

Pacific Journal of Mathematics

**A DECOMPOSITION FOR $B(X)^*$ AND UNIQUE
HAHN-BANACH EXTENSIONS**

JULIEN O. HENNEFELD

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For a Banach space X , let $B(X)$ be the space of all bounded linear operators on X , and \mathcal{C} the space of all compact linear operators on X . In general, the norm-preserving extension of a linear functional in the Hahn-Banach theorem is highly non-unique. The principal result of this paper is that, for $X = c_0$ or l^p with $1 < p < \infty$, each bounded linear functional on \mathcal{C} has a unique norm-preserving extension to $B(X)$. This is proved by using a decomposition theorem for $B(X)^*$, which takes on a special form for $X = c_0$ or l^p with $1 < p < \infty$.

1. DEFINITION 1.1. A basis $\{e_i\}$ for a Banach space X having coefficient functionals e_i^* in X^* is called unconditional if, for each x , $\sum_{i=1}^{\infty} e_i^*(x)e_i$ converges unconditionally. The basis is called monotone if $\|U_m x\| < \|x\|$ for all $x \in X$ and positive integers m , where $U_m x = \sum_{i=1}^m e_i^*(x)e_i$.

PROPOSITION 1.2. *If X has a monotone, unconditional basis $\{e_i\}$, then $B(X)^* = \mathcal{C}^* + \mathcal{C}^\perp$, where \mathcal{C}^* is a subspace of $B(X)^*$ isomorphically isometric to the space of bounded linear functionals on \mathcal{C} , and \mathcal{C}^\perp annihilates \mathcal{C} . Furthermore, the associated projection from $B(X)^*$ onto \mathcal{C}^* has unit norm.*

Proof. If $T \in B(X)$, then $T(x) = \sum_{i=1}^{\infty} f_i^T(x)e_i$ for each $x \in X$, where $f_i^T \in X^*$. For each T and i , let T_i be defined by $T_i(x) = f_i^T(x)e_i$ for all x . Also, for each $F \in B(X)^*$, define $G \in B(X)^*$ by $G(T) = \sum_{i=1}^{\infty} F(T_i)$. Note that this sum converges. Otherwise, we have $\sum_{i=1}^{\infty} |F(T_i)| = \lim_{n \rightarrow \infty} F[\sum_{i=1}^n SgF(T_i) \cdot T_i] = +\infty$, and then

$$\lim_{n \rightarrow \infty} \|\sum_{i=1}^n SgF(T_i) \cdot T_i\| = \infty.$$

Then by using an absolutely convergent series, it is easy to construct an element $y \in X$: $\lim_{n \rightarrow \infty} \|\sum_{i=1}^n SgF(T_i) \cdot T_i(y)\| = \infty$. Therefore, $\sum_{i=1}^{\infty} f_i^T(y)e_i$ converges while $\sum_{i=1}^{\infty} SgF(T_i) \cdot f_i^T(y)e_i$ does not, which contradicts the fact that an unconditionally convergent series is bounded multiplier convergent. See [3], p. 19.

Note that the norm of G restricted to \mathcal{C} is equal to the norm of G on $B(X)$, since by monotonicity $\|\sum_{i=1}^n T_i\| \leq \|T\|$ for each n and $T \in B(X)$. Also, F and G agree on \mathcal{C} , because \mathcal{C} is the closure of the set of all T for which only a finite number of the f_i^T are non-zero. Hence the projection defined by $PF = G$ has unit norm, since

$$\|F\|_{B(X)} \geq \|F\|_{\mathcal{E}} = \|G\|_{\mathcal{E}} = \|G\|_{B(X)}.$$

COROLLARY 1.3. *If X has an unconditional basis $\{e_i\}$, then there is a bounded projection from $B(X)^*$ onto a subspace isomorphic to \mathcal{E}^* .*

Proof. Renorm X so that the basis $\{e_i\}$ is monotone. See [1], p. 73.

