Pacific Journal of Mathematics

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LOUISE HAY

Vol. 46, No. 1

November 1973

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For any set α , let θA^{α} denote the index set of the class of all recursively enumerable (r.e.) subsets of α (i.e., if $\{W_x\}_{x\geq 0}$ is a standard enumeration of all r.e. sets, $\theta A^{\alpha} =$ $\{x \mid W_x \subset \alpha\}$.) The purpose of this paper is to examine the possible Turing degrees of the sets θA^{α} when α is r.e. It is proved that if *b* is any nonrecursive r.e. degree, the Turing degrees of sets θA^{α} for α r.e., $\alpha \in b$, are exactly the degrees c > 0' such that *c* is r.e. in *b*.

Index sets of form θA^{α} appear to have useful properties in the study of the partial ordering of all index sets under one-to-one reducibility. For instance, in the case where α is a nonrecursive incomplete r.e. set, the index set $\overline{\theta A^{\alpha}}$ was used in [1] to provide an example of an index set which is neither r.e. nor productive. In [2] it is shown that if the Turing degree of α is not $\geq 0'$, then the set θA^{α} is at the bottom of c discrete ω -sequences of index sets (i.e., linearly ordered chains of index sets such that no index sets are intermediate between the elements of the chain.) In particular, such a set θA^{α} has at least two nonisomorphic immediate successors in the partial ordering of index sets.

It is natural to ask: What relation, if any, exists between the Turing degree of α and that of ∂A^{α} ? In the case where α is co-r.e., it is easy to see that neither degree determines the other, since $\overline{\partial A^{\alpha}}$ is r.e. and hence has degree 0 or 0' (by Rice's theorem [5]), independently of the degree of α ; while both 0 and 0' contain sets θA^{α} for $\alpha \in \mathbf{0}$. In this paper it is shown that when α is r.e., the situation is similar, though more complicated. It was shown in [3,Theorem 1] that if β is a complete r.e. set, then θA^{β} is a complete On the other hand, C. G. Jockush, Jr. has constructed an Π_2° set. example (unpublished) of an effectively simple set γ such that θA^r has degree 0'. Since β and γ both have degree 0' [4], this shows that when α is r.e., the degree of α need not determine that of θA^{α} . The main result of this paper shows that these examples are extremal cases of the fact that when α is r.e., the degree of θA^{α} can take on all possible values within certain obvious restrictions. More precisely, we prove the following:

THEOREM. Let b be a nonrecursive r.e. degree. Let

 $\mathcal{B}_{1} = \{ c \mid (\exists \alpha) \ (\alpha \in b \ and \ \theta A^{\alpha} \in c \}, \\ \mathcal{B}_{2} = \{ c \mid (\exists \alpha) \ (\alpha \ is \ r.e. \ and \ \alpha \in b \ and \ \theta A^{\alpha} \in c \}, \\ \mathcal{B}_{3} = \{ c \mid c \ge 0' \ and \ c \ is \ r.e. \ in \ b \}. \\ Then \ \mathcal{B}_{1} = \mathcal{B}_{2} = \mathcal{B}_{3}.$

Proof. Clearly $\mathscr{B}_2 \subset \mathscr{B}_1$. It is thus sufficient to prove that $\mathscr{B}_1 \subset \mathscr{B}_3 \subset \mathscr{B}_2$.

 $\mathscr{B}_1 \subset \mathscr{B}_3$: Assume $\alpha \in b$ and $\theta A^{\alpha} \in c$. Since $\overline{\theta A^{\alpha}} = \{x \mid W_x \cap \overline{\alpha} \neq \emptyset\}, \ \overline{\theta A^{\alpha}}$ is r.e. in α so c is r.e. in b. Since $b > 0, \ \alpha \neq \emptyset$ or N, so θA^{α} is a nontrivial index set which, by the proof of Rice's theorem [5, Theorem 14-XIV] implies $K \leq {}_{T}\theta A^{\alpha}$ (where K denotes the complete r.e. set). So $c \geq 0'$. Since c was arbitrary, this shows $\mathscr{B}_1 \subset \mathscr{B}_3$.

The remainder of this paper is devoted to proving that $\mathscr{B}_3 \subset \mathscr{B}_2$. We assume that $c \geq 0'$, c r.e. in b, and describe the construction of an r.e. set α such that $\alpha \in b$ and $\theta A^{\alpha} \in c$.

2. Preliminaries. The notation is that of [5]. Given β r.e., nonrecursive, and γ r.e. in β , $0' \leq {}_{T}\gamma$, we require an r.e. set α such that $\alpha \equiv {}_{T}\beta$ and $\theta A^{\alpha} \equiv {}_{T}\gamma$. We attempt to achieve this as follows:

(a) to get $\beta \leq \alpha$, we "code" β into α ;

(b) to get $\alpha \leq {}_{r}\beta$, we arrange that an odd integer y is put into α only when some $x \leq y$ has just appeared in β . (The idea here is similar to that used in the proof of Theorem 2 of [7].);

(c) to get $\gamma \leq {}_{T}\theta A^{\alpha}$, we define a sequence $\{S_{e}\}_{e\geq 0}$ of r.e. sets such that the index of S_{e} is recursive in θA^{α} and $e \in \gamma \leftrightarrow S_{e} \cap \overline{\alpha} \neq \emptyset$;

(d) to get $\theta A^{\alpha} \leq {}_{T}\gamma$, we try to "preserve" nonempty intersections $W_{e} \cap \bar{\alpha}$ whenever they occur during the construction.

These requirements evidently conflict, and priorities must be assigned, in the manner of [6].

The fact that γ is r.e. in β will be used in the following way: Let $\{D_i\}_{i\geq 0}$ be the canonical indexing of finite sets; $\langle x, y \rangle$ is a standard recursive pairing function with recursive inverses π_1 , π_2 , and $\langle x, y, u, v \rangle = \langle \langle \langle x, y \rangle, u \rangle, v \rangle$.

LEMMA 1. If γ is r.e. in a set β , then there is a recursive function f such that for each $x, x \in \gamma \leftrightarrow (\exists z) \ (z \in W_{f(x)} \text{ and } D_{\pi_1(z)} \subset \beta$ and $D_{\pi_2(z)} \subset \overline{\beta})$.

Proof. Let $\gamma = W_e^{\beta}$. Then in the notation of Chapter 9 of [5], $x \in \gamma \leftrightarrow x \in W_e^{\beta} \leftrightarrow \mathcal{P}_e^{\beta}(x)$ is defined $\leftrightarrow (\exists y) (\exists u) (\exists v) (\langle x, y, u, v \rangle \in W_{\rho(e)})$ and $D_u \subset \beta$ and $D_v \subset \overline{\beta}$ where $\rho(e)$ is a recursive function of e. Let

$$V = \{ \langle u, v \rangle \, | \, (\exists y) (\langle x, y, u, v \rangle \in W_{
ho_{(e)}}) \}$$
 .

Then V is an r.e. set, whose index can be uniformly computed from x; so there is a recursive function f such that $V = W_{f(x)}$, and

$$x \in \gamma \longleftrightarrow (\exists u) (\exists v) (\langle u, v \rangle \in V \text{ and } D_u \subset \beta \text{ and } D_v \subset \beta)$$

 $\longleftrightarrow (\exists z) (z \in W_{f(x)} \text{ and } D_{\pi_1(z)} \subset \beta \text{ and } D_{\pi_2(z)} \subset \overline{\beta}).$

DEFINITION 2. Let g be a recursive function such that $\{D_{g(i)}\}_{i\geq 0}$ is a recursive partitioning of the positive even integers into disjoint finite sets such that $|D_{g(i)}| = i + 1$ for each i (e.g., let $D_{g(i)} =$ $\{i^2 + i + 2k \mid 0 < k \leq i + 1\}$). Let $e_x = e(x)$ be a recursive function such that e_x = the unique i for which $2x \in D_{g(i)}$.

