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It is the purpose of this paper to characterize countably subparacompact spaces in a number of ways and to point out similarities in the pathologies of countably subparacompact spaces and normal spaces. It will be shown *inter alia,* that a space is countably subparacompact if and only if it is countably σ-paracompact, and also if and only if it is countably metacompact and subnormal. The well known product of ordinal spaces, $W \times W^*$, is shown to be not countably subparacompact, despite the fact that W^* is compact and W is countably subparacompact and normal.

1. Introduction. Countably subparacompact spaces were first defined in the literature by R. E. Hodel in [3] as follows: a topological space is countably subparacompact iff every countable open cover of it has a σ-discrete closed refinement. The concept had been briefly studied in an earlier paper [7] by M. Mansfield. He showed that in normal spaces, countable subparacompactness is equivalent to countable metacompactness. Recall that a space is countably metacompact iff every countable open cover of it has a point finite open refinement. The following result of Hodel in the work cited above extended Mansfield's theorem: every countably subparacompact space is countably metacompact. A number of further results were developed independently by the author [6] and M. K. Singal and P. Jain [8].

We shall use the following conventions. The end of a proof is denoted by $\square$, the positive integers by N, and implication by $\Rightarrow$. "Iff" means "if and only if". X and Y are always topological spaces.

2. Characterizations of countably subparacompact spaces.

THEOREM 2.1. *The following are equivalent.*

(a) *Every countable open cover of X has a σ-discrete closed refinement* (i.e., *X is countably subparacompact*).

(b) *Every countable open cover of X has a σ-locally finite closed refinement.*

(c) *Every countable open cover of X has a σ-closure preserving closed refinement.*

(d) *Every countable open cover of X has a countable closed refinement.*

In [8] Singal and Jain give this theorem with only parts (a)—(c), and the proof offered is consequently somewhat intricate. The inclusion of part (d) simplifies matters.

Proof of Theorem 2.1. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (d) $\Rightarrow$ (a) are obvious. To see (c) $\Rightarrow$ (d), suppose $\{U_n: n \in N\}$ is a countable open cover of X with $\{F_{m\alpha}: \alpha \in A_m, m \in N\}$ as a σ-closure preserving closed refinement ($\{F_{m\alpha}: \alpha \in A_m\}$ is a closure-preserving collection of closed sets for each $m \in N$). Then letting $G_{mn} = \bigcup \{F_{m\alpha}: \alpha \in A_m, F_{m\alpha} \subset U_n\}$, it is clear that $\mathscr{G} = \{G_{mn}: m, n \in N\}$ is the required countable closed refinement. $\square$

In D. K. Burke's paper [1, p. 655] it was shown that σ-paracompactness (a definition introduced by Arhangelskii) is equivalent to subparacompactness. It is the case that if we define countable σ-paracompactness in the obvious way, the analogous theorem is true.

DEFINITION 2.2. X is *countably σ-paracompact* iff given a countable open cover $\mathscr{U}$ of X, there is a sequence $\{\mathscr{U}_n\}$ of open covers of X such that given $x \in X$, there are $n \in N$ and $U \in \mathscr{U}$ with $st(x, \mathscr{U}_n) \subset U$.

THEOREM 2.3. *X is countably subparacompact iff X is countably σ-paracompact.*

Proof of Theorem 2.3. "$\Rightarrow$" Let $\mathscr{U}$ be a countable open cover of X with countable closed refinement $\{F_n: n \in N\}$. For each $n \in N$ let U_n be an element of $\mathscr{U}$ with $F_n \subset U_n$ and let $\mathscr{U}_n = \{U_n, X - F_n\}$. Then given $x \in X$ there is $n \in N$ with $x \in F_n \subset st(x, \mathscr{U}_n) = U_n$.

"$\Leftarrow$" Let $\mathscr{U} = \{U_n: n \in N\}$ be a countable open cover of countably σ-paracompact space X. Let $\{\mathscr{U}_m\}$ be a sequence of open covers of X such that given $x \in X$ there are $U_n \in \mathscr{U}$ and $m \in N$ with $st(x, \mathscr{U}_m) \subset U_n$. We construct a countable closed refinement of $\mathscr{U}$ as follows.

Let $F_{mn} = \{x \in X: st(x, \mathscr{U}_m) \subset U_n\}$. Then $\{F_{mn}: m, n \in N\}$ is a countable closed refinement of $\mathscr{U}$.

