REAL PARTS OF UNIFORM ALGEBRAS

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This paper is concerned with identifying those uniform algebras B on $\Gamma = \{z \colon |z| = 1\}$ for which $\operatorname{Re} B$—the space of real parts of the functions in B—equals $\operatorname{Re} A$, where A denotes the disk algebra. It is shown that for any such algebra, there is an absolutely continuous homeomorphism Φ of Γ onto Γ so that $B = A(\Phi) = \{f(\Phi) \colon f \in A\}$. A partial converse to this theorem also holds: If Φ is a homeomorphism of Γ onto itself which is of class C^2 with nowhere vanishing derivative, then $\operatorname{Re} A(\Phi) = \operatorname{Re} A$.

For completeness, we recall that a uniform algebra on Γ is defined as a subalgebra of $C(\Gamma)$ which is closed in the norm $\|f\| = \max_\Gamma |f|$, contains the constants and separates the points of Γ. The disk algebra is the particular uniform algebra consisting of all functions in $C(\Gamma)$ which extend continuously to $\{z \colon |z| \leq 1\}$ to be analytic on $D = \{z \colon |z| < 1\}$. Note that if Φ is a homeomorphism of Γ onto itself, then $A(\Phi)$ is a uniform algebra. In this paper we will frequently use the convention of writing $f(\theta)$ for $f(e^{i\theta})$ when f belongs to $C(\Gamma)$.

The first result along the lines discussed in this paper is due to Hoffman and Wermer [7]. They prove that if B is a uniform algebra on a compact Hausdorff space X such that $\operatorname{Re} B$ is closed in the norm of uniform convergence, then $B = C(X)$; in particular, the theorem holds if $\operatorname{Re} B = C_r(X)$. A generalization of this fact by Sidney and Stout [10] says that if K is a closed subset of X and $\operatorname{Re} B_K$ is uniformly closed, then $B_K = C(K)$. Bernard [1, 2, 3] has provided extensions in a different direction. He shows that a Banach algebra $B \subseteq C(X)$, with any norm, for which $\operatorname{Re} B$ is closed under uniform convergence must equal $C(X)$. Further, he gives some sufficient conditions on two Banach algebras $B_1 \subseteq B_2 \subseteq C(X)$, with $\operatorname{Re} B_1 = \operatorname{Re} B_2$, to conclude that $B_1 = B_2$.

The theorems in this paper have previously been announced in Abstracts 71T-B10 and 71T-B94 of the *Notices of the American Mathematical Society*, 18 (1971). The author wishes to express his appreciation to Professor John Wermer for his many helpful suggestions and conversations during the research which led to these results.

II. **Algebras on Γ with the same real parts.**

THEOREM 1. *Let B be a uniform algebra on Γ with $\operatorname{Re} B = \operatorname{Re} A$. Then there exists a homeomorphism Φ of Γ onto Γ so that $B = A(\Phi)$.*

Proof. (1). Since $\cos(\theta)$ belongs to Re B, there exists $v(\theta)$ so that $\psi(\theta) = \cos(\theta) + iv(\theta)$ belongs to B. Re $\psi(\theta)$ is one-to-one and decreasing on $[0, \pi]$ and one-to-one increasing on $[-\pi, 0]$; hence $C - \psi(\Gamma)$ has a bounded component, or else $\psi(\Gamma)$ defines a single curve traced twice, once in each direction. In the latter case, $\psi(\Gamma)$ is a Jordan arc; it then follows by Mergelyan's theorem that the function Re z is a uniform limit of polynomials on $\psi(\Gamma)$ and we can conclude that Re $\psi(\theta) = \cos(\theta) \in B$. Similarly there exists $u(\theta)$ in Re B so that $\psi_1(\theta) = u(\theta) + i\sin(\theta)$ belongs to B; and either $C - \psi_1(\Gamma)$ has bounded component or else $\sin(\theta) \in B$.

