THE LEVI PROBLEM FOR A PRODUCT MANIFOLD

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Let $S$ be a Stein manifold, $T$ a one dimensional torus, $\pi$ a projection of the product $E = S \times T$ onto $S$ and $D$ a subdomain of $E$. The main object of this paper is to prove that $D$ is a Stein manifold if and only if $D$ is pseudoconvex in the sense of Cartan and $\pi^{-1}(x)$ is not contained in $D$ for any point $x$ of $S$.

1. A subdomain $D$ of a complex manifold $M$ is called pseudoconvex if, for any boundary point $x$ of $D$ in $M$, there is a Stein neighborhood $U$ of $x$ in $M$ such that $U \cap D$ is also a Stein manifold. A pair $(B, \beta)$ is called a domain over $M$ if $\beta$ is a locally biholomorphic mapping of a complex manifold $B$ in $M$. A domain $(B, \beta)$ over $C^n$ is called a domain of holomorphy if there exists a holomorphic function $f$ in $B$ such that the radius of convergence of $f$ at any point $x$ of $B$ is just the boundary distance $d(x)$ of $x$.

Moreover we recall another definition. Let $\varphi_i: M_i \rightarrow N_i$ be two mappings of a set $M_i$ into a set $N_i$ $(i = 1, 2)$. Then we define the product mapping $\varphi_1 \times \varphi_2$ of the product set $M_1 \times M_2$ into the product set $N_1 \times N_2$ by putting $(\varphi_1 \times \varphi_2)(x, y) = (\varphi_1(x), \varphi_2(y))$ for $(x, y) \in M_1 \times M_2$.

The proof of our theorem falls into two parts. We first prove it in the case of $S = C^n$, where we construct a strongly plurisubharmonic function by means of Hörmander [2] and reduce it to a result of Narasimhan [3]. In general case, using the imbedding of Docquier-Grauert [1], we reduce the theorem to the case of $C^n$.

2. Let $(B, \beta)$ be a domain of holomorphy over $C^n$. In the complex plane $C$ select any two complex numbers $\omega_1, \omega_2$ which are linearly independent over the real number field $R$. The numbers $\omega_1, \omega_2$ generate a subgroup $\Gamma'$ of $C$, namely

$$\Gamma' = \{m_1\omega_1 + m_2\omega_2; m_1, m_2 \in \text{additive group of integers}\}.$$

The quotient $\tau = C/\Gamma'$ is a one dimensional torus. $\tau$ has a natural complex structure and is a compact Riemann surface. The natural map $\tau: C \rightarrow T$ is a locally biholomorphic map. We denote by $E = B \times T$ the product of two complex manifolds $B$ and $T$, and by $\pi: E \rightarrow B$ the projection.

We first prove the following lemma:

**Lemma.** Let $D$ be a pseudoconvex open subset of $E$ such that $\pi^{-1}(x)$ is not contained in $D$ for any point $x$ of $B$. Then $D$ is a Stein manifold.
Proof. Let $1 \times \tau$ be the product map of the identity 1 of $B$ and the map $\tau$. The map $1 \times \tau$ is a locally biholomorphic map $B \times C$ onto $E$. If we denote by $A$ the inverse image $(1 \times \tau)^{-1}(D)$ of $D$, $A$ is pseudoconvex, because $D$ is pseudoconvex. $A$ is $\Gamma$-invariant, that is, for any fixed point $\gamma \in \Gamma$, $A$ is invariant under the transformation of $B \times C$: $(y, z) \mapsto (y, z + \gamma)$. Let $\alpha$ be the restriction to $A$ of the product map $\beta \times 1$ of the map $\beta$ and the identity map 1 of $C$, that is, $\alpha(y, z) = (\beta(y), z)$ for $(y, z) \in A$. $\alpha$ is a locally biholomorphic map of $A$ into $C^* \times C = C^{*+1}$ and $(A, \alpha)$ is a pseudoconvex domain over $C^{*+1}$. The distance function $d(y, z)$ of the domain $A$ over $C^{*+1}$ induces the function $d(y, t)$ in $D$. Indeed, for any point $(y, t) \in D$, $y \in B$, $t \in T$, select two representatives $z, z' \in C$ of the equivalence class $t$. Then there is $\gamma \in \Gamma$ such that $z' = z + \gamma$. But $A$ is $\Gamma$-invariant, and so $d(y, z') = d(y, z)$. Since $A$ is pseudoconvex, by Oka [4], the function $-\log d(y, z)$ is a continuous plurisubharmonic function in $A$. The function $-\log d(y, t)$ is therefore a continuous plurisubharmonic function in $D$, and so is the function