(i) Each F_{mn} is contained in some element of $\mathscr{U}$: clearly $F_{mn} \subset U_n$.

(ii) $\{F_{mn}: m, n \in N\}$ covers X: Given $x \in X$, there are m and n such that $st(x, \mathscr{U}_m) \subset U_n$, so $x \in F_{mn}$.

(iii) Each F_{mn} is closed: To show $X - F_{mn}$ is open, let $y \in X - F_{mn}$. Then $st(y, \mathscr{U}_m) \not\subset U_n$, so there is a $U \in \mathscr{U}_m$ with $y \in U$, but $U \not\subset U_n$. Then U is an open neighborhood of y not intersecting F_{mn}, for; if $z \in F_{mn}$ and $z \in U$, then $U \subset U_n$, which is not the case. $\square$

It is known, as previously mentioned, that countable subparacompactness and countable metacompactness are equivalent in normal spaces. The question arises, can we weaken normality and still get equivalence? We find an affirmative answer, and, in fact, arrive at another characterization of countably subparacompact spaces, by defining subnormality as follows.

DEFINITION 2.4. X is *subnormal* iff every finite open cover of X has a countable closed refinement.

To see that every normal space is subnormal, recall that X is normal iff every finite open cover of X has a finite closed refinement. Fortuitously, every countably subparacompact space is also subnormal, as may be seen from Theorem 2.1(d). We may now state:

THEOREM 2.5. *X is countably subparacompact iff X is countably metacompact and subnormal.*

Proof of Theorem 2.5 requires the use of a characterization of countably metacompact spaces due to F. Ishikawa [4]. That is, X is countably metacompact iff given a decreasing sequence $\{H_n\}$ of closed subsets of X such that $\bigcap\{H_n\} = \varnothing$, there is a decreasing sequence $\{V_n\}$ of open sets in X such that $H_n \subset V_n$ for all $n \in N$ and $\bigcap\{V_n\} = \varnothing$.

Proof of Theorem 2.5.[1] "$\Rightarrow$" This follows directly from the cited result of Hodel and a remark following Definition 2.4.

"$\Leftarrow$" Let $\mathscr{U} = \{U_n : n \in N\}$ be a countable open cover of X. We shall construct a countable closed refinement $\mathscr{F}$ of $\mathscr{U}$. For each $n \in N$ set $H_n = X - \bigcup_{j=1}^{n} U_j$. Then $\{H_n\}$ is a decreasing sequence of closed subsets of X such that $\bigcap\{H_n\} = \varnothing$. By Ishikawa's result there is thus a decreasing sequence $\{V_n\}$ of open sets in X such that $H_n \subset V_n$ for all $n \in N$ and $\bigcap\{V_n\} = \varnothing$.

For each n, it is easily seen that $\{U_1, \cdots, U_n, V_n\}$ is a finite open cover of X (if $x \notin \bigcup_{j=1}^{n} U_j$, then by construction $x \in H_n$, so $x \in V_n$ since $H_n \subset V_n$). As X is subnormal, we may let $\mathscr{F}_n'$ be a countable closed refinement of that cover.

Set $\mathscr{F}_n = \{F \cap (X - V_n) : F \in \mathscr{F}_n'\}$ and $\mathscr{F} = \bigcup_{n \in N} \mathscr{F}_n$. Then $\mathscr{F}$ is the required refinement, for:

(i) $\mathscr{F}$ is clearly a countable collection of closed sets in X.

(ii) Each element of $\mathscr{F}$ is contained in some element of $\mathscr{U}$:

[1] This proof is due to Phillip Zenor. The original was inelegant.

If $F \in \mathscr{F}$, then $F \in \mathscr{F}_n$ for some $n \in N$. Thus F is contained in $X - V_n$ and in some element of $\{U_1, \cdots, U_n, V_n\}$.

Clearly F cannot be contained in V_n, so F is contained in one of $\{U_1, \cdots, U_n\}$.

(iii) $\mathscr{F}$ covers X: Let $x \in X$. Pick n so that $x \notin V_n$. There is an element F of $\mathscr{F}_n'$ with $x \in F$. Clearly $x \in F \cap (X - V_n) \in \mathscr{F}$. □

3. **Pathology.** We have seen that countable subparacompactness is linked with normality via subnormality, which generalizes both. We shall see now that the pathological behavior of countable subparacompactness is similar to that of normality in products, and hence in inverse image theorems.