If both $\cos(\theta)$ and $\sin(\theta)$ are in B, then $B = C(\Gamma)$ and Re $B \neq$ Re A. Thus at least one of $C - \psi(\Gamma)$ and $C - \psi_1(\Gamma)$ has a bounded component. We will assume that $C - \psi(\Gamma)$ has a bounded component, which we call W. The remainder of the proof using ψ_1 involves only minor changes.

(2). The region W is bounded by two arcs on $\psi(\Gamma)$. Precisely, there are values θ_1 and θ_2 with $0 \leq \theta_1 < \theta_2 \leq \pi$ for which $\psi(\theta_i) = \psi(-\theta_i)$, $i = 1, 2$ and $\partial W = \psi([\theta_1, \theta_2]) \cup \psi([-\theta_2, -\theta_1])$. For θ not contained in $[\theta_1, \theta_2] \cup [-\theta_2, -\theta_1]$, Re $\psi(\theta) <$ Re $\psi(\theta_2)$ or Re $\psi(\theta) >$ Re $\psi(\theta_1)$.

Let τ be the Riemann map of W onto D; τ extends continuously to $\bar{W}$, mapping ∂W onto Γ in a one-to-one fashion. For convenience we may suppose $(\tau \circ \psi)(\theta_1) = 1$. We further extend τ by setting $\tau(z) = (\tau \circ \psi)(\theta_1)$ for z with Re $z >$ Re $\psi(\theta_1)$ and $\tau(z) = (\tau \circ \psi)(\theta_2)$ for z with Re $z <$ Re $\psi(\theta_2)$. Then, letting K denote $\psi(\Gamma)$ together with all its bounded complementary components, τ is continuous on K and is analytic on its interior. Appealing again to Mergelyn's theorem, we conclude that τ is the uniform limit on K of polynomials. Hence $\tau \circ \psi$ belongs to B. Put $\Phi = \tau \circ \psi$. Then Φ maps Γ onto Γ in the following manner: (see figure)
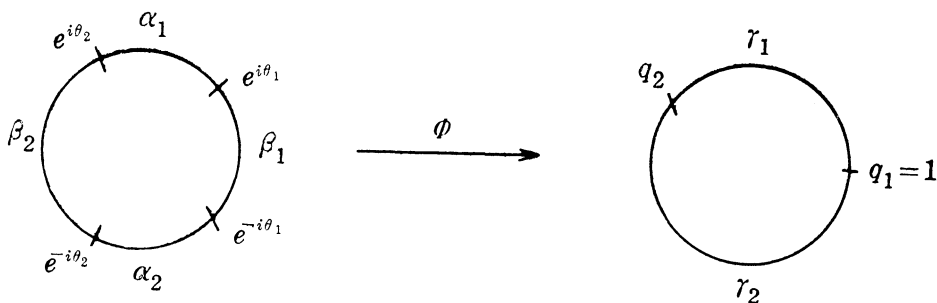
FIGURE

Φ takes the open arcs α_1 and α_2 homeomorphically onto γ_1 and γ_2, and the closed arcs β_1 and β_2 onto the points $q_1 = \Phi(\theta_1) = 1$, $q_2 = \Phi(\theta_2)$ respectively.

(3) For each measure ν on Γ we define a measure ν^* on Γ by $\nu^*(E) = \nu(\Phi^{-1}(E))$, and for g in B and λ on $\gamma_1 \cup \gamma_2$ we put $g^*(\lambda) = g(\Phi^{-1}(\lambda))$. Let μ be a measure on Γ with $\mu \perp B$ and $\mu^* \neq 0$. Such measures must exist as otherwise every $\mu \perp B$ would be the zero measure on $\alpha_1 \cup \alpha_2$. Then for every closed subarc E of $\alpha_1 \cup \alpha_2$ of positive linear measure, it would follow that $B|_E = C(E)$. Hence $\operatorname{Re} A|_E = C_R(E)$ which is impossible.