3. Construction. α will be constructed in stages, $\alpha = \bigcup_s \alpha_s$ where α_s is the finite set of integers which has been put into α by the end of stage s. If W is r.e., W^s will denote the result of performing s steps in some fixed enumeration of W; in particular $W^o = \emptyset$.

We define α_s and auxiliary recursive functions $y_e^s = y_e(s)$ and $z_e^s = z_e(s)$ and a partial recursive function h(y) by simultaneous recursion. If $y_e^s > 0$, y_e^s serves to witness that $e \in \gamma$, while z_e^s witnesses that $W_e \cap \overline{\alpha} \neq \emptyset$.

Stage 0. $lpha_{_0}=\{0\}, \ y_{_e}^{_0}=z_{_e}^{_0}=0.$

Let $C_s = \{z \mid z > 0 \text{ and } (\exists e) (\exists t)_{t < s} (z = y_e^s \lor z = z_e^t)\}$; so $C_1 = \emptyset$. Assume inductively that C_s is finite and that $y_e^{s-1} > 0$ implies y_e^{s-1} is odd and $h(y_e^{s-1})$ is defined, for all e.

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Stage s > 0, s \equiv 1 \pmod{3}.
Let
E_s = \{y \mid (\exists x) \ (y = 2x \text{ and } e_x \leq s \text{ and } e_x \in \beta^s \text{ and } (\forall i)_{i < e_x} (y \neq z_i^{s-1})\}
O_s = \{y \mid \exists e)_{e \leq s} (y = y_e^{s-1} \text{ and } D_{\pi_2(h(y))} \cap \beta^s \neq \emptyset\}.
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Let $\alpha_s = \alpha_{s-1} \cup E_s \cup O_s$. If $z_e^{s-1} \in E_s$, let $z_e^s = 0$. Otherwise, let $z_e^s = z_e^{s-1}$. If $y_e^{s-1} \in O_s$, let $y_e^s = 0$. Otherwise, let $y_e^s = y_e^{s-1}$.

 $\begin{array}{ll} Stage \ s > 0, \ s \equiv 2 \ (\mathrm{mod} \ 3).\\ \mathrm{Let} \ \alpha_s = \alpha_{s-1}. & \mathrm{For \ each} \ e \leq s \ (\mathrm{if \ any}) \ \mathrm{such \ that}\\ \mathrm{(a)} \quad z_{*}^{s-1} = 0 \ \mathrm{and}\\ \mathrm{(b)} \quad (\exists z) \ (z \in W_{*}^{s} \cap \bar{\alpha}_{s} \ \mathrm{and} \ (\forall i)_{i \leq e} \ (z \neq y_{i}^{s-1}) \ \mathrm{and}\\ \quad (\forall x)_{x < z} \ (z = 2x \rightarrow e_{x} > e)) \ , \end{array}$

let z_e^s = the least such z. For all other e, let $z_e^s = z_e^{s-1}$. If $y_j^{s-1} = z_e^s$

for some e < j, let $y_j^s = 0$. Otherwise let $y_j^s = y_j^{s-1}$.

Stage s > 0, $s \equiv 0 \pmod{3}$.

Let $\alpha_s = \alpha_{s-1}$, $z_e^s = z_e^{s-1}$ for all e. Let $F_s = \{e \mid e \leq s \text{ and } y_e^{s-1} = 0$ and $(\exists x) \ (x \in W_{f(e)}^s)$ and $D_{\pi_1(x)} \subset \beta^s$ and $D_{\pi_2(x)} \subset \overline{\beta^s}\}$. If $e \notin F_s$, let $y_e^s = y_e^{s-1}$. If $F_s \neq \emptyset$, let $F_s = \{k_0, k_1, \dots, k_n\}$, $n \geq 0$, $k_i < k_j$ for i < j. Define $y_{k_i}^s$ inductively as follows: assume $y_{k_j}^s$ has been defined for all j < i. Let $x_i = \text{least } x$ such that $x \in W_{f(k_i)}^s$ and $D_{\pi_1(x)} \subset \beta^s$ and $D_{\pi_2(x)} \subset \overline{\beta^s}$, $y_{k_i}^s = \text{least odd } y \in \overline{\alpha_s}$ such that

$$y>\max{(D_{\pi_1(x_i)}\cup D_{\pi_2(x_i)}\cup C_s\cup\{y^s_{k_j}\,|\,j< i\})}$$
 .

Define $h(y_{k_i}^s) = x_i$ for each $i \leq n$.

It is easily verified for all three types of stages that C_s is always finite, since new nonzero values are assigned to z_e^s and y_e^s for at most s + 1 values of e. For the second inductive assumption, it suffices to note that new nonzero values of y_e^s are defined only if $s \equiv 0 \pmod{3}$, and that each new value $y_e^s > 0$ is odd and $h(y_e^s)$ is defined. That h(y)is well defined will be proved below.

It is clear that $\alpha = \bigcup_s E_s \cup \bigcup_s O_s \cup \{0\}$ is r.e. We note for later use that O_s consists of odd numbers and E_s of even numbers, so that

(i)
$$0 < y = 2x \in \alpha \longleftrightarrow (\exists s) \ (s \equiv 1 \pmod{3} \text{ and } y \in E_s)$$
,

(ii) $y \text{ odd}, y \in \alpha \longleftrightarrow (\exists s) (s \equiv 1 \pmod{3}) \text{ and } y \in O_s)$.

4. Proof of Theorem.

LEMMA 3. For all e and s,

(a) If $y_{e}^{s-1} \neq y_{e}^{s}$ then either (i) $s \not\equiv 0 \pmod{3}$, $y_{e}^{s-1} > 0$ and $y_{e}^{s} = 0$, or (ii) $s \equiv 0 \pmod{3}$, $y_{e}^{s-1} = 0$ and $y_{e}^{s} > 0$.

(b) If $y = y_e^{s-1} > 0$ and $y_e^s \neq y$, then either $(\exists i)_{i < e}(y = z_i^s)$ or $s \equiv 1 \pmod{3}$ and $y \in O_s$.

(c) If $y_{e}^{s} > 0$, then either (i) $(\forall t) (t > s \rightarrow y_{e}^{t} = y_{e}^{s})$ or (ii) if $t' = least \ t > s$ such that $y_{e}^{t} \neq y_{e}^{s}$ then $y_{e}^{t'} = 0$ and $(\forall s') (s' > t' \rightarrow y_{e}^{s'} = 0$ or $y_{e}^{s'} > y_{e}^{s})$.

(d) If s < t and $0 < y_e^s = y_e^t$, then $(\forall t') (s < t' < t \rightarrow y_e^{t'} = y_e^s)$.

(e) If $\lim_{s} y_{e}^{s}$ exists and $\lim_{s} y_{e}^{s} = y > 0$, then $(\forall s) (\forall t) (y = y_{e}^{s})$ and $s < t \rightarrow y = y_{e}^{t}$.

Proof. (a) is clear from the construction.

(b) Assume $y = y_e^{s-1} > 0$, $y_e^s \neq y$. Then by (a), $s \not\equiv 0 \pmod{3}$. If $s \equiv 1 \pmod{3}$, then $y_e^s \neq y_e^{s-1}$ only if $y_e^{s-1} \in O_s$. If $s \equiv 2 \pmod{3}$, then $y_e^s \neq y_e^{s-1}$ only if $y = y_e^{s-1} = z_i^s$ for some i < e.

