

# Pacific Journal of Mathematics

**MOMENT SEQUENCES IN HILBERT SPACE**

GORDON G. JOHNSON

## MOMENT SEQUENCES IN HILBERT SPACE

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Suppose  $f$  is a real valued function of bounded variation on  $[0, 1]$ . Then for each nonnegative integer  $n$ , the Stieltjes integral  $\int_0^1 j^n df$  exists, where for each number  $x$ ,  $j(x) = x$ . A necessary and sufficient condition is given for  $f$  in order that the moment sequence for  $f$ ,  $\{C_n\}_{n=0}^\infty$ , is square summable. A second result establishes that the set of all such square summable moment sequences is dense in  $l^2$ .

LEMMA 1. If  $p$  is a number,  $1/2 < p < 1$ , and for each nonnegative integer  $n$ ,  $a_n = 1 - (n + 1)^{-p}$  then

1.  $\lim_{n \rightarrow \infty} a_n^n = 0$ ,
2.  $\sum_{n=0}^\infty a_n^{2n}$  exists

and

3.  $\sum_{n=0}^\infty (1 - a_n)^2$  exists.

*Proof.* To establish 1,

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n^n &= \lim_{n \rightarrow \infty} (1 - n^{-p})^n \\ &= \exp \left[ \lim_{n \rightarrow \infty} n \ln [1 - n^{-p}] \right]. \end{aligned}$$

Since  $1/2 < p < 1$ ,

$$\lim_{n \rightarrow \infty} n \ln [1 - n^{-p}] = -p \lim_{n \rightarrow \infty} n/[n^p - 1] = -\infty$$

and hence the result.

To establish 2, it will be sufficient to show that for sufficiently large  $n$

$$a_n^n \leq (1 + n)^{-p}$$

i.e., that  $[1 - n^{-p}]^{n-1} \leq n^{-p}$ .

Let  $n^p = k$  and  $g = p^{-1} - 1$  (note that  $g > 0$ ); we have then to show that

$$[[1 - k^{-1}]^k]^{k^g} \leq k^{-1} - k^{-2}.$$

Recall that

$$[1 - k^{-1}]^k \leq e^{-1}$$

and hence that

$$[[1 - k^{-1}]^k]^{k^g} \leq e^{-k^g}.$$

Now if  $k$  is large we have

$$e^{-k^g} \leq k^{-1} - k^{-2}$$

and the result is established.

The third part follows immediately from the definition of  $a_n$ .

**THEOREM 1.** *If  $f$  is a real valued function of bounded variation on  $[0, 1]$  and, for each nonnegative integer  $n$ ,  $\int_0^1 j^n df = C_n$  exists, then*

$$\sum_{n=0}^{\infty} C_n^2 < \infty$$

*if and only if*

$$\sum_{n=1}^{\infty} \left[ f(1) - \int_{a_n}^1 f dj^n (1 - a_n^n) \right]^2 < \infty$$

*where the sequence  $\{a_n\}_{n=0}^{\infty}$  is as given in Lemma 1.*

*Proof.* Let us first establish the necessity of the condition. Suppose  $\sum_{n=0}^{\infty} C_n^2 < \infty$ .

If  $n$  is a positive integer

$$\begin{aligned} C_n &= \int_0^1 j^n df \\ &= \int_0^{a_n} j^n df + \int_{a_n}^1 j^n df \\ &= a_n^n f(a_n) - \int_0^{a_n} f dj^n + \int_{a_n}^1 j^n df. \end{aligned}$$