If g belongs to B and $|z| \neq 1$, define

$$T(g, z) = \int_{|\lambda|=1} \frac{(gd\mu)^*}{\lambda - z}(\lambda) .$$

The function T is analytic on D and also on $\{z: |z| > 1\}$. We claim that $T(g, z) = 0$ if $|z| > 1$. For, on that set,

$$T(g, z) = \sum_{n=0}^{\infty} \frac{-1}{z^{n+1}} \int_{|\lambda|=1} \lambda^n (gd\mu)^*(\lambda) .$$

The term for $n = 0$ is $\int_{|\lambda|=1} (gd\mu)^* = \int_{\Gamma} gd\mu = 0.$

For each $n > 1$,

$$\begin{aligned}
\int_{|\lambda|=1} \lambda^n (gd\mu)^* &= \int_{\gamma_1 \cup \gamma_2} \lambda^n g^*(\lambda) d\mu^*(\lambda) \\
&\quad + q_1^n \int_{\beta_1} gd\mu + q_2^n \int_{\beta_2} gd\mu \\
&= \int_{\alpha_1 \cup \alpha_2} \Phi^n(t) g(t) d\mu(t) \\
&\quad + \int_{\beta_1} q_1^n g(t) d\mu(t) + \int_{\beta_2} q_2^n g(t) d\mu(t) \\
&= \int_{|t|=1} \Phi^n(t) g(t) d\mu(t) \\
&= 0, \text{ since } \Phi^n g \text{ belongs to } B.
\end{aligned}$$

(4) Let ν be any measure on Γ. Put $h(\theta) = $ the ν-measure of the arc $\widehat{1e^{i\theta}}$. At any point $\lambda_0 = e^{i\theta_0}$ where $dh/d\theta(\theta_0)$ exists, we set $d\nu/d\lambda(\lambda_0) = dh/d\theta(\theta_0)$. Then for almost all values of λ_0 on Γ, we have

$$\lim_{\substack{re^{i\theta} \to \lambda_0 \\ r<1}} \left[\int_{|\lambda|=1} \frac{d\nu(\lambda)}{\lambda - re^{i\theta}} - \int_{|\lambda|=1} \frac{d\nu(\lambda)}{\lambda - \frac{1}{r}e^{i\theta}} \right] = 2\pi i \frac{d\nu}{d\lambda}(\lambda_0) .$$

The limit is taken along non-tangential curves lying within the unit disk. The statement follows from Fatou's theorem by rewriting the terms inside the brackets as a Poisson integral.

By applying this equality to the measures $(gd\mu)^*$ and using that $T(g, z) = 0$ for $|z| > 1$ we conclude that, for almost all λ_0 on $\gamma_1 \cup \gamma_2$,

$$\lim_{\substack{z \to \lambda_0 \\ |z| < 1}} T(g, z) = \lim_{\substack{z \to \lambda_0 \\ |z| < 1}} \int_{|\lambda|=1} \frac{(gd\mu)^*}{\lambda - z}(\lambda)$$

$$= 2\pi i g^*(\lambda_0) \frac{d\mu^*}{d\lambda}(\lambda_0) .$$

The function $T(1, z)$ is the Cauchy transform of the nonzero measure μ^*. Since the Cauchy transform of a measure is zero almost everywhere-$dxdy$ if and only if the measure itself is zero[1], and since $T(1, z)$ vanishes identically for $|z| > 1$ it follows that $T(1, z)$ cannot vanish identically on D. Also, $T(1, z)$ has the boundary value $2\pi i(d\mu^*/d\lambda)(\lambda_0)$ for almost all λ_0 on $\gamma_1 \cup \gamma_2$. (Here boundary value means the limiting value along non-tangential curves approaching λ_0 from inside the unit disk). It is a theorem of Privalov and Lusin[2] that a function which is analytic on D and which has the non-tangential limiting value zero on a set of positive measure on Γ must vanish identically on D. Applying this result to $T(1, z)$ we can conclude that $d\mu^*/d\lambda(\lambda_0) \neq 0$ for almost all λ_0 on $\gamma_1 \cup \gamma_2$.