(c) Assume $y = y_e^s > 0$. If (i) fails to hold, let t' be the least

t > s such that $y_e^t \neq y_e^s$; thus t' > s and $0 < y = y_e^s = y_e^{t'-1} \neq y_e^{t'}$. Then by (a), $y_e^{t'} = 0$. Suppose (ii) fails to hold; then for some s' > t', $0 < y_e^{s'} \leq y$. Let s' be least. Then $s' - 1 \geq t'$ and $y_e^{s'-1} = 0$ or $y_e^{s'-1} > y > 0$. So $y_e^{s'} \neq y_e^{s'-1}$, and it follows by (a) that $s \equiv 0 \pmod{3}$ and $y_e^{s'-1} = 0$. But s' > t' > s implies that $y = y_e^s \in C_{s'}$, while $y_e^{s'} \neq y_e^{s'-1}$ implies $y_e^{s'} > \max C_{s'}$. So $y_e^{s'} > y$, which is a contradiction. So (ii) must hold.

(d) Let s < t and $0 < y_e^s = y_e^t$. Suppose that $y_e^{t'} \neq y_e^s$ for some t', s < t' < t and let t' be least. Then by (c), $y_e^{t'} = 0$, and t > t' > s implies $y_e^t = 0$ or $y_e^t > y_e^s$, both contrary to hypothesis. So $y_e^s = y_e^t$ implies $y_e^{t'} = y_e^s$ for all t', s < t' < t.

(e) Assume $\lim_{s} y_{\epsilon}^{s} = y > 0$ and $y = y_{\epsilon}^{s}$. Suppose that for some t > s, $y_{\epsilon}^{t} \neq y$, and let t be least. Then by (c), $y_{\epsilon}^{s'} \neq y$ for all s' > t which contradicts the assumption that $y = \lim_{s} y_{\epsilon}^{s}$.

LEMMA 4. For all e and s,

(a) $y_e^s > 0$ implies $y_e^s \in \overline{\alpha_s}$.

(b) If $y_e^s > 0$ and $e' \neq e$, then $y_e^s \neq y_{e'}^t$ for all t.

(c) If $y = y_e^s > 0$, then h(y) is well-defined.

Proof. (a) Assume $y = y_e^s > 0$, and let s' be the least t such that $y = y_e^t$. Then $0 < s' \leq s$, and $y_e^{s'-1} \neq y_e^{s'} = y > 0$. So by Lemma $3(a), y_e^{s'-1} = 0$ and $s' \equiv 0 \pmod{3}$. By the construction, $y_e^{s'} > y_e^{s'-1}$ implies $y_e^{s'} \in \overline{\alpha_{s'}}$. Now assume $y \in \alpha_s$, and consider the least t such that $y \in \alpha_t$. Clearly $s' < t \leq s$, $y \in \overline{\alpha_{t-1}}$ and $t \equiv 1 \pmod{3}$. Now $y = y_e^s > 0$ is odd, so $y \notin E_t \cup \alpha_{t-1}$. So $y \in \alpha_t$ implies $y_e^{t-1} = y_e^t = y$. By Lemma $3(d), y_e^s = y_e^{s'}$ and $s' \leq t - 1 < t \leq s$ implies $y_e^{t-1} = y_e^t = y$. But by the construction, $y = y_e^{t-1} \in O_t$ implies $y_e^t = 0 \neq y$, which is a contradiction. So $y \in \overline{\alpha_s}$.

(b) Assume $y_e^s > 0$ and $e' \neq e$. Clearly if $y_{e'}^t = 0$ then $y_{e'}^t \neq y_e^s$; so assume $y_{e'}^t > 0$. Consider the least s' such that $y_e^s = y_e^{s'}$ and the least t' such that $y_{e'}^t = y_{e'}^{t'}$. Then $s' \equiv t' \equiv 0 \pmod{3}$, $e \in F_{s'}$ and $e' \in F_{t'}$. If s' < t' then $y_e^{s'} \in C_{t'}$, so by the construction,

$$y_{\scriptscriptstyle e'}^{\scriptscriptstyle t} = y_{\scriptscriptstyle e'}^{\scriptscriptstyle t'} > y_{\scriptscriptstyle e}^{\scriptscriptstyle s'} = y_{\scriptscriptstyle e}^{\scriptscriptstyle s}$$
 .

If s' > t', then $y_{e'}^{s'} \in C_{s'}$, so $y_e^s = y_{e'}^{s'} > y_{e'}^{t'} = y_e^t$. If s' = t', then $e, e' \in F_{s'}$, $e = k_i, e' = k_j$ for $i \neq j$. If e < e' then i < j and $y_e^s = y_e^{s'} \in \{y_{k_s}^{s'} | i < j\}$ while $y_{e'}^t = y_{e'}^{s'} > \max\{y_{k_i}^{s'} | i < j\}$, so $y_{e'}^t > y_e^s$. By symmetry, if e' < ethen $y_e^s > y_e^t$. Thus in any case, $e \neq e'$ implies $y_e^s \neq y_{e'}^t$.

(c) First note that by the construction, h(y) is defined if and only if there exist e, s such that $y_e^{s-1} = 0$ and $y = y_e^s > 0$. In particular, if $y = y_e^s > 0$, h(y) is defined since if s' is the least t such that $y = y_e^t$, then $y_e^{s'-1} = 0$. To show h(y) is well-defined, it suffices to

show that there exists at most one pair e, s such that $y = y_e^s$ and $y_e^{s-1} = 0$. Suppose $y = y_e^s = y_i^t$ where $y_e^{s-1} = y_i^{t-1} = 0$. Since y > 0, $y_e^s = y_i^t$ implies i = e, by part (b) of this lemma. If $s \neq t$, say s < t, then $s \leq t-1 < t$. But then by Lemma 3(d), $y_e^{t-1} = y_i^{t-1} = 0 \neq y_e^s$ implies $y_e^s \neq y_e^t = y_i^t$, contrary to hypothesis. This completes the proof.

DEFINITION 5. Let $\tau(x)$ = the least t such that

$$(\forall z)_{z \leq x} (z \in \beta \longrightarrow z \in \beta^t)$$
.

Then $\tau(x)$ is defined for all x, and $\tau(x)$ is evidently recursive in β .

LEMMA 6. If $t \equiv 1 \pmod{3}$, $y = y_e^{t-1} > 0$ and $D_{\pi_2 h(y)} \cap \beta^t \neq \emptyset$, then $t < \tau(y) + 3$.

Proof. Assume the hypothesis. Let s' be the least s such that $y = y_e^s$ and let x = h(y); x is well-defined by Lemma 4(c). Then $e \leq s' < t$, $s' \equiv 0 \pmod{3}$ and, by the construction, $D_{\pi_2(x)} \subset \overline{\beta^{s'}}$ and $y = y_e^{s'} > \max D_{\pi_2(x)}$. By hypothesis, $D_{\pi_2(x)} \cap \beta^t \neq \emptyset$; let z be any element of $D_{\pi_2(x)} \cap \beta^t$. Then $z \in D_{\pi_2(x)}$ implies z < y, so that $z \in \beta$, $z \notin \beta^{s'}$ implies $s' < \tau(y)$. Now suppose $t \geq \tau(y) + 3$, and let s = t - 3. Then $s \geq \tau(y) > s' \geq e$ and $s \equiv t \equiv 1 \pmod{3}$. Also $s \geq \tau(y)$ implies $z \in \beta^s$, so $D_{\pi_2(x)} \cap \beta^s \neq \emptyset$. By Lemma 3(d), $0 < y = y_e^{s'} = y_e^{t-1}$ and $s' \leq s - 1 < s < t - 1$ implies $y_e^{s-1} = y_e^s = y$. But by the construction, $e \leq s$ and $y = y_e^{s-1}$ and $D_{\pi_2h(y)} \cap \beta^s \neq \emptyset$ implies $y_e^{s-1} \in O_s$, so that $y_e^s = 0 \neq y$. Since this is a contradiction, we conclude $t < \tau(y) + 3$.