Let  $\gamma_n = \int_0^{a_n} f dj^n / a_n^n$ , then

$$C_n = a_n^n [f(a_n) - \gamma_n] + f(1) - f(a_n) a_n^n - \int_{a_n}^1 f dj^n.$$

Let  $\delta_n = \int_{a_n}^1 f dj^n / (1 - a_n^n)$ , then

$$\begin{aligned} C_n &= a_n^n [f(a_n) - \gamma_n] + f(1) - f(a_n) a_n^n - (1 - a_n^n) \delta_n \\ C_n &= a_n^n [\delta_n - \gamma_n] + [f(1) - \delta_n] \end{aligned}$$

and

$$C_n^2 = (\alpha_n[\delta_n - \gamma_n] + [f(1) - \delta_n])^2 .$$

Since the sequence  $\{[\delta_n - \gamma_n]\}_{n=1}^\infty$  is bounded it follows from Lemma 1 that

$$\sum_{n=1}^\infty \alpha_n^{2n} [\delta_n - \gamma_n]^2 < \infty .$$

Hence, since  $\sum_{n=0}^\infty C_n^2 < \infty$ , we have that

$$\sum_{n=1}^\infty [f(1) - \delta_n]^2 < \infty$$

i.e.,

$$\sum_{n=1}^\infty \left| f(1) - \int_{\alpha_n}^1 f dj^n / (1 - \alpha_n) \right|^2 < \infty$$

and therefore the condition is necessary.

Now let us establish the sufficiency, i.e., suppose that

$$\sum_{n=1}^\infty \left| f(1) - \int_{\alpha_n}^1 f dj^n / (1 - \alpha_n) \right|^2$$

exists.

Now  $C_n = \left[ f(1) - \int_{\alpha_n}^1 f dj^n \right] - \int_0^{\alpha_n} f dj^n$  for  $n = 0, 1, 2, \dots$

As before

$$\sum_{n=1}^\infty \left( \int_0^{\alpha_n} f dj^n \right)^2$$

exists and hence we have only to consider

$$\begin{aligned} & \sum_{n=1}^\infty \left( f(1) - \int_{\alpha_n}^1 f dj^n \right)^2 \\ &= \sum_{n=1}^\infty \left( \left[ f(1) - \int_{\alpha_n}^1 f dj^n \right] / [1 - \alpha_n] \right)^2 (1 - \alpha_n)^2 \\ &\leq \sum_{n=1}^\infty \left( \left[ f(1) - \int_{\alpha_n}^1 f dj^n \right] / [1 - \alpha_n] \right)^2 \\ &= \sum_{n=1}^\infty \left( f(1) - \int_{\alpha_n}^1 f dj^n / [1 - \alpha_n] + f(1) \alpha_n / [1 - \alpha_n] \right)^2 . \end{aligned}$$

Recall the assumption that

$$\sum_{n=1}^\infty \left( f(1) - \int_{\alpha_n}^1 f dj^n / [1 - \alpha_n] \right)^2$$

exists and hence we need only consider

$$\begin{aligned} & \sum_{n=1}^{\infty} (f(1)a_n^n/[1 - a_n^n])^2 \\ &= \sum_{n=1}^{\infty} (f(1))^2 a_n^{2n}/[1 - a_n^n]^2 \end{aligned}$$

which also exists. Hence it follows that  $\sum_{n=0}^{\infty} C_n^2$  exists.

As an immediate consequence of this result we have the following results, which are stated here without proof.

**PROPOSITION 1.** *If there is a  $\delta, 0 < \delta < 1$ , such that  $f(1) - f(x) \leq 1 - x$  if  $\delta \leq x \leq 1$  then  $\sum_{n=0}^{\infty} C_n^2 < \infty$ .*

**PROPOSITION 2.** *If there is a  $\delta, 0 < \delta < 1$ , such that  $f$  has a continuous derivative on  $[\delta, 1]$  then  $\sum_{n=0}^{\infty} C_n^2 < \infty$ .*