For each g contained in B, $T(g, z)$ is analytic on D and has the boundary values $2\pi i g^*(\lambda_0) d\mu^*/d\lambda(\lambda_0)$ almost everywhere on $\gamma_1 \cup \gamma_2$. Consequently, the function $G(z) = T(g, z)/T(1, z)$ is meromorphic on D and has the boundary values $g^*(\lambda_0)$ almost everywhere on $\gamma_1 \cup \gamma_2$.

(5) Now we show that each function $G(z)$ obtained in the manner just described is analytic on D. To this end, take z_0 in D so that $T(1, z_0) \neq 0$. The functional S defined on B by $S(g) = G(z_0)$ is linear. For any two functions g_1 and g_2 belonging to B, the function $t(z) = T(1, z)T(g_1 g_2, z) - T(g_1, z)T(g_2, z)$ is analytic on D and has zero boundary values almost everywhere. Hence, by the theorem Privalov and Lusin cited in the last paragraph, $t \equiv 0$ and the meromorphic functions

$$\frac{T(g_1 g_2, z)}{T(1, z)} \text{ and } \frac{T(g_1, z)}{T(1, z)} \frac{T(g_2, z)}{T(1, z)}$$

are identically equal whenever defined. Thus S is a multiplicative linear functional on the commutative Banach algebra B and as such has norm one. That is, $|G(z_0)| \leq \|g\|$ for every z_0 with $T(1, z_0) \neq 0$. As the zeros of $T(1, z)$ form a discrete set, all the singularities of G are removable and G extends analytically to D. Furthermore, G is bounded on D with $|G(z)| \leq \|g\|$ for all z contained in D.

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[1] See Browder [4] or Gamelin [5] for details.
[2] See Goluzin [6], p. 428, or Privalov [9].

(6). The next step is to show G belongs to A. G can be defined from g^* using the Poisson integral formula.   A well-known property of that formula says that G has the nontangential boundary value $g^*(\lambda_0)$ at every point of continuity of g^*.   Thus G has the boundary values g^* everywhere on $\gamma_1 \cup \gamma_2$.   The following result may be used to prove that G has continuous boundary values everywhere on Γ.

THEOREM.[3]   *If $f(z) \to a$ as $z \to \infty$ along a straight line and $f(z) \to b$ as $z \to \infty$ along another straight line, and $f(z)$ is regular and bounded in the angle between, then $a = b$ and $f(z) \to a$ uniformly in the angle.*

In the present case, the four limits $\lim_{\lambda \to q_i} g^*(\lambda)$, for $i, j = 1, 2$, all exist, because of the continuity of the function g.   By considering neighborhoods of q_1 and q_2 in $\bar{D}$, mapping them to the upper half plane sending q_1 and q_2 in turn to the point at ∞, we can conclude that $\lim_{z \to q_1} G(z)$ and $\lim_{z \to q_2} G(z)$ both exist and that $G \in A$.   In fact, we must have that $\lim_{z \to q_1} G(z) = g(\theta_1) = g(-\theta_1)$ and $\lim_{z \to q_2} G(z) = g(\theta_2) = g(-\theta_2)$, for every function g belonging to B.   But as B separates points on Γ, this is impossible unless $\theta_1 = 0$, $\theta_2 = \pi$.   There, Φ is a homeomorphism of Γ onto Γ.

(7). Since Φ belongs to B, $A(\Phi) \subseteq B$ or equivalently $A \subseteq B(\Phi^{-1})$. As $\operatorname{Re} B = \operatorname{Re} A \neq C_R(\Gamma)$, then $B(\Phi^{-1}) \neq C(\Gamma)$.   Hence by Wermer's Maximality Theorem, $B(\Phi^{-1}) = A$ or $B = A(\Phi)$.

