1}(SX)$  to  $\pm c\eta \in H^{n+1}(BT)$ , as was to be shown.

This finishes the proof of the theorem.

REFERENCES

1. A. Borel, et al., *Seminar on Transformation Groups*, Ann. of Math. Studies No. 46, Princeton University Press, 1960.
2. G. Bredon, *Sheaf Theory*, McGraw-Hill, New York, 1967.

3. D. Golber, *Torus actions on a product of two odd spheres*, *Topology*, **10** (1971), 313-316.
4. Wu-Yi Hsiang, *On Generalizations of a Theorem of A. Borel and Their Applications in the Study of Topological Actions*, *Topology of Manifolds*, Cantrell and Edwards, editors, Markham Publishing Co., Chicago, 1970, pp. 276-290.

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