A DECOMPOSITION FOR $B(X)^*$ AND UNIQUE HAHN-BANACH EXTENSIONS

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For a Banach space $X$, let $B(X)$ be the space of all bounded linear operators on $X$, and $C$ the space of all compact linear operators on $X$. In general, the norm-preserving extension of a linear functional in the Hahn-Banach theorem is highly non-unique. The principal result of this paper is that, for $X = c_0$ or $l^p$ with $1 < p < \infty$, each bounded linear functional on $C$ has a unique norm-preserving to $B(X)$. This is proved by using a decomposition theorem for $B(X)^*$, which takes on a special form for $X = c_0$ or $l^p$ with $1 < p < \infty$.

1. DEFINITION 1.1. A basis $\{e_i\}$ for a Banach space $X$ having coefficient functionals $e_i^*$ in $X^*$ is called unconditional if, for each $x$, $\sum_{i=1}^{\infty} e_i^*(x)e_i$ converges unconditionally. The basis is called monotone if $\|U_m x\| < \|x\|$ for all $x \in X$ and positive integers $m$, where $U_m x = \sum_{i=1}^{m} e_i^*(x)e_i$.

PROPOSITION 1.2. If $X$ has a monotone, unconditional basis $\{e_i\}$, then $B(X)^* = C^* + C^\perp$, where $C^*$ is a subspace of $B(X)^*$ isometrically isometric to the space of bounded linear functionals on $C$, and $C^\perp$ annihilates $C$. Furthermore, the associated projection from $B(X)^*$ onto $C^*$ has unit norm.

Proof. If $T \in B(X)$, then $T(x) = \sum_{i=1}^{\infty} f_i^*(x)e_i$ for each $x \in X$, where $f_i^* \in X^*$. For each $T$ and $i$, let $T_i$ be defined by $T_i(x) = f_i^*(x)e_i$ for all $x$. Also, for each $F \in B(X)^*$, define $G \in B(X)^*$ by $G(T) = \sum_{i=1}^{\infty} F(T_i)$. Note that this sum converges. Otherwise, we have $\sum_{i=1}^{\infty} F(T_i) = \lim_{n \to \infty} F[\sum_{i=1}^{n} SgF(T_i) \cdot T_i] = +\infty$, and then

$$\lim_{n \to \infty} \|\sum_{i=1}^{n} SgF(T_i) \cdot T_i\| = \infty.$$ 

Then by using an absolutely convergent series, it is easy to construct an element $y \in X$: $\lim_{n \to \infty} \|\sum_{i=1}^{n} SgF(T_i) \cdot T_i(y)\| = \infty$. Therefore, $\sum_{i=1}^{\infty} f_i^*(y)e_i$ converges while $\sum_{i=1}^{\infty} SgF(T_i) \cdot f_i^*(y)e_i$ does not, which contradicts the fact that an unconditionally convergent series is bounded multiplier convergent. See [3], p. 19.

Note that the norm of $G$ restricted to $C$ is equal to the norm of $G$ on $B(X)$, since by monotonicity $\|\sum_{i=1}^{n} T_i\| \leq \|T\|$ for each $n$ and $T \in B(X)$. Also, $F$ and $G$ agree on $C$, because $C$ is the closure of the set of all $T$ for which only a finite number of the $f_i^*$ are non-zero. Hence the projection defined by $PF = G$ has unit norm, since
\[ ||F||_{B(X)} \geq ||F||_\psi = ||G||_\psi = ||G||_{B(X)^*}. \]

**Corollary 1.3.** If \( X \) has an unconditional basis \( \{e_i\} \), then there is a bounded projection from \( B(X)^* \) onto a subspace isomorphic to \( \mathcal{C}^* \).

