

# Pacific Journal of Mathematics

**THE LEVI PROBLEM FOR A PRODUCT MANIFOLD**

YASOU MATSUGU

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Let  $S$  be a Stein manifold,  $T$  a one dimensional torus,  $\pi$  a projection of the product  $E = S \times T$  onto  $S$  and  $D$  a subdomain of  $E$ . The main object of this paper is to prove that  $D$  is a Stein manifold if and only if  $D$  is pseudoconvex in the sense of Cartan and  $\pi^{-1}(x)$  is not contained in  $D$  for any point  $x$  of  $S$ .

1. A subdomain  $D$  of a complex manifold  $M$  is called pseudoconvex if, for any boundary point  $x$  of  $D$  in  $M$ , there is a Stein neighborhood  $U$  of  $x$  in  $M$  such that  $U \cap D$  is also a Stein manifold. A pair  $(B, \beta)$  is called a domain over  $M$  if  $\beta$  is a locally biholomorphic mapping of a complex manifold  $B$  in  $M$ . A domain  $(B, \beta)$  over  $C^n$  is called a domain of holomorphy if there exists a holomorphic function  $f$  in  $B$  such that the radius of convergence of  $f$  at any point  $x$  of  $B$  is just the boundary distance  $d(x)$  of  $x$ .

Moreover we recall another definition. Let  $\varphi_i: M_i \rightarrow N_i$  be two mappings of a set  $M_i$  into a set  $N_i$  ( $i = 1, 2$ ). Then we define the product mapping  $\varphi_1 \times \varphi_2$  of the product set  $M_1 \times M_2$  into the product set  $N_1 \times N_2$  by putting  $(\varphi_1 \times \varphi_2)(x, y) = (\varphi_1(x), \varphi_2(y))$  for  $(x, y) \in M_1 \times M_2$ .

The proof of our theorem falls into two parts. We first prove it in the case of  $S = C^n$ , where we construct a strongly plurisubharmonic function by means of Hörmander [2] and reduce it to a result of Narasimhan [3]. In general case, using the imbedding of Docquier-Grauert [1], we reduce the theorem to the case of  $C^n$ .

2. Let  $(B, \beta)$  be a domain of holomorphy over  $C^n$ . In the complex plane  $C$  select any two complex numbers  $\omega_1, \omega_2$  which are linearly independent over the real number field  $R$ . The numbers  $\omega_1, \omega_2$  generate a subgroup  $\Gamma$  of  $C$ , namely

$$\Gamma = \{m_1\omega_1 + m_2\omega_2; m_1, m_2 \in Z = \text{additive group of integers}\}.$$

The quotient  $\tau = C/\Gamma$  is a one dimensional torus.  $\tau$  has a natural complex structure and is a compact Riemann surface. The natural map  $\tau: C \rightarrow \tau$  is a locally biholomorphic map. We denote by  $E = B \times T$  the product of two complex manifolds  $B$  and  $T$ , and by  $\pi: E \rightarrow B$  the projection.

We first prove the following lemma:

LEMMA. *Let  $D$  be a pseudoconvex open subset of  $E$  such that  $\pi^{-1}(x)$  is not contained in  $D$  for any point  $x$  of  $B$ . Then  $D$  is a Stein manifold.*

*Proof.* Let  $1 \times \tau$  be the product map of the identity 1 of  $B$  and the map  $\tau$ . The map  $1 \times \tau$  is a locally biholomorphic map  $B \times C$  onto  $E$ . If we denote by  $A$  the inverse image  $(1 \times \tau)^{-1}(D)$  of  $D$ ,  $A$  is pseudoconvex, because  $D$  is pseudoconvex.  $A$  is  $\Gamma$ -invariant, that is, for any fixed point  $\gamma \in \Gamma$ ,  $A$  is invariant under the transformation of  $B \times C: (y, z) \mapsto (y, z + \gamma)$ . Let  $\alpha$  be the restriction to  $A$  of the product map  $\beta \times 1$  of the map  $\beta$  and the identity map 1 of  $C$ , that is,  $\alpha(y, z) = (\beta(y), z)$  for  $(y, z) \in A$ .  $\alpha$  is a locally biholomorphic map of  $A$  into  $C^n \times C = C^{n+1}$  and  $(A, \alpha)$  is a pseudoconvex domain over  $C^{n+1}$ . The distance function  $d(y, z)$  of the domain  $A$  over  $C^{n+1}$  induces the function  $d(y, t)$  in  $D$ . Indeed, for any point  $(y, t) \in D$ ,  $y \in B$ ,  $t \in T$ , select two representatives  $z, z' \in C$  of the equivalence class  $t$ . Then there is  $\gamma \in \Gamma$  such that  $z' = z + \gamma$ . But  $A$  is  $\Gamma$ -invariant, and so  $d(y, z') = d(y, z)$ . Since  $A$  is pseudoconvex, by Oka [4], the function  $-\log d(y, z)$  is a continuous plurisubharmonic function in  $A$ . The function  $-\log d(y, t)$  is therefore a continuous plurisubharmonic function in  $D$ , and so is the function