LEMMA 7. Assume y is odd. Then $y \in \alpha$ if and only if

 $(\exists t)_{t<\tau(y)+3}(\exists e)_{e\leq t} \ (t\equiv 1 \pmod{3} \ and \ y=y_e^{t-1} \ and \ D_{\pi_2h(y)}\cap \beta^t\neq \varnothing).$

Proof. By the construction, if y is odd then $y \in \alpha \longleftrightarrow (\exists t) \ (t \equiv 1 \pmod{3} \text{ and } y \in O_t)$ $\longleftrightarrow (\exists t) \ (\exists e)_{e \leq t} \ (t \equiv 1 \pmod{3} \text{ and } y = y_e^{t-1} \text{ and}$ $D_{\pi_2h(y)} \cap \beta^t \neq \emptyset)$.

By Lemma 6, such a t can be bounded by $\tau(y) + 3$, which proves the lemma.

LEMMA 8. For all i and s,

(a) If $z_i^{s-1} > 0$ and $z_i^s \neq z_i^{s-1}$, then $s \equiv 1 \pmod{3}$, $z_i^s = 0$ and $z_i^{s-1} \in E_s \subset \alpha_s$.

(b) If
$$z = 2x > 0$$
, $i < e_x$ and $z = z_i^s$, then $z = z_i^t$ for all $t \ge s$.

Proof. (a) Assume $z_i^{s-1} > 0$ and $z_i^s \neq z_i^{s-1}$. If $s \equiv 0 \pmod{3}$, then $z_i^s = z_i^{s-1}$ for all *i*. If $s \equiv 2 \pmod{3}$ then $z_i^s \neq z_i^{s-1}$ only if $z_i^{s-1} = 0$. It follows that $s \equiv 1 \pmod{3}$. But then $z_i^s \neq z_i^{s-1}$ only if $z_i^{s-1} \in E_s \subset \alpha_s$, and in that case $z_i^s = 0$.

(b) Assume z = 2x > 0, $i < e_x$ and $z = z_i^s$. Suppose $(\exists t) (t > s and z_i^t \neq z_i^s)$, and let t be least. Then $0 < z = z_i^{t-1} \neq z_i^t$; so by (a), $t \equiv 1 \pmod{3}$ and $z \in E_t$. But this implies that $(\exists x') (z = 2x' \text{ and } (\forall i)_{i < e_x'} (z \neq z_i^{t-1}))$. Clearly x' = x, so $(\forall i)_{i < e_x} (z \neq z_i^{t-1})$ which is a contradiction. So $z = z_i^t$ for all $t \geq s$.

DEFINITION 9. Define functions $\sigma(y)$, $\sigma'(y)$ as follows:

(a) If y is odd, $\sigma'(y) = 0$.

(b) If y = 2x and $e_x \notin \beta$, $\sigma'(y) = 0$.

- (c) If y = 2x and $e_x \in \beta$, $\sigma'(y) = \text{least } s$ such that $e_x \in \beta^s$.
- (d) If $\sigma'(y) = 0$ then $\sigma(y) = 0$.

(e) If $\sigma'(y) > 0$, then $\sigma(y) = \text{least } s \ge \max \{e_x, \sigma'(y)\}$ such that $s \equiv 1 \pmod{3}$.

It is clear that σ , σ' are defined for all y and that $\sigma(y) > 0$ if and only if y = 2x and $e_x \in \beta$. Since e_x is a recursive function of x, σ' and σ are recursive in β .

LEMMA 10. Assume y = 2x > 0. Then $y \in \alpha$ if and only if $e_x \in \beta$ and $\sigma(y) > 0$ and $(\forall i)_{i < e_x} (y \neq z_i^{\sigma(y)-1})$.

Proof. (-). Assume $e_x \in \beta$ and $\sigma(y) > 0$ and $(\forall i)_{i < e_x} (y \neq z_i^{\sigma(y)-1})$. Then by Definition 9, $e_x \leq \sigma(y)$ and $e_x \in \beta^{\sigma'(y)} \subset \beta^{\sigma(y)}$. Then by the construction, since $\sigma(y) \equiv 1 \pmod{3}$, $y \in E_{\sigma(y)} \subset \alpha_{\sigma(y)}$. So $y \in \alpha$.

 (\rightarrow) . Assume $y \in \alpha$. Then since y > 0 is even, $y \in E_s$ for some $s, s \equiv 1 \pmod{3}$; so $e_x \in \beta^s$, $e_x \leq s$ and $(\forall i)_{i < e_x} (y \neq z_i^{s-1})$. So in particular $e_x \in \beta$, and, by definition of $\sigma(y)$, $0 < \sigma(y) \leq s$. Suppose that for some $i < e_x$, $y = z_i^{\sigma(y)-1}$. Then by Lemma 8(b), $y = z_i^{s-1}$, since

$$s-1 \geq \sigma(y)-1$$
.

But this is a contradiction, which proves the lemma.

Lemma 11. $\alpha \leq T\beta$.

Proof. We show how to decide membership in α , recursively in β . If y = 0, then $y \in \alpha$, by Stage 0 of the construction. Suppose y > 0.

Case 1. y is odd.

Then by Lemma 7, $y \in \alpha$ if and only if $(\exists t)_{t < \tau(y)+3} (\exists e)_{e \le t} (t \equiv 1 \pmod{3})$ and $y = y_e^{t-1}$ and $D_{\pi_2 h(y)} \cap \beta^t \ne \emptyset$). Now for fixed t, y it can be decided recursively whether $t \equiv 1 \pmod{3}$ and whether $y = y_e^{t-1}$ for some $e \le t$, since y_e^s is a recursive function of s. If $y = y_e^{t-1}$, then h(y) is well defined by Lemma 4(c), and it can be decided recursively whether $D_{\pi_2 h(y)} \cap \beta^t \ne \emptyset$. So $y \in \alpha \leftrightarrow (\exists t)_{t < \tau(y)+3} R(t, y)$ where R(t, y) is a recursive predicate. Since, as noted in Definition 5, $\tau(y)$ is recursive in β , the question of whether $y \in \alpha$ can be decided, recursively in β .

Case 2. y is even.

Then by Lemma 10, y = 2x implies that $y \in \alpha$ if and only if $e_x \in \beta$ and $\sigma(y) > 0$ and $(\forall i)_{i < e_x} (y \neq z_i^{\sigma(y)-1})$. For y = 2x, whether $e_x \in \beta$ can be decided recursively in β , since e_x is a recursive function of x. If $e_x \notin \beta$, then $y \notin \alpha$. If $e_x \in \beta$, then $\sigma(y) > 0$ can be computed recursively in β , as noted in Definition 9. Since z_i^s is a recursive function of s, the membership of the finite set $D = \{z_i^{\sigma(y)-1} | i < e_x\}$ can be completely determined, once $\sigma(y)$ is known. Then

$$y \in lpha \longleftrightarrow y
otin D$$
 .

LEMMA 12. For all $e, e \in \beta \leftrightarrow D_{g(e)} \cap \alpha \neq \emptyset$.