THEOREM 2.   *Let Φ be a homeomorphism of Γ onto Γ satisfying $\operatorname{Re} A(\Phi) = \operatorname{Re} A$.   Then Φ is absolutely continuous.*

Proof.   We also have that $\operatorname{Re} A(\Phi^{-1}) = \operatorname{Re} A$.   Let m denote Lebesgue measure on Γ.   We claim that if K is a closed set with $m(K) = 0$, then $m(\Phi(K)) = 0$.   For, let $g \in C(\Phi(K))$.   Then $g \circ \Phi|_K \in C(K)$. Since we assume that K is closed with $m(K) = 0$, it follows that $A|_K = C(K)$[4], so that there exists G belonging to A with $G|_K = g \circ \Phi|_K$. Thus $g|_{\Phi(K)} = G \circ \Phi^{-1}|_{\Phi(K)}$.   In particular, if $g \in C_R(\Phi(K))$, then $g \in \operatorname{Re} A(\Phi^{-1})|_{\Phi(K)} = \operatorname{Re} A|_{\Phi(K)}$.   Hence $\operatorname{Re} A|_{\Phi(K)}$ is closed in the uniform norm.   This implies that $A|_{\Phi(K)} = C(\Phi)K)$[5].   If $m(\Phi(K)) > 0$, A would necessarily contain a nonzero function which vanished on a set of positive measure on Γ.   This is impossible since the functions in A satisfy the Jensen inequality $\log|f(0)| \leqq 1/2\pi \int_\Gamma \log|f| d\theta$.   Hence we must have $m(\Phi(K)) = 0$.

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[3] Titchmarsh [11], 5,64.

[4] This is a theorem of Rudin and Carleson. See [5] for details.

[5] This result is due to Sidney and Stout [10].

Now if we define a measure μ on Γ by $\mu(E) = m(\Phi(E))$ we have that μ is absolutely continuous with respect to m. To see this, let $m(E) = 0$. There exists an F_σ set $F \subseteq E$ such that $\mu(E \setminus F) = 0$. We can write $F = \bigcup_{n=1}^{\infty} K_n$ where K_n is closed and $K_n \subseteq K_{n+1}$ for each n. From the last paragraph, it follows that $\mu(K_n) = 0$. Hence $\mu(F) = 0$ and $m(\phi(E)) = \mu(E) = 0$.

For an arc $\widehat{\alpha\beta}$, $\mu(\widehat{\alpha\beta}) = |\Phi(\beta) - \Phi(\alpha)|$. The absolute continuity of μ implies that for $\varepsilon > 0$, there is a $\delta > 0$ so that $m(E) < \delta \Rightarrow \mu(E) < \varepsilon$. Hence if $\widehat{\alpha_i\beta_i}$, $1 \leq i \leq N$, are disjoint arcs with $\sum_{i=1}^{N} m(\widehat{\alpha_i\beta_i}) < \delta$ then $\sum_{i=1}^{N} |\Phi(\beta_i) - \Phi(\alpha_i)| < \varepsilon$. Therefore Φ is absolutely continuous.

REMARKS. (1) Theorem 1 cannot be extended freely: there exist examples of sets X, and two uniform algebras A and B defined on X satisfying $\operatorname{Re} A = \operatorname{Re} B$, but for which no homeomorphism Φ yields $B = A(\Phi)$.

(2) If, on Γ, $B = \bar{A}$, the set of complex conjugates of the functions in A, the map $z \to \bar{z}$ is a suitable choice for Φ.

III. Sufficient conditions on Φ to conclude $\operatorname{Re} A(\Phi) = \operatorname{Re} A$.

DEFINITION. Let $\operatorname{Re} A^\dagger = \{f : f$ is continuous on R, with period 2π, and so that the function F, defined on Γ by $F(e^{i\theta}) = f(\theta)$, belongs to $\operatorname{Re} A\}$.

DEFINITION. For u in $\operatorname{Re} A$, define $|||u||| = ||u|| + ||\tilde{u}||$ where $||\cdot||$ is the uniform norm on Γ, $u + i\tilde{u}$ belongs to A and $\tilde{u}(0) = 0$. For f in $(\operatorname{Re} A)^\dagger$, $\tilde{f}$ denotes the function in $(\operatorname{Re} A)^\dagger$ such that $G(e^{i\theta}) = f(\theta) + i\tilde{f}(\theta)$ belongs to A and $\operatorname{Im} G(0) = 0$. We put $||f|| = \max_R |f|$ and

$$|||f||| = ||f|| + ||\tilde{f}|| .$$

With $||| \cdot |||$ as a norm, $\operatorname{Re} A$ and $(\operatorname{Re} A)^\dagger$ have the structure of real Banach spaces in which smooth functions are dense.

