Pacific Journal of Mathematics

A DECOMPOSITION FOR $B(X)^*$ AND UNIQUE HAHN-BANACH EXTENSIONS

JULIEN O. HENNEFELD

Vol. 46, No. 1 November 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 1, 1973

A DECOMPOSITION FOR $B(X)^*$ AND UNIQUE HAHN-BANACH EXTENSIONS

JULIEN HENNEFELD

For a Banach space X, let $B(X)$ be the space of all bounded linear operators on X, and $\mathscr{C}$ the space of all compact linear operators on X. In general, the norm-preserving extension of a linear functional in the Hahn-Banach theorem is highly non-unique. The principal result of this paper is that, for $X = c_0$ or l^p with $1 < p < \infty$, each bounded linear functional on $\mathscr{C}$ has a unique norm-preserving to $B(X)$. This is proved by using a decomposition theorem for $B(X)^*$, which takes on a special form for $X = c_0$ or l^p with $1 < p < \infty$.

1. DEFINITION 1.1. A basis $\{e_i\}$ for a Banach space X having coefficient functionals e_i^* in X^* is called unconditional if, for each x, $\sum_{i=1}^{\infty} e_i^*(x)e_i$ converges unconditionally. The basis is called monotone if $||U_m x|| < ||x||$ for all $x \in X$ and positive integers m, where $U_m x = \sum_{i=1}^{m} e_i^*(x)e_i$.

PROPOSITION 1.2. *If X has a monotone, unconditional basis $\{e_i\}$, then $B(X)^* = \mathscr{C}^* + \mathscr{C}^\perp$, where $\mathscr{C}^*$ is a subspace of $B(X)^*$ isomorphically isometric to the space of bounded linear functionals on $\mathscr{C}$, and $\mathscr{C}^\perp$ annihilates $\mathscr{C}$. Furthermore, the associated projection from $B(X)^*$ onto $\mathscr{C}^*$ has unit norm.*

Proof. If $T \in B(X)$, then $T(x) = \sum_{i=1}^{\infty} f_i^T(x)e_i$ for each $x \in X$, where $f_i^T \in X^*$. For each T and i, let T_i be defined by $T_i(x) = f_i^T(x)e_i$ for all x. Also, for each $F \in B(X)^*$, define $G \in B(X)^*$ by $G(T) = \sum_{i=1}^{\infty} F(T_i)$. Note that this sum converges. Otherwise, we have $\sum_{i=1}^{\infty} |F(T_i)| = \lim_{n\to\infty} F[\sum_{i=1}^{\infty} SgF(T_i)\cdot T_i] = +\infty$, and then

$$\lim_{n\to\infty} ||\textstyle\sum_{i=1}^{n} SgF(T_i)\cdot T_i|| = \infty .$$

Then by using an absolutely convergent series, it is easy to construct an element $y \in X$: $\lim_{n\to\infty} ||\sum_{i=1}^{n} SgF(T_i)\cdot T_i(y)|| = \infty$. Therefore, $\sum_{i=1}^{\infty} f_i^T(y)e_i$ converges while $\sum_{i=1}^{\infty} SgF(T_i)\cdot f_i^T(y)e_i$ does not, which contradicts the fact that an unconditionally convergent series is bounded multiplier convergent. See [3], p. 19.

Note that the norm of G restricted to $\mathscr{C}$ is equal to the norm of G on $B(X)$, since by monotonicity $||\sum_{i=1}^{n} T_i|| \leqq ||T||$ for each n and $T \in B(X)$. Also, F and G agree on $\mathscr{C}$, because $\mathscr{C}$ is the closure of the set of all T for which only a finite number of the f_i^T are nonzero. Hence the projection defined by $PF = G$ has unit norm, since

$||F||_{B(X)} \geqq ||F||_{\mathscr{C}} = ||G||_{\mathscr{C}} = ||G||_{B(X)}$.

COROLLARY 1.3. *If X has an unconditional basis $\{e_i\}$, then there is a bounded projection from $B(X)^*$ onto a subspace isomorphic to $\mathscr{C}^*$.*

Proof. Renorm X so that the basis $\{e\}_i$ is monotone. See [1], p. 73.