Proof. (\leftarrow). Assume $D_{g(e)} \cap \alpha \neq \emptyset$. Let $y \in D_{g(e)} \cap \alpha$. Then $y \in D_{g(e)}$ implies y is even, y > 0, and, as remarked above, $y \in \alpha$ implies $y \in E_s$ for some s. So y = 2x for some x such that $e_x \in \beta$, i.e., $y = 2x \in D_{g(i)}$ where $i \in \beta$. But by definition of g, $D_{g(e)} \cap D_{g(i)} \neq \emptyset$ implies i = e; so $e \in \beta$.

 (\rightarrow) . Assume $e \in \beta$, and let $t \equiv 1 \pmod{3}$ be so large that $t \ge e$ and $e \in \beta^t$. Now by the definition of g, $D_{g(e)}$ has e + 1 elements, while $\{z_i^{t-1} | i < e\}$ has at most e elements. So $D_{g(e)} - \{z_i^{t-1} | i < e\} \neq \emptyset$. Let $y = 2x \in D_{g(e)} - \{z_i^{t-1} | i < e_x\}$. Then y > 0, and by definition of $e_x, e_x = e$. So $e_x \le t$, $e_x \in \beta^t$ and $(\forall i)_{i < e_x}(y \ne z_i^{t-1})$. This implies $y \in E_t \subset \alpha_t$, and $y \in D_{g(e)} \cap \alpha$.

Lemma 13. $\beta \leq T \alpha$.

Proof. By Lemma 12, β is in fact truth-table reducible to α .

LEMMA 14. For all i, e and s, $z_{e}^{s} > 0$ implies $z_{e}^{s} \neq y_{i}^{s}$.

Proof. By induction on s. Since $z_e^0 = 0$, the lemma holds

vacuously for s = 0. Assume that for all i and e, $z_e^{s-1} > 0$ implies $z_e^{s-1} \neq y_i^{s-1}$, and assume $y_e^s > 0$.

Case 1. $s \equiv 0 \pmod{3}$.

Then $z_e^s = z_e^{s-1}$, which implies $z_e^{s-1} > 0$. Then by the induction hypothesis, $z_e^{s-1} \neq y_i^{s-1}$ for all *i*. If $y_i^s = y_i^{s-1}$, then

$$z^{s}_{e} = z^{s-1}_{e}
eq y^{s-1}_{i} = y^{s}_{i}$$
 .

If $y_i^s \neq y_i^{s-1}$, then by the construction, $z_e^s = z_e^{s-1} \in C_s$ while $y_i^s > \max C_s$. So in either case, $z_e^s \neq y_i^s$.

Case 2. $s \equiv 1 \pmod{3}$.

By the construction, $z_e^s \neq z_e^{s-1}$ implies $z_e^s = 0$ and $y_i^s \neq y_i^{s-1}$ implies $y_i^s = 0$. So $z_e^s > 0$ implies $z_e^s = z_e^{s-1} \neq y_i^{s-1}$, by the induction hypothesis. If $y_i^{s-1} = y_i^s$, clearly $z_e^s \neq y_i^s$. If $y^s \neq y_i^{s-1}$ then $y_i^s = 0$ so $y_i^s \neq z_e^s$ since $z_e^s > 0$.

Case 3. $s \equiv 2 \pmod{3}$.

For each *i*, either $y_i^s = y_i^{s-1}$ or $y_i^s = 0$. If $y_i^s = 0$ then $y_i^s \neq z_e^s$. If $y_i^s = y_i^{s-1}$, then $y_i^s = z_e^s$ implies $y_i^{s-1} = z_e^s$. If $z_e^s = z_e^{s-1}$ this contradicts the induction hypothesis, since in that case $z_e^{s-1} > 0$. So $z_e^s = y_i^s = y_i^{s-1}$ implies $z_e^s \neq z_e^{s-1}$ which by the construction implies z_e^s satisfies

$$(orall i)_{i \leq e} (z^s_e
eq y^{s-1}_i)$$
 .

So $z_{\epsilon}^{s} = y_{i}^{s^{-1}}$ implies i > e. But by the construction, $z_{\epsilon}^{s} = y_{i}^{s^{-1}}$ for i > e implies $y_{i}^{s} = 0$ which contradicts the hypothesis that $y_{i}^{s} = z_{\epsilon}^{s} > 0$. So $y_{i}^{s} = y_{i}^{s^{-1}}$ implies $z_{\epsilon}^{s} \neq y_{i}^{s}$. This completes the proof of the lemma.

LEMMA 15. For all e and t, if $z_e^t > 0$ then (a) $z_e^t \in W_e \cap \overline{\alpha_t}$, and (b) $z_e^t = 2x$ implies $e_x > e$.

Proof. Assume $z = z_e^t > 0$, and let s' be the least s such that $z = z_e^s$. Then $z_e^{s'} = z$, $s' \leq t$ and, by the construction, $s' \equiv 2 \pmod{3}$, $z \in W_e^{s'} \cap \overline{\alpha_{s'}}$ and z = 2x implies $e_x > e$. It remains to show $z \in \overline{\alpha_t}$. Suppose not, and let t' = the least s such that $z \in \alpha_{t'}$, $s' < t' \leq t$. Then $t' \equiv 1 \pmod{3}$ and $z \in E_{t'} \cup O_{t'}$. We claim that if $z \neq z_e^{t'}$ then $z \neq z_e^s$ for all $s \geq t'$. Assume otherwise; i.e., $z \neq z_e^{t'}$ but $z = z_e^s$ for some s > t'. Let s be least. Then s > t', $z_e^{s-1} \neq z = z_e^s > 0$. Then $s \equiv 2 \pmod{3}$ and $z_e^s = z \in W_e^s \cap \overline{\alpha_s}$. But s > t' implies $z \in \alpha_{t'} \subset \alpha_s$, which is a contradiction. So $z \neq z_e^{t'}$ implies $z \neq z_e^s$ for all $s \geq t'$; so

in particular $t \ge t'$ and $z \ne z_e^{t'}$ implies $z \ne z_e^t$. But since by hypothesis $z = z_e^t$, it follows that $z = z_e^{t'}$. Now by the construction, $t' \equiv 1 \pmod{3}$ and $z_e^{t'-1} \ne z_e^{t'}$ implies $z_e^{t'} = 0$. Since $z = z_e^{t'} > 0$, if follows that $z_e^{t'-1} = z_e^{t'} = z$. So $z_e^{t'-1} = z \notin E_{t'}$, which implies $z \in O_{t'}$. But then $z = z_e^{t'-1} = y_t^{t'-1}$ for some *i*, which contradicts Lemma 14. So *t'* cannot exist, and $z \in \overline{\alpha_t}$.

DEFINITION 16. z is permanently restrained by e if z > 0 and $z = \lim_{s} z_{e}^{s}$.

LEMMA 17. For all e,

- (a) If z is permanently restrained by e, then $z \in \overline{\alpha}$.
- (b) If z is permanently restrained by e, then

 $(\forall s) \ (\forall t) \ (z = z_e^s \ and \ t > s \longrightarrow z = z_e^t)$.

- (c) At most one z is permanently restrained by e.
- (d) If $z = z_e^*$ and $z \in \overline{\alpha}$, then z is permanently restrained by e.

Proof. (a) Assume z is permanently restrained by e. Then z > 0 and $z = z_e^s$ for cofinitely many s. By Lemma 15(a), this implies $z \in \overline{\alpha}_s$ for cofinitely many s. So $z \in \overline{\alpha}$.

(b) Assume z is permanently restrained by e and $z = z_e^s$. Suppose $(\exists t) (t > s \text{ and } z \neq z_e^t)$ and let t be least. Then t > s and $z = z_e^{t-1} \neq z_e^t$. Then by Lemma 8(a), since $z_e^{t-1} = z > 0$ this implies $z \in \alpha$, which contradicts (a). So $(\forall t) (t > s \rightarrow z = z_e^t)$.

