

# Pacific Journal of Mathematics

**A DECOMPOSITION FOR  $B(X)^*$  AND UNIQUE  
HAHN-BANACH EXTENSIONS**

JULIEN O. HENNEFELD

## A DECOMPOSITION FOR $B(X)^*$ AND UNIQUE HAHN-BANACH EXTENSIONS

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For a Banach space  $X$ , let  $B(X)$  be the space of all bounded linear operators on  $X$ , and  $\mathcal{C}$  the space of all compact linear operators on  $X$ . In general, the norm-preserving extension of a linear functional in the Hahn-Banach theorem is highly non-unique. The principal result of this paper is that, for  $X = c_0$  or  $l^p$  with  $1 < p < \infty$ , each bounded linear functional on  $\mathcal{C}$  has a unique norm-preserving extension to  $B(X)$ . This is proved by using a decomposition theorem for  $B(X)^*$ , which takes on a special form for  $X = c_0$  or  $l^p$  with  $1 < p < \infty$ .

1. DEFINITION 1.1. A basis  $\{e_i\}$  for a Banach space  $X$  having coefficient functionals  $e_i^*$  in  $X^*$  is called unconditional if, for each  $x$ ,  $\sum_{i=1}^{\infty} e_i^*(x)e_i$  converges unconditionally. The basis is called monotone if  $\|U_m x\| < \|x\|$  for all  $x \in X$  and positive integers  $m$ , where  $U_m x = \sum_{i=1}^m e_i^*(x)e_i$ .

PROPOSITION 1.2. If  $X$  has a monotone, unconditional basis  $\{e_i\}$ , then  $B(X)^* = \mathcal{C}^* + \mathcal{C}^\perp$ , where  $\mathcal{C}^*$  is a subspace of  $B(X)^*$  isomorphically isometric to the space of bounded linear functionals on  $\mathcal{C}$ , and  $\mathcal{C}^\perp$  annihilates  $\mathcal{C}$ . Furthermore, the associated projection from  $B(X)^*$  onto  $\mathcal{C}^*$  has unit norm.

*Proof.* If  $T \in B(X)$ , then  $T(x) = \sum_{i=1}^{\infty} f_i^T(x)e_i$  for each  $x \in X$ , where  $f_i^T \in X^*$ . For each  $T$  and  $i$ , let  $T_i$  be defined by  $T_i(x) = f_i^T(x)e_i$  for all  $x$ . Also, for each  $F \in B(X)^*$ , define  $G \in B(X)^*$  by  $G(T) = \sum_{i=1}^{\infty} F(T_i)$ . Note that this sum converges. Otherwise, we have  $\sum_{i=1}^{\infty} |F(T_i)| = \lim_{n \rightarrow \infty} F[\sum_{i=1}^n SgF(T_i) \cdot T_i] = +\infty$ , and then

$$\lim_{n \rightarrow \infty} \|\sum_{i=1}^n SgF(T_i) \cdot T_i\| = \infty.$$

Then by using an absolutely convergent series, it is easy to construct an element  $y \in X$ :  $\lim_{n \rightarrow \infty} \|\sum_{i=1}^n SgF(T_i) \cdot T_i(y)\| = \infty$ . Therefore,  $\sum_{i=1}^{\infty} f_i^T(y)e_i$  converges while  $\sum_{i=1}^{\infty} SgF(T_i) \cdot f_i^T(y)e_i$  does not, which contradicts the fact that an unconditionally convergent series is bounded multiplier convergent. See [3], p. 19.

Note that the norm of  $G$  restricted to  $\mathcal{C}$  is equal to the norm of  $G$  on  $B(X)$ , since by monotonicity  $\|\sum_{i=1}^n T_i\| \leq \|T\|$  for each  $n$  and  $T \in B(X)$ . Also,  $F$  and  $G$  agree on  $\mathcal{C}$ , because  $\mathcal{C}$  is the closure of the set of all  $T$  for which only a finite number of the  $f_i^T$  are non-zero. Hence the projection defined by  $PF = G$  has unit norm, since

$$\|F\|_{B(X)} \geq \|F\|_{\mathcal{E}} = \|G\|_{\mathcal{E}} = \|G\|_{B(X)}.$$

**COROLLARY 1.3.** *If  $X$  has an unconditional basis  $\{e_i\}$ , then there is a bounded projection from  $B(X)^*$  onto a subspace isomorphic to  $\mathcal{E}^*$ .*

*Proof.* Renorm  $X$  so that the basis  $\{e_i\}$  is monotone. See [1], p. 73.