2. THEOREM 2.1. *Let X have an unconditional, shrinking basis $\{e_i\}$, for which there is a function N of two real variables such that:*
(i) *$N(a, b) \leqq N(\alpha, \beta)$ if $0 \leqq a \leqq \alpha$ and $0 \leqq b \leqq \beta$;*
(ii) *$N(||x||, ||y||) = ||x + y||$ for which $x = \sum_{i=1}^{n} a_i e_i$ and $y = \sum_{n+1}^{\infty} a_i e_i$. Then for each $F \in B(X)^*$, $||F|| = ||G|| + ||H||$, where $F = G + H$ with $G \in \mathscr{C}^*$ and $H \in \mathscr{C}^{\perp}$.*

Proof. Note that the existence of N implies that the basis is monotone, and so we have a decomposition for $B(X)^*$. The operators whose matrices have a finite number of nonzero entries form a dense subset of $\mathscr{C}$. Hence, for $\varepsilon > 0$, there exists an operator D of unit norm whose image lies in the subspace $[e_1, e_2, \cdots, e_m]$, and whose kernel contains $[e_{m+1}, e_{m+2}, \cdots]$: $G(D) > ||G|| - \varepsilon/3$. Also, there exists an operator $T \in B(X)$ of unit norm: $H(T) > ||H|| - \varepsilon/3$. Let Q_r be the projection onto $[e_{r+1}, e_{r+2}, \cdots]$. Define $T^{(r)}x = \sum_{i=r+1}^{\infty} f_i^T(Qx)e_i$. Note that the matrix for $T^{(r)}$ is simply the matrix for T, with the first r-rows and r-columns replaced by zeros.

Then $\lim_{r\to\infty} G(T^{(r)}) = 0$. To see this, first note that the existence of N and the basis being shrinking imply that the functionals in $\mathscr{C}^*$ with a finite number of nonzero entries form a dense subset of $\mathscr{C}^*$. See [2], Propositions 3.1 and 3.3. Thus, for any $\delta > 0$, $\exists J \in B(X)^*$, for which $||J - G|| < \delta$ and: $\lim_{r\to\infty} J(T^{(r)}) = 0$. Hence $\lim_{r\to\infty} G(T^{(r)}) = 0$.

Then pick $r > m$: $|G(T^{(r)})| < \varepsilon/3$. Observe that $||D + T^{(r)}|| = 1$, since and $z \in X$ can be written as $z = x + y$ where $x \in [e_1, \cdots, e_r]$ and $y \in [e_{r+1}, \cdots]$. Then

$$||(D + T^{(r)})(x + y)|| = ||Dx + T^{(r)}y|| = N(||Dx||, ||T^{(r)}y||)$$
$$\leqq N(||x||, ||y||) = ||x, y|| .$$

Using the fact that H annihilates $\mathscr{C}$, we have

$$F(D + T^{(r)}) = G(D) + G(T^{(r)}) + H(T^{(r)}) > ||G|| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} + ||H|| - \frac{\varepsilon}{3}$$
$$= ||G|| + ||H|| - \varepsilon .$$

Hence $||F|| = ||G|| + ||H||$.

COROLLARY 2.2. *If* X *is* (c_0) *or* l^p *with* $1 < p < \infty$, *then, for each* $F \in B(X)^*$, $\|F\| = \|G\| + \|H\|$, *where* $F = G + H$ *with* $G \in \mathscr{C}^*$ *and* $H \in \mathscr{C}^\perp$.

Proof. Let $\{e_i\}$ be the standard basis. Let $N(a, b) = [|a^p| + |b^p|]^{1/p}$ for l^p. Let $N(a, b) = \mathrm{Max}(|a|, |b|)$ for c_0.

THEOREM 2.3. *Each bounded linear functional on* $\mathscr{C}$ *has a unique normpreserving extension to* $B(X)$ *for* $X = c_0$ *or* l^p *with* $1 < p < \infty$.

REFERENCES

1. M. M. Day, *Normed Linear Spaces*, Academic Press, New York, 1962.
2. J. Hennefeld, *The Arens products and an imbedding theorem*, Pacific J. Math., **29** (1969), 551-563.
3 J. T. Marti, *Introduction to the Theory of Basis*, Springer-Verlag, New York, 1969.

Received December 15, 1971. The author wishes to thank the referee for streamlining the proof of the basic result.

BOSTON COLLEGE

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 46, No. 1 November, 1973