LEMMA 1. *Suppose ϕ is a homeomorphism of $[-\pi, \pi]$ onto $[\alpha, 2\pi + \alpha]$, with inverse ψ, and satisfying*
   (i) $\phi(0) = 0$,
   (ii) *ϕ is of class C^2, $\phi'(-\pi) = \phi'(\pi)$, $\psi'(\alpha) = \psi'(\alpha + 2\pi)$,*
   (iii) *$|\phi'(t)| \geq \gamma > 0$, $|\psi'(t)| \geq \delta > 0$ and $|\psi''(t)| \leq M$ for all t in* $[-\pi, \pi]$.
Then, for a smooth function f in $(\operatorname{Re} A)^\dagger$, $f \circ \phi$ which is defined on $[-\pi, \pi]$, if extended periodically to R, lies in $(\operatorname{Re} A)^\dagger$. There exists a constant $C > 0$, depending only on γ, δ and M, so that

$$|\widetilde{f \circ \phi}(0) - \tilde{f}(0)| \leq C||f|| .$$

Consequently,

$$|\widetilde{f \circ \phi}(0)| \leqq (C+1)|||f|||\ .$$

Proof. We can assume, with no loss of generality, that $\phi'(t) \geqq \gamma$ on $[-\pi, \pi]$. The proof if ϕ is a decreasing function is almost identical. Our assumption means that $\phi(-\pi) = \alpha < 0$.

Let f be a smooth function in $(\mathrm{Re}\,A)^\dagger$. As $\phi(-\pi) + 2\pi = \phi(\pi)$, then $f \circ \phi$ can be extended to $\boldsymbol{R}$, continuously with period 2π. Since f and ϕ are smooth and $\phi'(-\pi) = \phi'(\pi)$, $f \circ \phi$ is smooth and thus belongs to $(\mathrm{Re}\,A)^\dagger$.

Assume first that $f(0) = 0$. The integral formula for evaluating harmonic conjugates gives[6]

$$\tilde{f}(0) = -\frac{1}{2\pi}\int_0^\pi \frac{f(t) - f(-t)}{\tan \dfrac{t}{2}}dt$$

$$(1) \qquad\qquad = -\frac{1}{2\pi}\int_{-\pi}^\pi \frac{f(t)}{\tan \dfrac{t}{2}}dt$$

$$= -\frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} \frac{f(t)}{\tan \dfrac{t}{2}}dt\ .$$

Also $\widetilde{f \circ \phi}(0) = -\frac{1}{2\pi}\int_0^\pi \frac{(f \circ \phi)(t) - (f \circ \phi)(-t)}{\tan \dfrac{t}{2}}dt$

$$= -\frac{1}{2\pi}\int_0^\pi \frac{(f \circ \phi)(t)}{\tan \dfrac{t}{2}}dt - \frac{1}{2\pi}\int_{-\pi}^0 \frac{(f \circ \phi)(t)}{\tan \dfrac{t}{2}}dt$$

$$(2) \qquad\qquad = -\frac{1}{2\pi}\int_0^{2\pi+\alpha} \frac{f(u)}{\tan \dfrac{\psi(u)}{2}}\psi'(u)du$$

$$-\frac{1}{2\pi}\int_\alpha^0 \frac{f(u)}{\tan \dfrac{\psi(u)}{2}}\psi'(u)du$$

$$= -\frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} \frac{f(u)\psi'(u)}{\tan \dfrac{\psi(u)}{2}}du\ .$$

All these integrals are absolutely convergent because of the smoothness of f and ϕ, and because $f(0) = \phi(0) = 0$. We use the fact that $x \cot x$ has a series expansion

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[6] Zygmund [13] vol. I, p. 131.