$$\frac{1}{d(y, t)} = e^{-\log d(y, t)}.$$

On the other hand, since $B$ is Stein, there is a real analytic strongly plurisubharmonic function $q > 0$ with the following property: for any real number $c > 0$,

$$B_c = \{y \in B; q(y) < c\} \subset B.$$

The function

$$\gamma(y, t) = \frac{1}{d(y, t)} + q(y)$$

defined in $D$ is a continuous plurisubharmonic function. It holds that

$$D_c = \{(y, t) \in D; \gamma(y, t) < c\} \subset B_c \times T \cap \{(y, t) \in D; d(y, t) > \frac{1}{c}\} \subset D$$

for any real number $c > 0$.

Since $D = \bigcup_{c > 0} D_c$, if we show that $D_c$ is a Stein manifold, we know by Docquier-Grauert [1], that $D$ is itself a Stein manifold.

Fix an arbitrary real number $c > 0$. For any point $y \in B$, we set

$$A(y) = \{z \in C; (y, \tau(z)) \in D\}.$$

By the hypothesis of the lemma, it follows that $A(y) \subset C$. Select a complex-valued measurable function $a(y)$ in $B$ such that

$$a(y) \in C - A(y)$$

for any point $y \in B$. 

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For sufficiently small number $\varepsilon$ with $0 < \varepsilon < 1/(c + 1) < 1/c$, we define the function $s(y, t)$ in $D_{c+1}$ as follows:

$$s(y, t) = \frac{1}{\varepsilon^{2n}} \int_{\mathbb{R}^n} \rho \left( \frac{y - \bar{z}}{\varepsilon} \right) \sum_{m_1, m_2 \to \infty} \frac{d\lambda(\bar{z})}{|z - a(\bar{z}) - m_1 \omega_1 - m_2 \omega_2|^2},$$

where $\rho$ is Friedrichs' modifier, and $z$ in the summation $\sum$ is a representative of $t$. Clearly the sum $\sum$ converges uniformly, and does not depend on any choice of representative $z$.

Moreover, we define a function $p(y, t)$ in $D_{c+1}$ by putting

$$p(y, t) = s(y, t) + Kq(y)$$

where $K$ is a sufficient large constant. Since $D_c \subset D_{c+1}$ and $q$ is a strongly plurisubharmonic function in $B$, it follows that the function $p(y, t)$ is strongly plurisubharmonic in $D_c$. By Narasimhan [3], we can conclude that $D_c$ is a Stein manifold.

3. Now we shall prove our main theorem.

**Theorem.** Let $E$ be the product $S \times T$ of a Stein manifold $S$ and a complex torus $T$, and $\pi$ be the projection $E \to S$. Let $D$ be an open subset of $E$. Then $D$ is a Stein manifold if and only if $D$ is pseudoconvex and $\pi^{-1}(x)$ is not contained in $D$ for any point $x \in S$.

**Proof.** By Docquier-Grauert [1], there are a biholomorphic map $\sigma$ of $S$ onto a regular analytic set of a domain of holomorphy $(B, \beta)$ over $C^n$ and a holomorphic mapping $\rho$ of $B$ onto $\sigma(S)$ such that the restriction $\rho|_{\sigma(S)}$ is the identity of $\sigma(S)$. We define a mapping $\xi$ of the product $G = B \times T$ onto $E = S \times T$ by putting $\xi(x, t) = (\sigma^{-1}(\rho(x)), t)$ for $(x, t) \in G$. The inverse image $\xi^{-1}(D)$ of $D$ under the map is a pseudoconvex open subset of $G$ and satisfies the hypothesis of the lemma. $\xi^{-1}(D)$ is therefore a Stein manifold. Since $D$ is a regular analytic subset of the Stein manifold $\xi^{-1}(D)$, $D$ is also a Stein manifold.

**References**


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