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MOMENT SEQUENCES IN HILBERT SPACE

GORDON G. JOHNSON

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Suppose f is a real valued function of bounded variation on $[0, 1]$. Then for each nonnegative integer n , the Stieltjes integral $\int_0^1 j^n df$ exists, where for each number x , $j(x) = x$. A necessary and sufficient condition is given for f in order that the moment sequence for f , $\{C_n\}_{n=0}^\infty$, is square summable. A second result establishes that the set of all such square summable moment sequences is dense in l^2 .

LEMMA 1. If p is a number, $1/2 < p < 1$, and for each nonnegative integer n , $a_n = 1 - (n + 1)^{-p}$ then

1. $\lim_{n \rightarrow \infty} a_n^n = 0$,
2. $\sum_{n=0}^\infty a_n^{2n}$ exists

and

3. $\sum_{n=0}^\infty (1 - a_n)^2$ exists.

Proof. To establish 1,

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n^n &= \lim_{n \rightarrow \infty} (1 - n^{-p})^n \\ &= \exp \left[\lim_{n \rightarrow \infty} n \ln [1 - n^{-p}] \right]. \end{aligned}$$

Since $1/2 < p < 1$,

$$\lim_{n \rightarrow \infty} n \ln [1 - n^{-p}] = -p \lim_{n \rightarrow \infty} n/[n^p - 1] = -\infty$$

and hence the result.

To establish 2, it will be sufficient to show that for sufficiently large n

$$a_n^n \leq (1 + n)^{-p}$$

i.e., that $[1 - n^{-p}]^{n-1} \leq n^{-p}$.

Let $n^p = k$ and $g = p^{-1} - 1$ (note that $g > 0$); we have then to show that

$$[[1 - k^{-1}]^k]^{k^g} \leq k^{-1} - k^{-2}.$$

Recall that

$$[1 - k^{-1}]^k \leq e^{-1}$$

and hence that

$$[[1 - k^{-1}]^k]^{k^g} \leq e^{-k^g}.$$

Now if k is large we have

$$e^{-k^g} \leq k^{-1} - k^{-2}$$

and the result is established.

The third part follows immediately from the definition of a_n .

THEOREM 1. *If f is a real valued function of bounded variation on $[0, 1]$ and, for each nonnegative integer n , $\int_0^1 j^n df = C_n$ exists, then*

$$\sum_{n=0}^{\infty} C_n^2 < \infty$$

if and only if

$$\sum_{n=1}^{\infty} \left[f(1) - \int_{a_n}^1 f dj^n (1 - a_n^n) \right]^2 < \infty$$

where the sequence $\{a_n\}_{n=0}^{\infty}$ is as given in Lemma 1.

Proof. Let us first establish the necessity of the condition. Suppose $\sum_{n=0}^{\infty} C_n^2 < \infty$.

If n is a positive integer

$$\begin{aligned} C_n &= \int_0^1 j^n df \\ &= \int_0^{a_n} j^n df + \int_{a_n}^1 j^n df \\ &= a_n^n f(a_n) - \int_0^{a_n} f dj^n + \int_{a_n}^1 j^n df. \end{aligned}$$

Let $\gamma_n = \int_0^{a_n} f dj^n / a_n^n$, then

$$C_n = a_n^n [f(a_n) - \gamma_n] + f(1) - f(a_n) a_n^n - \int_{a_n}^1 f dj^n.$$

Let $\delta_n = \int_{a_n}^1 f dj^n / (1 - a_n^n)$, then

$$\begin{aligned} C_n &= a_n^n [f(a_n) - \gamma_n] + f(1) - f(a_n) a_n^n - (1 - a_n^n) \delta_n \\ C_n &= a_n^n [\delta_n - \gamma_n] + [f(1) - \delta_n] \end{aligned}$$

and

$$C_n^2 = (a_n^n [\delta_n - \gamma_n] + [f(1) - \delta_n])^2.$$

Since the sequence $\{[\delta_n - \gamma_n]\}_{n=1}^\infty$ is bounded it follows from Lemma 1 that

$$\sum_{n=1}^\infty a_n^{2n} [\delta_n - \gamma_n]^2 < \infty.$$

Hence, since $\sum_{n=0}^\infty C_n^2 < \infty$, we have that

$$\sum_{n=1}^\infty [f(1) - \delta_n]^2 < \infty$$

i.e.,

$$\sum_{n=1}^\infty \left| f(1) - \int_{a_n}^1 f dj^n / (1 - a_n^n) \right|^2 < \infty$$

and therefore the condition is necessary.