(c) Suppose z is permanently restrained by e. Then $z = z_e^s$ for some s, and by (b), $z = z_e^t$ for all t > s. If z' is permanently restrained by e, then $z' = z_e^t$ for cofinitely many t. Then in particular $z' = z_e^t$ for some t > s, which implies z' = z.

(d) Assume $z = z_e^s$ and $z \in \overline{\alpha}$. Since $0 \in \alpha$ it follows that z > 0. Suppose that $(\exists t) \ (t > s \text{ and } z \neq z_e^t)$ and let t be least. Then $0 < z = z_e^{t-1} \neq z_e^t$. Then by Lemma 8(a), $z = z_e^{t-1} \in \alpha$, contrary to hypothesis. So $(\forall t) \ (t > s \rightarrow z = z_e^t)$, i.e., $z = \lim_s z_e^s$ and z is permanently restrained by e.

LEMMA 18. For all e and z,

(a) $\lim_{s} z_{e}^{s}$ exists.

(b) If $z(e) = \lim_{s} z_{e}^{s}$, z(e) is recursive in 0'.

(c) z is permanently restrained by e if and only if z = z(e) and z(e) > 0.

Proof. Fix e. Since $D_{g(i)}$ is finite for each *i*, there exists *t* such that $(\forall i)_{i \leq e} (D_{g(i)} \cap \alpha^t = D_{g(i)} \cap \alpha)$. Let t(e) be the least such *t*;

t(e) is recursive in 0' since α is r.e. and $\bigcup_{i \leq e} D_{g(i)}$ is completely known from the canonical indexing. Define a function r(e) by

$$r(e) = 1$$
 if $(\exists s) \ (s \ge t(e) \text{ and } z_e^s > 0)$
= 0 otherwise.

Then since t(e) can be computed recursively in 0' and z_e^s is a recursive function of s, r(e) is also recursive in 0'.

Case 1. r(e) = 0. Then $(\forall s) (s \ge t(e) \rightarrow z_e^s = 0)$. Then $0 = \lim_s z_e^s = z_e^{t(e)}$.

Case 2. r(e) = 1.

Then $(\exists s) (s \geq t(e) \text{ and } z_e^s > 0)$. Let s(e) be the least such s. Then s(e) is recursive in 0', since s(e) can in fact be computed recursively, given t(e). Suppose $(\exists t) (t > s(e) \text{ and } z_e^t \neq z_e^{s(e)})$, and let t be least. Then t > s(e), $0 < z_e^{s(e)} = z_e^{t-1} \neq z_e^t$. By Lemma 8(a), this implies $z_e^{s(e)} = z_e^{t-1} \in E_t$. Then by definition of E_t it follows that $(\exists x)(z_e^{s(e)} = 2x$ and $(\forall i)_{i < e_x}(z_e^{s(e)} \neq z_i^{t-1}))$. Since $z_e^{s(e)} = z_e^{t-1}$, this implies $e \geq e_x$. But by Lemma 15(b), $z_e^{s(e)} = 2x$ implies $e_x > e$, which gives a contradiction. So $z_e^{s(e)} = z_e^t$ for all $t \geq s(e)$, and $z_e^{s(e)} = \lim_s z_e^s$.

Thus (a) holds in either case. Define

$$egin{array}{rll} z(e) &= 0 & ext{if} & r(e) &= 0 \ &= s(e) & ext{if} & r(e) &= 1 \end{array}.$$

As shown above, $z(e) = \lim_{s} z_{e}^{s}$ for each e, and z(e) is recursive in 0' since r(e) and s(e) are, which proves (b). (c) is an immediate consequence of Definition 16 and the definition of z(e).

LEMMA 19. For all $e, e \in \gamma$ if and only if $\lim_{s} y_{e}^{s} = y$ exists and $y \in \overline{\alpha}$.

Proof. (\leftarrow) Assume $y = \lim_{s} y_{e}^{s}$ and $y \in \overline{\alpha}$. Then y > 0 since $0 \in \alpha$. Let t be the least s such that $y = y_{e}^{s}$. Then by the construction, $t \equiv 0 \pmod{3}$, $e \leq t$ and x = h(y) implies $D_{\pi_{1}(x)} \subset \beta^{t}$ and $D_{\pi_{2}(x)} \subset \overline{\beta^{t}}$. Suppose $e \notin \gamma$. Then by Lemma 1, $x \in W_{f(e)}$ and $D_{\pi_{1}(x)} \subset \beta$ implies $D_{\pi_{2}(x)} \cap \beta \neq \emptyset$, so there exists s' > t such that $D_{\pi_{2}(x)} \cap \beta^{s'} \neq \emptyset$. Let s' be least, and choose $s'' \geq s'$ such that $s'' \equiv 1 \pmod{3}$. By Lemma 3(e), $0 < y = \lim_{s} y_{e}^{s}$ and $y = y_{e}^{t}$ implies $y = y_{e}^{s}$ for all $s \geq t$; so in particular $s'' > s'' - 1 \geq s' - 1 \geq t$ implies $y = y_{e}^{s''-1} = y_{e}^{s''}$. But $s'' \geq s' > t$ also implies e < s'' and $D_{\pi_{2}(x)} \cap \beta^{s''} \neq \emptyset$. So by the construction $y = y_{e}^{s''-1} \in O_{s''}$ and $y_{e}^{s''} = 0 \neq y$, which is a contradiction. So $e \in \gamma$.

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 (\rightarrow) Assume $e \in \gamma$. Then by Lemma 1,

 $(\exists x) \ (x \in W_{f(e)} \text{ and } D_{\pi_1(x)} \subset \beta \text{ ane } D_{\pi_2(x)} \subset \overline{\beta}).$ Let x_e be the least such x, and let s' be the least s such that $x_e \in W_{f(e)}^s$ and $D_{\pi_1}(x_e) \subset \beta^s$. Clearly $x_e \in W_{f(e)}^s$ and $D_{\pi_1(x_e)} \subset \beta^{\beta}$ and $D_{\pi_2(x_e)} \subset \overline{\beta^s}$ for all s > s'. Since x_e is least,

$$(\forall x)_{x < x_e} (x \in W_{f(e)} \text{ and } D_{\pi_1(x)} \subset \beta \longrightarrow D_{\pi_2(x)} \cap \beta \neq \emptyset)$$
.

Let $s'' \geq s'$ be large that

$$(\forall x)_{x < x_e} (x \in W_{f(e)} \text{ and } D_{\pi_1(x)} \subset \beta \longrightarrow D_{\pi_2(x)} \cap \beta^{s''} \neq \emptyset)$$
.

Now by Lemma 18(a), $\lim_{s} z_{i}^{s}$ exists for all *i*. Let m(i) be a function such that $z_{i}^{m(i)} = \lim_{s} z_{i}^{s}$, i.e.,

$$(\forall i) \ (\forall s) \ (s \ge m(i) \longrightarrow z_i^s = z_i^{m(i)}) \ ,$$

and choose $t \geq s''$, $\max_{i < e} m(i)$.

Case 1. $(\forall s) (s \ge t \rightarrow y_e^s > 0)$.

Then $y_{\epsilon}^{t} > 0$ so by Lemma 3(c), for all t > s either $y_{\epsilon}^{s} = y_{\epsilon}^{t}$ or $y_{\epsilon}^{t'} = 0$ for some t', $s < t' \leq t$. Since by assumption the latter does not happen, it follows that $y_{\epsilon}^{t} = \lim_{s} y_{\epsilon}^{s}, y_{\epsilon}^{t} > 0$.