EXAMPLE 3.1. Let W be the well known space consisting of all ordinals less than Ω, the first uncountable ordinal. Let W^* be $W \cup \{\Omega\}$. Both W and W^* are given the order topology. W^* is known to be a compact T_2 space and W a countably compact, normal T_2 space. We shall show $W \times W^*$ is not subnormal (it has been known for some time that $W \times W^*$ is not normal). For a good presentation of W and W^*, look up "ordinal examples" in the index of Greever's book [2].

Three facts about W and W^* given in the next lemma will be needed. An outline of the proof of this lemma is given in [5, Problem 4E].

LEMMA 3.2. (a) *If A is a countable subset of W, then* sup (A) *exists and belongs to W.*

(b) *If $x \in W^*$ and $x > 1$, then $\{(\alpha, x]: \alpha < x\}$ is a fundamental system of open neighborhoods of x. $\{1\}$ is itself an open set in W^*.*

(c) *If $\{x_n\}$ and $\{y_n\}$ are sequences in W such that $x_n \leqq y_n \leqq x_{n+1}$ for all $n \in N$, then there is an element z of W such that $\{x_n\}$ and $\{y_n\}$ both converge to z.*

Verification that $W \times W^*$ is not subnormal:

Let $H = \{(x, \Omega): x \in W\}$ and $K = \{(x, x): x \in W\}$. H and K are disjoint closed sets in $W \times W^*$, so $\{(W \times W^*) - H, (W \times W^*) - K\}$ is an open cover of $W \times W^*$. Suppose, as is false, that there is a countable closed refinement $\mathscr{M}$ of this cover. Call the elements of $\mathscr{M}$ F_n if they intersect H and G_n if they do not intersect H.

For each $x > 1$ in W and each $n \in N$, $(x, \Omega) \notin G_n$, so there are z_{nx} and y_{nx} such that both are in W and $(z_{nx}, x] \times (y_{nx}, \Omega] \cap G_n = \varnothing$. If $x = 1$, there is y_{nx} such that $\{1\} \times (y_{nx}, \Omega] \cap G_n = \varnothing$.

Let $y_x = \sup\{y_{nx}: n \in N\}$. Then $\{(x, y): y \in W^*, y > y_x\}$ is disjoint from G_n for all $n \in N$ and nonempty because $y_x \in W$ by 3.2(a).

Assertion. For some $n_0 \in N$, given $x \in W$ there are w and y in W with $x < w < y$ and $(w, y) \in F_{n_0}$.

Proof of the Assertion. Suppose it were false, then for each n, there would be an x_n in W such that $(w, y) \notin F_n$ whenever $x_n < w < y$. We could then let $x_0 = \sup\{x_n: n \in N)$ and pick $y_0 > y_{x_0}$ in W. The point (x_0, y_0) would belong to no F_n or G_n, an impossibility.

Construct a sequence $\{(x_n, y_n)\}$ in $(W \times W) \cap F_{n_0}$ as follows. Let (x_1, y_1) be any point in $(W \times W) \cap F_{n_0}$ with $y_1 > x_1$. In general, pick (x_{n+1}, y_{n+1}) in $(W \times W) \cap F_{n_0}$ with $y_n < x_{n+1} < y_{n+1}$. The assertion assures us we can do this.

Then $\{x_n\}$ and $\{y_n\}$ are sequences in W such that $x_n \leqq y_n \leqq x_{n+1}$ for all $n \in N$. By 3.2(c) there is $z \in W$ such that $\{x_n\}$ and $\{y_n\}$ converge to z. Thus (z, z) is a limit point of F_{n_0}, implying $(z, z) \in F_{n_0}$. Hence $K \cap F_{n_0} \neq \emptyset$, a contradiction. □

Example 3.1 shows that Theorem 3.3 of Singal and Jain [8] is false; i.e., it is not true that if $f: X \to Y$ is a closed, continuous mapping from a regular space X onto a countably subparacompact space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then X is countably subparacompact. The mistake in their proof lies in the next-to-last sentence, which is untrue.

A number of product and inverse image theorems for countably subparacompact spaces are given in [6].

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