Now let us establish the sufficiency, i.e., suppose that

$$\sum_{n=1}^\infty \left| f(1) - \int_{a_n}^1 f dj^n / (1 - a_n^n) \right|^2$$

exists.

$$\text{Now } C_n = \left[f(1) - \int_{a_n}^1 f dj^n \right] - \int_0^{a_n} f dj^n \text{ for } n = 0, 1, 2, \dots$$

As before

$$\sum_{n=1}^\infty \left(\int_0^{a_n} f dj^n \right)^2$$

exists and hence we have only to consider

$$\begin{aligned} & \sum_{n=1}^\infty \left(f(1) - \int_{a_n}^1 f dj^n \right)^2 \\ &= \sum_{n=1}^\infty \left(\left[f(1) - \int_{a_n}^1 f dj^n \right] / [1 - a_n^n] \right)^2 (1 - a_n^n)^2 \\ &\leq \sum_{n=1}^\infty \left(\left[f(1) - \int_{a_n}^1 f dj^n \right] / [1 - a_n^n] \right)^2 \\ &= \sum_{n=1}^\infty \left(f(1) - \int_{a_n}^1 f dj^n / [1 - a_n^n] + f(1) a_n^n / [1 - a_n^n] \right)^2. \end{aligned}$$

Recall the assumption that

$$\sum_{n=1}^\infty \left(f(1) - \int_{a_n}^1 f dj^n / [1 - a_n^n] \right)^2$$

exists and hence we need only consider

$$\begin{aligned} & \sum_{n=1}^{\infty} (f(1)\alpha_n^n/[1 - \alpha_n^n])^2 \\ &= \sum_{n=1}^{\infty} (f(1))^2\alpha_n^{2n}/[1 - \alpha_n^n]^2 \end{aligned}$$

which also exists. Hence it follows that $\sum_{n=0}^{\infty} C_n^2$ exists.

As an immediate consequence of this result we have the following results, which are stated here without proof.

PROPOSITION 1. *If there is a $\delta, 0 < \delta < 1$, such that $f(1) - f(x) \leq 1 - x$ if $\delta \leq x \leq 1$ then $\sum_{n=0}^{\infty} C_n^2 < \infty$.*

PROPOSITION 2. *If there is a $\delta, 0 < \delta < 1$, such that f has a continuous derivative on $[\delta, 1]$ then $\sum_{n=0}^{\infty} C_n^2 < \infty$.*

PROPOSITION 3. *If there is a number $\delta, 0 < \delta < 1$, a number $\alpha > 1/2$ and a number $B > 0$ such that*

$$|f(1) - f(x)| \leq B|1 - x|^\alpha \text{ for } x \text{ in } [\delta, 1]$$

then $\sum_{n=0}^{\infty} C_n^2 < \infty$.

Consider the following example. Let $f = 1 - (1 - j)^{1/2}$ on $[0, 1]$, then $C_n = \int_0^1 j^n df = 2n \int_0^1 j^2(1 - j^2)^{n-1}$ if $n \geq 1$, and hence $C_{n+1} = 2(n + 1) \int_0^1 j^2(1 - j^2)^n$. It then follows that $(2n + 3)C_{n+1} = (2n + 2)C_n$ and this yields the following for $n = 1, 2, \dots, C_{n+1} = C_1 \prod_{t=0}^{n-1} [(2t + 4)((2t + 5))]$. By the use of Stirlings formula we have that $C_{n+1}^2 \geq 6\sqrt{\pi} (n + 3/2)^{-1/2}$ and hence $\sum_{n=0}^{\infty} C_n^2$ does not exist.

The following lemma is stated without proof.