Case 2. $(\exists s) (s \ge t \text{ and } y_e^s = 0).$

Let t'-1 be the least such s; so t' > t, $y_e^{t'-1} = 0$. By the construction, if $t' \not\equiv 0 \pmod{3}$ then $y_e^{t'} = 0$ also; so we may as well assume $t' \equiv 0 \pmod{3}$. Then since $t' > t \geq s''$, x_e is the least x satisfying: $x \in W_{f(e)}^{t'}$ and $D_{\pi_1(x)} \subset \beta^{t'}$ and $D_{\pi_2(x)} \subset \overline{\beta^{t'}}$. Since $y_e^{t'-1} = 0$ then by the construction $x_e^{t'} = x_e$, $y_e^{t'} > 0$ and $x_e = h(y_e^{t'})$; let $y = y_e^{t'}$. Assume $y \neq \lim_s y_e^{s}$; i.e., $y_e^{s} \neq y_e^{t'}$ for some s > t'. Let s be least; then s > t', and $y = y_e^{t'} = y_e^{s-1} \neq y_e^{s}$. Then by Lemma 3 (b) either $y \in O_s$ or $y = z_s^s$ for some i < e. But $y \in O_s$ only if $D_{\pi_2(h(y))} \cap \beta^s \neq \emptyset$. Since $h(y) = x_e$ and $D_{\pi_2(x)} \subset \overline{\beta}$ this cannot happen. So $y \notin O_s$ which implies $y = z_s^s$ for some i < e. But by Lemma 14, $0 < y = y_e^{s-1}$ implies $y \neq z_s^{s-1}$. So i < e and $z_s^{s-1} \neq z_s^s$. But $s > t' > t > \max_{i < e} m(i)$, so $s - 1 \ge m(i)$ which implies $z_s^s = z_s^{s-1} = z_s^{m(i)}$ which is a contradiction. It follows that $y = \lim_s y_e^s$, where y > 0.

Thus in either case $e \in \gamma$ implies $\lim_s y_e^s$ exists and $\lim_s y_e^s = y > 0$. To show $y \in \overline{\alpha}$, it suffices to observe that by Lemma 4(a), $y_e^s > 0$ implies $y_e^s \in \overline{\alpha_s}$; so $y \in \overline{\alpha_s}$ for cofinitely many s, which can happen only if $y \in \overline{\alpha}$.

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LEMMA 20. Let

$$egin{array}{ll} y(e) &= 0 & ext{if} \; e
otin \gamma, \ &= \lim_s y^s_s \; \; ext{if} \; e
otin \gamma \,. \end{array}$$

Then y(e) is defined for all e, and y(e) is recursive in γ .

Proof. It follows from Lemma 19 that y(e) is defined for all e. To compute y(e) recursively in γ , ask first whether $e \in \gamma$. If not, y(e) = 0. If $e \in \gamma$, it remains to compute $\lim_{s} y_{e}^{s}$. But this can be done recursively in 0', since y_{e}^{s} is a recursive function of s, and

$$y = \lim_{s} y_{e}^{s} \longleftrightarrow (\exists s) (\forall t) (t \ge s \longrightarrow y = y_{e}^{t})$$
$$\longleftrightarrow (\forall s) (\exists t) (t \ge s \text{ and } y = y_{e}^{t})$$

Since by hypothesis $\gamma \in c$ and $0' \leq c$, y(e) can be computed recursively in γ .

LEMMA 21. $W_e \cap \overline{\alpha} \neq \emptyset$ if and only if there is a z in W_e satisfying one of the following conditions:

- (a) $(\exists i)_{i \leq e} (z \text{ is permanently restrained by } i),$
- (b) $(\exists i)_{i\leq e} (i \in \gamma \text{ and } z = \lim_{s} y_i^s),$
- (c) $(\exists i)_{i\leq e} \ (i\in \overline{\beta} \ and \ z\in D_{g(i)}).$

Proof. (\leftarrow) Assume that for some z in W_e , (a), (b) or (c) holds. If z is permanently restrained by i, then $z \in \overline{\alpha}$ by Lemma 17(a). If $z = \lim_s y_i^s$ for some $i \in \gamma$, then $z \in \overline{\alpha}$ by Lemma 19. If $z \in D_{g(i)}$ for $i \in \overline{\beta}, z = 2x$ for some x > 0 and $i = e_x \in \overline{\beta}$ by Definition 2. Now $0 < z \in \alpha$ only if $z \in E_s \cup O_s$ for some s. But $z \in E_s$ implies $e_x \in \beta$, and $z \in O_s$ implies z is odd. Since neither is the case, $z \notin E_s \cup O_s$ and $z \in \overline{\alpha}$. Thus in all cases, $z \in \overline{\alpha} \cap W_e$ and $W_e \cap \overline{\alpha} \neq \emptyset$.

 (\rightarrow) Assume $W_e \cap \overline{\alpha} \neq \emptyset$. Let z be the least element of $W_e \cap \overline{\alpha}$, and let s' be the least s such that $z \in W_e^s$, then $z \in W_e^s \cap \overline{\alpha}_s$ for all $s \ge s'$. Since $0 \in \alpha$, z > 0 and since z is least, it follows that $(\forall y)_{y < z} (y \in W_e \rightarrow y \in \alpha)$. Thus for each y < z such that $y \in W_e$ there is a stage s(y) such that $y \in \alpha^{s(y)}$. Choose $t \ge s'$, $\max_{y < z} s(y)$.

Case 1. $(\forall s) (s \ge t \rightarrow z_e^s > 0)$.

Let $y = z_{\epsilon}^{t}$. Then y > 0 and by Lemma 15(a), $y \in W_{\epsilon}$. Suppose that $z_{\epsilon}^{s} \neq y$ for some s > t, and let s be least. Then

$$0 < y = z^{s-1}_e
eq z^s_e$$

which by Lemma 8(a) implies $z_e^s = 0$, which contradicts the assump-

tion since s > t. It follows that $y = \lim_{s} z_{e}^{s}$ so that $y \in W_{e}$ is permanently restrained by e. So (a) holds.

Case 2. $(\exists s) (s \ge t \text{ and } z_e^s = 0).$

Let t'-1 be any such s; so t' > t and $z_e^{t'-1} = 0$. If $t' \not\equiv 2 \pmod{3}$ then by the construction $z_e^{t'} = 0$ also, so we may as well assume $t' \equiv 2 \pmod{3}$. Since $t'-1 \ge t \ge s'$ we have $z \in W_e^{t'} \cap \overline{\alpha_{t'}}$. We consider several subcases.

Subcase 2.1. $(\forall i)_{i \leq e} \ (z \neq y_i^{t'-1})$ and $(\forall x)_{x < z} \ (z = 2x \rightarrow e_x > e)$.

Now since $t' > t > \max_{y < z} s(y)$ it follows that no y < z is in $W_{\epsilon}^{t'} \cap \overline{\alpha_{t'}}$. So z is the least z satisfying the requirements of the construction for choosing a new value of $z_{\epsilon}^{t'}$ when $t' \equiv 2 \pmod{3}$. So $z = z_{\epsilon}^{t'}$. Since by hypothesis $z \in \overline{\alpha}$, this implies by Lemma 17(d) that z is permanently restrained by e. So (a) holds.

Subcase 2.2. $(\exists i)_{i \leq e} (z = y_i^{t'-1})$ or $(\exists x)_{x < z} (z = 2x \text{ and } e_x \leq e)$.