**PROPOSITION 3.** *If there is a number  $\delta, 0 < \delta < 1$ , a number  $\alpha > 1/2$  and a number  $B > 0$  such that*

$$|f(1) - f(x)| \leq B|1 - x|^\alpha \text{ for } x \text{ in } [\delta, 1]$$

*then  $\sum_{n=0}^{\infty} C_n^2 < \infty$ .*

Consider the following example. Let  $f = 1 - (1 - j)^{1/2}$  on  $[0, 1]$ , then  $C_n = \int_0^1 j^n df = 2n \int_0^1 j^2(1 - j^2)^{n-1}$  if  $n \geq 1$ , and hence  $C_{n+1} = 2(n + 1) \int_0^1 j^2(1 - j^2)^n$ . It then follows that  $(2n + 3)C_{n+1} = (2n + 2)C_n$  and this yields the following for  $n = 1, 2, \dots, C_{n+1} = C_1 \prod_{t=0}^{n-1} [(2t + 4)((2t + 5))]$ . By the use of Stirlings formula we have that  $C_{n+1}^2 \geq 6\sqrt{\pi}(n + 3/2)^{-1/2}$  and hence  $\sum_{n=0}^{\infty} C_n^2$  does not exist.

The following lemma is stated without proof.

**LEMMA 2.** *If  $t$  is a positive integer and  $n$  is a nonnegative integer less than  $t$ , then*

$$\sum_{m=0}^t \binom{t}{m} m^t (-1)^m = (-1)^t t!$$

and

$$\sum_{m=0}^t \binom{t}{m} m^n (-1)^m = 0.$$

**DEFINITION 1.** Suppose  $\{C_m\}_{m=0}^{\infty}$  is a real number sequence and  $n$  is a positive integer. Let  $\varphi_n(0) = 0, \varphi_n(1) = C_0$  and if  $x$  is in  $(0, 1) \cap [k/n, (k + 1)/n]$  where  $k = 0, 1, 2, \dots, n - 1$  let

$$\mathcal{P}_n(x) = \sum_{t=0}^k \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i C_{i+t}.$$

**THEOREM 2.** *The set of all square summable moment sequences is dense in  $l^2$ .*

*Proof.* Let, for each nonnegative integer  $t$ ,  $\varepsilon_t = \{\delta_{ij}\}_{i=0}^\infty$  where  $\delta_{ij}$  is the Kronecker  $\delta$ . Associated with each such sequence  $\varepsilon_t$ , there is a function sequence  $\{\mathcal{P}_{k,t}\}_{k=1}^\infty$  as given in Definition 1. For each nonnegative integer  $t$  and each positive integer  $k$  there is a number sequence  $C_{k,t} = \{C_{n,k,t}\}_{n=0}^\infty$  associated, where  $C_{n,k,t} = \int_0^1 j^n d\varphi_{k,t}$ .

A straight forward computation yields

$$\begin{aligned} C_{n,k,t} &= \sum_{m=0}^t (-1)^{t-m} (m/k)^n \binom{k}{m} \binom{k-m}{t-m} \\ &= (-1)^t \binom{k}{t} \sum_{m=0}^t \binom{t}{m} (-1)^m (m/k)^n \end{aligned}$$

and therefore

$$\sum_{n=0}^\infty C_{n,k,t}^2 = \sum_{n=0}^\infty \binom{k}{t}^2 \left[ \sum_{m=0}^t \binom{t}{m} (-1)^m (m/k)^n \right]^2.$$