LEMMA 2. *If t is a positive integer and n is a nonnegative integer less than t , then*

$$\sum_{m=0}^t \binom{t}{m} m^t (-1)^m = (-1)^t t!$$

and

$$\sum_{m=0}^t \binom{t}{m} m^n (-1)^m = 0.$$

DEFINITION 1. Suppose $\{C_m\}_{m=0}^{\infty}$ is a real number sequence and n is a positive integer. Let $\varphi_n(0) = 0, \varphi_n(1) = C_0$ and if x is in $(0, 1) \cap [k/n, (k + 1)/n]$ where $k = 0, 1, 2, \dots, n - 1$ let

$$\mathcal{P}_n(x) = \sum_{t=0}^k \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i C_{i+t}.$$

THEOREM 2. *The set of all square summable moment sequences is dense in l^2 .*

Proof. Let, for each nonnegative integer t , $\varepsilon_t = \{\delta_{it}\}_{i=0}^{\infty}$ where δ_{ij} is the Kronecker δ . Associated with each such sequence ε_t , there is a function sequence $\{\mathcal{P}_{k,t}\}_{k=1}^{\infty}$ as given in Definition 1. For each nonnegative integer t and each positive integer k there is a number sequence $C_{k,t} = \{C_{n,k,t}\}_{n=0}^{\infty}$ associated, where $C_{n,k,t} = \int_0^1 j^n d\mathcal{P}_{k,t}$.

A straight forward computation yields

$$\begin{aligned} C_{n,k,t} &= \sum_{m=0}^t (-1)^{t-m} (m/k)^n \binom{k}{m} \binom{k-m}{t-m} \\ &= (-1)^t \binom{k}{t} \sum_{m=0}^t \binom{t}{m} (-1)^m (m/k)^n \end{aligned}$$

and therefore

$$\sum_{n=0}^{\infty} C_{n,k,t}^2 = \sum_{n=0}^{\infty} \binom{k}{t}^2 \left[\sum_{m=0}^t \binom{t}{m} (-1)^m (m/k)^n \right]^2.$$

This, using Lemma 2, becomes

$$\begin{aligned} &\sum_{n=t}^{\infty} \binom{k}{t}^2 \left[\sum_{m=0}^t \binom{t}{m} (-1)^m (m/k)^n \right]^2 \\ &= \sum_{n=0}^{\infty} \binom{k}{t}^2 \left[\sum_{m=0}^t \binom{t}{m} (-1)^m (m/k)^n (m/k)^t \right]^2 \\ &= \sum_{n=0}^{\infty} \binom{k}{t}^2 k^{-2t} \left[\sum_{m=0}^t \binom{t}{m} (m^2/k^2)^n m^{2t} \right. \\ &\quad \left. + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (m/k)^n (-1)^m \sum_{i=m+1}^t \binom{t}{i} i^t (i/k)^n (-1)^i \right] \\ &= \binom{k}{t}^2 k^{-2t} \left[\sum_{m=0}^t \binom{t}{m} m^{2t} k^2 / (k^2 - m^2) \right. \\ &\quad \left. + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (-1)^m \sum_{i=m+1}^t \binom{t}{i} i^t (-1)^i k^2 / (k^2 - mi) \right] \\ &= \sum_{m=0}^t \binom{t}{m}^2 m^{2t} \binom{k}{t}^2 k^{-2t} k^2 / (k^2 - m^2) \\ &\quad + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (-1)^m \sum_{i=m+1}^t \binom{t}{i} i^t (-1)^i \binom{k}{t}^2 k^{-2t} k^2 / (k^2 - mi) \end{aligned}$$

if $k > t$.

Note that

$$\lim_{k \rightarrow \infty} k^{-2t} k^2 \binom{k}{t}^2 / (k^2 - m^2) = (t!)^{-2}$$

and that

$$\lim_{k \rightarrow \infty} k^{-2t} k^2 \binom{k}{t}^2 / (k^2 - mi) = (t!)^{-2}.$$

Then it follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} C_{n,k,t}^2 &= \sum_{m=0}^t \binom{t}{m}^2 m^{2t} (t!)^{-2} \\ &\quad + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (-1)^m \sum_{i=m+1}^t \binom{t}{i} i^t (-1)^i (t!)^{-2} \\ &= (t!)^{-2} \left[\sum_{m=0}^t \binom{t}{m} m^t (-1)^m \right]^2 \\ &= 1. \end{aligned}$$

Hence, if t is a nonnegative integer

$$\lim_{k \rightarrow \infty} \|C_{k,t}\| = 1. (\|\cdot\| \text{ is } l^2 \text{ norm})$$

Let us now show that

$$\lim_{k \rightarrow \infty} \|\varepsilon_t - \varepsilon_{k,t}\| = 0.$$

Suppose t is a nonnegative integer and k is a positive integer greater than t .