**2. THEOREM 2.1.** *Let  $X$  have an unconditional, shrinking basis  $\{e_i\}$ , for which there is a function  $N$  of two real variables such that:*

- (i)  $N(a, b) \leq N(\alpha, \beta)$  if  $0 \leq a \leq \alpha$  and  $0 \leq b \leq \beta$ ;
- (ii)  $N(\|x\|, \|y\|) = \|x + y\|$  for which  $x = \sum_{i=1}^n a_i e_i$  and  $y = \sum_{i=n+1}^{\infty} a_i e_i$ . Then for each  $F \in B(X)^*$ ,  $\|F\| = \|G\| + \|H\|$ , where  $F = G + H$  with  $G \in \mathcal{E}^*$  and  $H \in \mathcal{E}^\perp$ .

*Proof.* Note that the existence of  $N$  implies that the basis is monotone, and so we have a decomposition for  $B(X)^*$ . The operators whose matrices have a finite number of nonzero entries form a dense subset of  $\mathcal{E}$ . Hence, for  $\varepsilon > 0$ , there exists an operator  $D$  of unit norm whose image lies in the subspace  $[e_1, e_2, \dots, e_m]$ , and whose kernel contains  $[e_{m+1}, e_{m+2}, \dots]$ :  $G(D) > \|G\| - \varepsilon/3$ . Also, there exists an operator  $T \in B(X)$  of unit norm:  $H(T) > \|H\| - \varepsilon/3$ . Let  $Q_r$  be the projection onto  $[e_{r+1}, e_{r+2}, \dots]$ . Define  $T^{(r)}x = \sum_{i=r+1}^{\infty} f_i^T(Qx)e_i$ . Note that the matrix for  $T^{(r)}$  is simply the matrix for  $T$ , with the first  $r$ -rows and  $r$ -columns replaced by zeros.

Then  $\lim_{r \rightarrow \infty} G(T^{(r)}) = 0$ . To see this, first note that the existence of  $N$  and the basis being shrinking imply that the functionals in  $\mathcal{E}^*$  with a finite number of nonzero entries form a dense subset of  $\mathcal{E}^*$ . See [2], Propositions 3.1 and 3.3. Thus, for any  $\delta > 0$ ,  $\exists J \in B(X)^*$ , for which  $\|J - G\| < \delta$  and:  $\lim_{r \rightarrow \infty} J(T^{(r)}) = 0$ . Hence  $\lim_{r \rightarrow \infty} G(T^{(r)}) = 0$ .

Then pick  $r > m$ :  $|G(T^{(r)})| < \varepsilon/3$ . Observe that  $\|D + T^{(r)}\| = 1$ , since and  $z \in X$  can be written as  $z = x + y$  where  $x \in [e_1, \dots, e_r]$  and  $y \in [e_{r+1}, \dots]$ . Then

$$\begin{aligned} \|(D + T^{(r)})(x + y)\| &= \|Dx + T^{(r)}y\| = N(\|Dx\|, \|T^{(r)}y\|) \\ &\leq N(\|x\|, \|y\|) = \|x, y\|. \end{aligned}$$

Using the fact that  $H$  annihilates  $\mathcal{E}$ , we have

$$\begin{aligned} F(D + T^{(r)}) &= G(D) + G(T^{(r)}) + H(T^{(r)}) > \|G\| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} + \|H\| - \frac{\varepsilon}{3} \\ &= \|G\| + \|H\| - \varepsilon. \end{aligned}$$

Hence  $\|F\| = \|G\| + \|H\|$ .

**COROLLARY 2.2.** *If  $X$  is  $(c_0)$  or  $l^p$  with  $1 < p < \infty$ , then, for each  $F \in B(X)^*$ ,  $\|F\| = \|G\| + \|H\|$ , where  $F = G + H$  with  $G \in \mathcal{E}^*$  and  $H \in \mathcal{E}^\perp$ .*

*Proof.* Let  $\{e_i\}$  be the standard basis. Let  $N(a, b) = [ |a|^p + |b|^p ]^{1/p}$  for  $l^p$ . Let  $N(a, b) = \text{Max}(|a|, |b|)$  for  $c_0$ .

**THEOREM 2.3.** *Each bounded linear functional on  $\mathcal{E}$  has a unique normpreserving extension to  $B(X)$  for  $X = c_0$  or  $l^p$  with  $1 < p < \infty$ .*

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