This, using Lemma 2, becomes

$$\begin{aligned} &\sum_{n=t}^\infty \binom{k}{t}^2 \left[ \sum_{m=0}^t \binom{t}{m} (-1)^m (m/k)^n \right]^2 \\ &= \sum_{n=0}^\infty \binom{k}{t}^2 \left[ \sum_{m=0}^t \binom{t}{m} (-1)^m (m/k)^n (m/k)^t \right]^2 \\ &= \sum_{n=0}^\infty \binom{k}{t}^2 k^{-2t} \left[ \sum_{m=0}^t \binom{t}{m} (m^2/k^2)^n m^{2t} \right. \\ &\quad \left. + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (m/k)^n (-1)^m \sum_{i=m+1}^t \binom{t}{i} i^t (i/k)^n (-1)^i \right] \\ &= \binom{k}{t}^2 k^{-2t} \left[ \sum_{m=0}^t \binom{t}{m}^2 m^{2t} k^2 / (k^2 - m^2) \right. \\ &\quad \left. + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (-1)^m \sum_{i=m+1}^t \binom{t}{i} i^t (-1)^i k^2 / (k^2 - mi) \right] \\ &= \sum_{m=0}^t \binom{t}{m}^2 m^{2t} \binom{k}{t}^2 k^{-2t} k^2 / (k^2 - m^2) \\ &\quad + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (-1)^m \sum_{i=m+1}^t \binom{t}{i} i^t (-1)^i \binom{k}{t}^2 k^{-2t} k^2 / (k^2 - mi) \end{aligned}$$

if  $k > t$ .

Note that

$$\lim_{k \rightarrow \infty} k^{-2t} k^2 \binom{k}{t} / (k^2 - m^2) = (t!)^{-2}$$

and that

$$\lim_{k \rightarrow \infty} k^{-2t} k^2 \binom{k}{t} / (k^2 - mi) = (t!)^{-2}.$$

Then it follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} C_{n,k,t}^2 &= \sum_{m=0}^t \binom{t}{m}^2 m^{2t} (t!)^{-2} \\ &\quad + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (-1)^m \sum_{i=m+1}^t \binom{t}{i} i^t (-1)^i (t!)^{-2} \\ &= (t!)^{-2} \left[ \sum_{m=0}^t \binom{t}{m} m^t (-1)^m \right]^2 \\ &= 1. \end{aligned}$$

Hence, if  $t$  is a nonnegative integer

$$\lim_{k \rightarrow \infty} \|C_{k,t}\| = 1. \quad (\|\cdot\| \text{ is } l^2 \text{ norm})$$

Let us now show that

$$\lim_{k \rightarrow \infty} \|\varepsilon_t - \varepsilon_{k,t}\| = 0.$$

Suppose  $t$  is a nonnegative integer and  $k$  is a positive integer greater than  $t$ .

$$\begin{aligned} &\sum_{n=0}^{\infty} (\delta_{n,t} - C_{n,k,t})^2 \\ &= \sum_{n=t}^{\infty} (\delta_{n,t} - C_{n,k,t})^2 \\ &= (\delta_{t,t} - C_{t,k,t})^2 + \sum_{n=t+1}^{\infty} C_{n,k,t}^2 \\ &= (1 - C_{t,k,t})^2 + \sum_{n=t+1}^{\infty} C_{n,k,t}^2. \end{aligned}$$

Now

$$(1 - C_{t,k,t})^2 = \left( 1 - \binom{k}{t} k^{-t} \sum_{m=0}^t \binom{t}{m} m^t (-1)^{m+t} \right)^2$$

and

$$\lim_{k \rightarrow \infty} (1 - C_{t,k,t})^2 = \left[ 1 - (t!)^{-1} \sum_{m=0}^t \binom{t}{m} m^t (-1)^{m+t} \right]^2$$

since

$$\lim_{k \rightarrow \infty} \binom{k}{t} k^{-t} = (t!)^{-1}$$

and hence by Lemma 2

$$\lim_{k \rightarrow \infty} (1 - C_{t,k,t})^2 = 0$$

Combining this with the fact that

$$\sum_{n=0}^{\infty} C_{n,k,t}^2 = 1$$

yields,  $\lim_{k \rightarrow \infty} \|\varepsilon_t - \varepsilon_{k,t}\| = 0$  for each nonnegative integer  $t$ .

Since  $\{\varepsilon_t: t = 0, 1, 2, \dots\}$  is a complete orthonormal set for  $l^2$  and each point can be approximated by a square summable moment sequence, it follows that the set of all square summable moment sequences is dense in  $l^2$  and hence the theorem is established.

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