$$\begin{aligned} \sum_{n=0}^{\infty} (\delta_{n,t} - C_{n,k,t})^2 &= \sum_{n=t}^{\infty} (\delta_{n,t} - C_{n,k,t})^2 \\ &= (\delta_{t,t} - C_{t,k,t})^2 + \sum_{n=t+1}^{\infty} C_{n,k,t}^2 \\ &= (1 - C_{t,k,t})^2 + \sum_{n=t+1}^{\infty} C_{n,k,t}^2. \end{aligned}$$

Now

$$(1 - C_{t,k,t})^2 = \left(1 - \binom{k}{t} k^{-t} \sum_{m=0}^t \binom{t}{m} m^t (-1)^{m+t}\right)^2$$

and

$$\lim_{k \rightarrow \infty} (1 - C_{t,k,t})^2 = \left[1 - (t!)^{-1} \sum_{m=0}^t \binom{t}{m} m^t (-1)^{m+t} \right]^2$$

since

$$\lim_{k \rightarrow \infty} \binom{k}{t} k^{-t} = (t!)^{-1}$$

and hence by Lemma 2

$$\lim_{k \rightarrow \infty} (1 - C_{t,k,t})^2 = 0$$

Combining this with the fact that

$$\sum_{n=0}^{\infty} C_{n,k,t}^2 = 1$$

yields, $\lim_{k \rightarrow \infty} \|\varepsilon_t - \varepsilon_{k,t}\| = 0$ for each nonnegative integer t .

Since $\{\varepsilon_t: t = 0, 1, 2, \dots\}$ is a complete orthonormal set for l^2 and each point can be approximated by a square summable moment sequence, it follows that the set of all square summable moment sequences is dense in l^2 and hence the theorem is established.

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Allan Francis Abrahamse, <i>Uniform integrability of derivatives on σ-lattices</i>	1
Ronald Alter and K. K. Kubota, <i>The diophantine equation $x^2 + D = p^n$</i>	11
Grahame Bennett, <i>Some inclusion theorems for sequence spaces</i>	17
William Cutler, <i>On extending isotopies</i>	31
Robert Jay Daverman, <i>Factored codimension one cells in Euclidean n-space</i>	37
Patrick Barry Eberlein and Barrett O'Neill, <i>Visibility manifolds</i>	45
M. Edelstein, <i>Concerning dentability</i>	111
Edward Graham Evans, Jr., <i>Krull-Schmidt and cancellation over local rings</i>	115
C. D. Feustel, <i>A generalization of Kneser's conjecture</i>	123
Avner Friedman, <i>Uniqueness for the Cauchy problem for degenerate parabolic equations</i>	131
David Golber, <i>The cohomological description of a torus action</i>	149
Alain Goulet de Rugy, <i>Un théorème du genre "Andô-Edwards" pour les Fréchet ordonnés normaux</i>	155
Louise Hay, <i>The class of recursively enumerable subsets of a recursively enumerable set</i>	167
John Paul Helm, Albert Ronald da Silva Meyer and Paul Ruel Young, <i>On orders of translations and enumerations</i>	185
Julien O. Hennefeld, <i>A decomposition for $B(X)^*$ and unique Hahn-Banach extensions</i>	197
Gordon G. Johnson, <i>Moment sequences in Hilbert space</i>	201
Thomas Rollin Kramer, <i>A note on countably subparacompact spaces</i>	209
Yves A. Lequain, <i>Differential simplicity and extensions of a derivation</i>	215
Peter Lorimer, <i>A property of the groups $\text{Aut PU}(3, q^2)$</i>	225
Yasou Matsugu, <i>The Levi problem for a product manifold</i>	231
John M.F. O'Connell, <i>Real parts of uniform algebras</i>	235
William Lindall Paschke, <i>A factorable Banach algebra without bounded approximate unit</i>	249
Ronald Joel Rudman, <i>On the fundamental unit of a purely cubic field</i>	253
Tsuan Wu Ting, <i>Torsional rigidities in the elastic-plastic torsion of simply connected cylindrical bars</i>	257
Philip C. Tonne, <i>Matrix representations for linear transformations on analytic sequences</i>	269
Jung-Hsien Tsai, <i>On E-compact spaces and generalizations of perfect mappings</i>	275
Alfons Van Daele, <i>The upper envelope of invariant functionals majorized by an invariant weight</i>	283
Giulio Varsi, <i>The multidimensional content of the frustum of the simplex</i>	303