2. THEOREM 2.1. *Let X have an unconditional, shrinking basis $\{e_i\}$, for which there is a function N of two real variables such that:*

(i) $N(a, b) \leq N(\alpha, \beta)$ if $0 \leq a \leq \alpha$ and $0 \leq b \leq \beta$;

(ii) $N(\|x\|, \|y\|) = \|x + y\|$ for which $x = \sum_{i=1}^n a_i e_i$ and $y = \sum_{i=n+1}^{\infty} a_i e_i$. Then for each $F \in B(X)^*$, $\|F\| = \|G\| + \|H\|$, where $F = G + H$ with $G \in \mathcal{E}^*$ and $H \in \mathcal{E}^1$.

Proof. Note that the existence of N implies that the basis is monotone, and so we have a decomposition for $B(X)^*$. The operators whose matrices have a finite number of nonzero entries form a dense subset of \mathcal{E} . Hence, for $\varepsilon > 0$, there exists an operator D of unit norm whose image lies in the subspace $[e_1, e_2, \dots, e_m]$, and whose kernel contains $[e_{m+1}, e_{m+2}, \dots]$: $G(D) > \|G\| - \varepsilon/3$. Also, there exists an operator $T \in B(X)$ of unit norm: $H(T) > \|H\| - \varepsilon/3$. Let Q_r be the projection onto $[e_{r+1}, e_{r+2}, \dots]$. Define $T^{(r)}x = \sum_{i=r+1}^{\infty} f_i^T(Qx)e_i$. Note that the matrix for $T^{(r)}$ is simply the matrix for T , with the first r -rows and r -columns replaced by zeros.

Then $\lim_{r \rightarrow \infty} G(T^{(r)}) = 0$. To see this, first note that the existence of N and the basis being shrinking imply that the functionals in \mathcal{E}^* with a finite number of nonzero entries form a dense subset of \mathcal{E}^* . See [2], Propositions 3.1 and 3.3. Thus, for any $\delta > 0$, $\exists J \in B(X)^*$, for which $\|J - G\| < \delta$ and: $\lim_{r \rightarrow \infty} J(T^{(r)}) = 0$. Hence $\lim_{r \rightarrow \infty} G(T^{(r)}) = 0$.

Then pick $r > m$: $|G(T^{(r)})| < \varepsilon/3$. Observe that $\|D + T^{(r)}\| = 1$, since and $z \in X$ can be written as $z = x + y$ where $x \in [e_1, \dots, e_r]$ and $y \in [e_{r+1}, \dots]$. Then

$$\begin{aligned} \|(D + T^{(r)})(x + y)\| &= \|Dx + T^{(r)}y\| = N(\|Dx\|, \|T^{(r)}y\|) \\ &\leq N(\|x\|, \|y\|) = \|x, y\|. \end{aligned}$$

Using the fact that H annihilates \mathcal{E} , we have

$$\begin{aligned} F(D + T^{(r)}) &= G(D) + G(T^{(r)}) + H(T^{(r)}) > \|G\| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} + \|H\| - \frac{\varepsilon}{3} \\ &= \|G\| + \|H\| - \varepsilon. \end{aligned}$$

Hence $\|F\| = \|G\| + \|H\|$.

COROLLARY 2.2. *If X is (c_0) or l^p with $1 < p < \infty$, then, for each $F \in B(X)^*$, $\|F\| = \|G\| + \|H\|$, where $F = G + H$ with $G \in \mathcal{E}^*$ and $H \in \mathcal{E}^\perp$.*

Proof. Let $\{e_i\}$ be the standard basis. Let $N(a, b) = [|a|^p + |b|^p]^{1/p}$ for l^p . Let $N(a, b) = \text{Max}(|a|, |b|)$ for c_0 .

THEOREM 2.3. *Each bounded linear functional on \mathcal{E} has a unique normpreserving extension to $B(X)$ for $X = c_0$ or l^p with $1 < p < \infty$.*

REFERENCES

1. M. M. Day, *Normed Linear Spaces*, Academic Press, New York, 1962.
2. J. Hennefeld, *The Arens products and an imbedding theorem*, *Pacific J. Math.*, **29** (1969), 551-563.
3. J. T. Marti, *Introduction to the Theory of Basis*, Springer-Verlag, New York, 1969.

Received December 15, 1971. The author wishes to thank the referee for streamlining the proof of the basic result.

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