Subcase 2.2.1. $(\exists i)_{i \leq e} (z = y_i^{t'-1}).$

Let *i* be least, $i \leq e$. If $z = \lim_{s} y_i^s$ then by Lemma 19, $z \in \overline{\alpha}$ implies $i \in \gamma$, in which case (b) holds. If $z \neq \lim_{s} y_i^s$, then $(\exists s) (s \geq t'$ and $z \neq y_i^s)$; let *s* be least. Then $0 < z = y_i^{s-1} \neq y_i^s$ which by Lemma 3(b) implies $z \in O_s \subset \alpha$ or $z = z_j^s$ for some j < i. The former cannot happen, since $z \in \overline{\alpha}$; so $z = z_j^s$ for $j < i \leq e$. Then by Lemma 17(d), *z* is permanently restrained by j < e, and (a) holds.

Subcase 2.2.2. $(\forall i)_{i \leq e} (z \neq y_i^{t'-1}) \text{ and } (\exists x)_{x < z} (z = 2x \text{ and } e_x \leq e).$ Let x = z/2; then $e_x \leq e$ and $z \in D_{g(e_x)}$ by Definition 2. If $e_x \in \overline{\beta}$, then (c) holds with $i = e_x$. Now assume $e_x \in \beta$. Then $e_x \in \beta^s$ for some s; let s" be the least such s. Choose t" so large that

$$t'' > \max\{s'', t', e_x\}$$

and $t'' \equiv 1 \pmod{3}$. Suppose that $(\forall i)_{i < e_x} (z \neq z_i^{t''-1})$. Then since $e_x \leq t''$ and $e_x \in \beta^{s''} \subset \beta^{t''}$, if follows that $z \in E_{t''} \subset \alpha$, which contradicts the hypothesis that $z \in \overline{\alpha}$. So $z = z_i^{t''-1}$ for some $i < e_x$. But then it again follows by Lemma 17(d) that z is permanently restrained by $i < e_x \leq e$, and (a) holds.

This completes the proof of the lemma.

LEMMA 22. $\theta A^{\alpha} \leq T^{\gamma}$.

Proof. We show how to decide for each e whether $W_e \subset \alpha$, recursively in γ . By Lemma 18, there is a function z(i) recursive in

0' such that z is permanently restrained by i if and only if

$$z = z(i) > 0$$
.

By Lemma 20, there is a function y(i) recursive in γ such that for each *i*, $y(i) = \lim_{s} y_{i}^{s}$ if $i \in \gamma$. Let

$$T_e = \{z(i) \mid i \leq e \text{ and } z(i) > 0\},\ U_e = \{y(i) \mid i < e \text{ and } i \in \gamma\},\ V_e = \{z \mid (\exists i)_{i < e} (i \in \overline{eta} \text{ and } z \in D_{g(i)})\}.$$

The sets T_e , U_e , V_e are evidently finite. The membership of T_e can be completely determined recursively in 0' and hence in γ since $\gamma \in c$ and $0' \leq c$. The membership of U_e can be determined recursively in γ . The membership of V_e can be obtained recursively in β and hence recursively in γ , since $\beta \in b \leq 0' \leq c$. Since this can all be done uniformly given e, there is a function q(e), recursive in γ , such that $T_e \cup U_e \cup V_e = D_{q(e)}$, for each e. Now by Lemma 21,

$$W_e \cap \overline{lpha} \neq arnothing \longleftrightarrow (\exists z) \ (z \in W_e \quad ext{and} \quad z \in T_e \cup U_e \cup V_e) \ \longleftrightarrow V_e \cap D_{q(e)} \neq arnothing \mathbf{.}$$

But once q(e) is known, the question of whether $W_e \cap D_{q(e)} = \emptyset$ can be answered recursively in 0', and hence recursively in γ .

LEMMA 23. For each e, let $T_e = \{z \mid z \text{ is permanently restrained by some } i < e\},$ $R_e = \{y \mid y > 0 \text{ and } (\exists s) (y = y_e^s)\}.$ Then $e \in \gamma$ if and only if $R_e \cap \overline{\alpha} \cap \overline{T}_e \neq \emptyset.$

Proof. (\rightarrow) . Asume $e \in \gamma$. Then by Lemma 19, $\lim_s y_e^s = y$ exists and $y \in \overline{\alpha}$. Clearly $y \in R_e$, so it suffices to show $y \notin T_e$. Suppose y is permanently restrained by some i < e. Then $y = z_i^s$ for cofinitely many s. But $y = \lim_s y_e^s$ implies $y = y_e^s$ for cofinitely many s; while by Lemma 14, $0 < y = z_i^s$ implies $y \neq y_e^s$ for all s, which is a contradiction. So $y \in R_e \cap \overline{\alpha} \cap \overline{T}_e$.

 $(\leftarrow) \text{ Assume } R_e \cap \bar{\alpha} \cap \bar{T}_e \neq \emptyset, \text{ and let } y \text{ be the least element} \\ \text{ of } R_e \cap \bar{\alpha} \cap \bar{T}_e. \text{ Now } y \in R_e \text{ implies } 0 < y = y_e^t \text{ for some } t. \text{ Since } \\ y \in \bar{\alpha} \text{ it suffices by Lemma 19 to show } y = \lim_s y_e^s. \text{ Suppose not; then} \\ \text{ for some } s > t, \ y_e^s \neq y_e^t = y; \text{ and if } s \text{ is least, } 0 < y = y_e^{s-1} \neq y_e^s. \\ \text{ Then by Lemma 3(b), either } y \in O_s \subset \alpha_s \text{ or } y = z_i^s \text{ for some } i < e. \text{ But } \\ y \in \alpha_s \text{ contradicts the hypothesis that } y \in \bar{\alpha}; \text{ while } 0 < y = z_i^s \text{ for } i < e. \end{cases}$

implies by Lemma 17(d) that y is permanently restrained by i < e, contradicting the hypothesis that $y \notin T_e$. So $y = \lim_s y_e^s$ and $y \in \overline{\alpha}$ which implies $e \in \gamma$.

LEMMA 24. $\gamma \leq T \theta A^{\alpha}$.

Proof. Let

 $T_e = \{z \text{ is permanently restrained by some } i < e\}$,

 $R_{\scriptscriptstyle e} = \{y \,|\, y > 0 \quad ext{and} \quad (\exists s) \; (y = y_{\scriptscriptstyle e}^{\scriptscriptstyle s})\}$.

By Lemma 23, $e \in \gamma$ if and only if $R_e \cap \overline{\alpha} \cap \overline{T}_e \neq \emptyset$. So it suffices to show that we can decide whether $R_e \cap \overline{\alpha} \cap \overline{T}_e = \emptyset$, recursively in θA^{α} . By Lemma 13, $\beta \leq {}_{T}\alpha$. This implies α is nonrecursive since β is nonrecursive. In particular, $\alpha \neq \emptyset$ or N, so A^{α} is nontrivial and by Rice's theorem, [4], 0' is recursive in θA^{α} . Since y_e^s is a recursive function of s, R_e is an r.e. set. By Lemma 18,

$$T_e = \{z(i) \mid i < e \text{ and } z(i) > 0\}$$

where z(i) is recursive in 0' and thus in θA^{α} . Thus the membership of the finite set T_e can be completely determined, recursively in θA^{α} , and since this can be done uniformly in e, there is a function p(e)recursive in θA^{α} such that $W_{p(e)} = R_e - T_e$, for each e. Then

which can be decided recursively in θA^{α} once p(e) is known. This completes the proof of the lemma.

It follows from Lemmas 11 and 13 that $\alpha \in b$, and by Lemmas 22 and 24 that $\theta A^{\alpha} \in c$. Since α is r.e., this completes the proof of the theorem.

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Received January 2, 1972. Research supported in part by National Science Foundation Grant GP-19958.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

University of Washington Seattle, Washington 98105

R. A. BEAUMONT

Pacific Journal of Mathematics Vol. 46, No. 1 November, 1973

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