$$1/d(y, t) = e^{-\log d(y, t)}.$$

On the other hand, since  $B$  is Stein, there is a real analytic strongly plurisubharmonic function  $q > 0$  with the following property: for any real number  $c > 0$ ,

$$B_c = \{y \in B; q(y) < c\} \Subset B.$$

The function

$$\gamma(y, t) = \frac{1}{d(y, t)} + q(y)$$

defined in  $D$  is a continuous plurisubharmonic function. It holds that

$$\begin{aligned} D_c &= \{(y, t) \in D; \gamma(y, t) < c\} \\ &\subset B_c \times T \cap \left\{ (y, t) \in D; d(y, t) > \frac{1}{c} \right\} \subset D \end{aligned}$$

for any real number  $c > 0$ .

Since  $D = \bigcup_{c>0} D_c$ , if we show that  $D_c$  is a Stein manifold, we know by Docquier-Grauert [1], that  $D$  is itself a Stein manifold.

Fix an arbitrary real number  $c > 0$ . For any point  $y \in B$ , we set

$$A(y) = \{z \in C; (y, \tau(z)) \in D\}.$$

By the hypothesis of the lemma, it follows that  $A(y) \Subset C$ . Select a complex-valued measurable function  $a(y)$  in  $B$  such that

$$a(y) \in C - A(y) \text{ for any point } y \in B.$$

For sufficiently small number  $\varepsilon$  with  $0 < \varepsilon < 1/(c + 1) < 1/c$ , we define the function  $s(y, t)$  in  $D_{c+1}$  as follows:

$$s(y, t) = \frac{1}{\varepsilon^{2n}} \int_{\xi \in B} \rho\left(\frac{y - \xi}{\varepsilon}\right) \sum_{m_1, m_2 = -\infty}^{+\infty} \frac{d\lambda(\xi)}{|z - \alpha(\xi) - m_1\omega_1 - m_2\omega_2|^2},$$

where  $\rho$  is Friedrichs' modifier, and  $z$  in the summation  $\sum$  is a representative of  $t$ . Clearly the sum  $\sum$  converges uniformly, and does not depend on any choice of representative  $z$ .

Moreover, we define a function  $p(y, t)$  in  $D_{c+1}$  by putting

$$p(y, t) = s(y, t) + Kq(y)$$

where  $K$  is a sufficient large constant. Since  $D_c \subset D_{c+1}$  and  $q$  is a strongly plurisubharmonic function in  $B$ , it follows that the function  $p(y, t)$  is strongly plurisubharmonic in  $D_c$ . By Narasimhan [3], we can conclude that  $D_c$  is a Stein manifold.

3. Now we shall prove our main theorem.

**THEOREM.** *Let  $E$  be the product  $S \times T$  of a Stein manifold  $S$  and a complex torus  $T$ , and  $\pi$  be the projection  $E \rightarrow S$ . Let  $D$  be an open subset of  $E$ . Then  $D$  is a Stein manifold if and only if  $D$  is pseudoconvex and  $\pi^{-1}(x)$  is not contained in  $D$  for any point  $x \in S$ .*

*Proof.* By Docquier-Grauert [1], there are a biholomorphic map  $\sigma$  of  $S$  onto a regular analytic set of a domain of holomorphy  $(B, \beta)$  over  $C^n$  and a holomorphic mapping  $\rho$  of  $B$  onto  $\sigma(S)$  such that the restriction  $\rho|_{\sigma(S)}$  is the identity of  $\sigma(S)$ . We define a mapping  $\xi$  of the product  $G = B \times T$  onto  $E = S \times T$  by putting  $\xi(x, t) = (\sigma^{-1}(\rho(x)), t)$  for  $(x, t) \in G$ . The inverse image  $\xi^{-1}(D)$  of  $D$  under the map is a pseudoconvex open subset of  $G$  and satisfies the hypothesis of the lemma.  $\xi^{-1}(D)$  is therefore a Stein manifold. Since  $D$  is a regular analytic subset of the Stein manifold  $\xi^{-1}(D)$ ,  $D$  is also a Stein manifold.

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