**Proof.** Renorm \( X \) so that the basis \( \{e_i\} \) is monotone. See [1], p. 73.

**Theorem 2.1.** Let \( X \) have an unconditional, shrinking basis \( \{e_i\} \), for which there is a function \( N \) of two real variables such that:

(i) \( N(a, b) \leq N(\alpha, \beta) \) if \( 0 \leq a \leq \alpha \) and \( 0 \leq b \leq \beta \);

(ii) \( N(||x||, ||y||) = ||x + y|| \) for which \( x = \sum_{i=1}^n a_i e_i \) and \( y = \sum_{i=n+1}^\infty a_i e_i \). Then for each \( F \in B(X)^* \), \( ||F|| = ||G|| + ||H|| \), where \( F = G + H \) with \( G \in \mathcal{C}^* \) and \( H \in \mathcal{C}^1 \).

**Proof.** Note that the existence of \( N \) implies that the basis is monotone, and so we have a decomposition for \( B(X)^* \). The operators whose matrices have a finite number of nonzero entries form a dense subset of \( \mathcal{C}^* \). Hence, for \( \varepsilon > 0 \), there exists an operator \( D \) of unit norm whose image lies in the subspace \( [e_1, e_2, \ldots, e_m] \), and whose kernel contains \( [e_{m+1}, e_{m+2}, \ldots] \): \( ||D|| < \varepsilon/3 \). Also, there exists an operator \( T \in B(X) \) of unit norm: \( ||T|| > ||H|| - \varepsilon/3 \). Let \( Q_r \) be the projection onto \( [e_{r+1}, e_{r+2}, \ldots] \). Define \( T^{(r)}(x) = \sum_{i=r+1}^\infty f_i(Q_r x) e_i \). Note that the matrix for \( T^{(r)} \) is simply the matrix for \( T \), with the first \( r \)-rows and \( r \)-columns replaced by zeros.

Then \( \lim_{r \to \infty} G(T^{(r)}) = 0 \). To see this, first note that the existence of \( N \) and the basis being shrinking imply that the functionals in \( \mathcal{C}^* \) with a finite number of nonzero entries form a dense subset of \( \mathcal{C}^* \). See [2], Propositions 3.1 and 3.3. Thus, for any \( \delta > 0 \), \( \exists J \in B(X)^* \), for which \( ||J - G|| < \delta \) and: \( \lim_{r \to \infty} J(T^{(r)}) = 0 \). Hence \( \lim_{r \to \infty} G(T^{(r)}) = 0 \).

Then pick \( r > m: ||G(T^{(r)})|| < \varepsilon/3 \). Observe that \( ||D + T^{(r)}|| = 1 \), since and \( z \in X \) can be written as \( z = x + y \) where \( x \in [e_1, \ldots, e_r] \) and \( y \in [e_{r+1}, \ldots] \). Then

\[ ||(D + T^{(r)})(x + y)|| = ||Dx + T^{(r)}y|| = N(||Dx||, ||T^{(r)}y||) \leq N(||x||, ||y||) = ||x, y||. \]

Using the fact that \( H \) annihilates \( \mathcal{C}^* \), we have

\[ F(D + T^{(r)}) = G(D) + G(T^{(r)}) + H(T^{(r)}) > ||G|| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} + ||H|| - \frac{\varepsilon}{3} \]

\[ = ||G|| + ||H|| - \varepsilon. \]

Hence \( ||F|| = ||G|| + ||H||. \)
Corollary 2.2. If X is $(c_0)$ or $l^p$ with $1 < p < \infty$, then, for each $F \in B(X)^*$, $\|F\| = \|G\| + \|H\|$, where $F = G + H$ with $G \in \mathscr{G}$ and $H \in \mathbb{C}$.

Proof. Let $\{e_i\}$ be the standard basis. Let $N(a, b) = (\|a^p\| + \|b^p\|)^{1/p}$ for $l^p$. Let $N(a, b) = \max(|a|, |b|)$ for $c_0$.

Theorem 2.3. Each bounded linear functional on $\mathscr{G}$ has a unique norm-preserving extension to $B(X)$ for $X = c_0$ or $l^p$ with $1 < p < \infty$.

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