$$x \cot x = \sum_{n=0}^{\infty} A_n x^n$$

which converges for $|x| < \pi$. Hence

(3) $$\frac{1}{\tan x} = \frac{1}{x} + a(2x)$$

where $a(t)$ is real analytic for $|t| < 2\pi$. Substitution of (3) into (1) and (2) yields

$$\tilde{f}(0) = -\frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} f(t)\left[\frac{1}{\dfrac{t}{2}} + a(t)\right]dt$$

and

$$\widetilde{f\circ\phi}(0) = -\frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} f(t)\left[\frac{1}{\dfrac{\psi(t)}{2}} + a(\psi(t))\right]\psi'(t)dt \ .$$

Since $-2\pi < \alpha < 0$, $a(t)$ is continuous on $[\alpha, 2\pi + \alpha]$, and since $-\pi \leqq \psi(t) \leqq \pi$, these two integrals are absolutely convergent. Subtracting them, we get

(4) $$|\widetilde{f\circ\phi}(0) - \tilde{f}(0)| \leqq \frac{1}{2\pi}\|f\|\int_{\alpha}^{2\pi+\alpha}\left\{2\left|\frac{\psi'(t)}{\psi(t)} - \frac{1}{t}\right|\right.$$
$$\left. + |a(\psi(t))|\,\psi'(t) + |a(t)|\right\}dt \ .$$

We estimate each term in the integrand. For the first term, we write $\psi(t) = t\psi_1(t)$. ψ_1 is smooth on $[\alpha, 2\pi + \alpha]$ and $\psi_1(0) \neq 0$. For each t in $[\alpha, 2\pi + \alpha]$,

$$\psi(t) = \int_0^1 \frac{d}{ds}\psi(st)ds$$
$$= \int_0^1 t\psi'(st)ds$$
$$= t\int_0^1 \psi'(st)ds \ .$$

Thus

$$\psi_1(t) = \int_0^1 \psi'(st)ds \geqq \delta \ .$$

Also,

$$\psi_1'(t) = \int_0^1 \frac{d}{dt} \psi'(st) ds$$

$$= \int_0^1 s\psi''(st) ds$$

(5)
$$\Longrightarrow |\psi_1'(t)| \leqq M \int_0^1 sds = \frac{M}{2}$$

$$\Longrightarrow \left| \frac{\psi'(t)}{\psi(t)} - \frac{1}{t} \right| = \left| \frac{\psi_1'(t)}{\psi_1(t)} \right| \leqq \frac{M}{2\delta}$$

$$\Longrightarrow \int_\alpha^{2\pi+\alpha} \left| \frac{\psi'(t)}{\psi(t)} - \frac{1}{t} \right| dt \leqq \frac{M\pi}{\delta} \ .$$

For the second term in (4),

(6)
$$\int_\alpha^{2\pi+\alpha} |a(\psi(t))| \psi'(t) dt = \int_{-\pi}^\pi |a(t)| dt \ .$$

For the last term, we note that

$$2\pi + \alpha = \phi(\pi)$$

$$= \phi(\pi) - \phi(0)$$

$$= \pi\phi'(r) \text{ for some } r \in [0, \pi] \ .$$

$$\Longrightarrow 2\pi + \alpha \geqq \gamma\pi$$

$$\Longrightarrow \alpha \geqq \gamma\pi - 2\pi \ .$$

Similarly,

$$-\alpha = -\phi(-\pi)$$

$$= \phi(0) - \phi(-\pi)$$

$$= \pi\phi'(r_1) \text{ for some } r_1 \in [-\pi, 0] \ .$$

$$\Longrightarrow -\alpha \geqq \gamma\pi$$

$$\Longrightarrow 2\pi + \alpha \leqq 2\pi - \gamma\pi \ .$$

Thus,

(7)
$$\int_\alpha^{2\pi+\alpha} |a(t)| dt \leqq \int_{-2\pi+\gamma\pi}^{2\pi-\gamma\pi} |a(t)| dt \ .$$

The inequalities (4), (5), (6) and (7) imply

(8)
$$|\widetilde{f\circ\phi}(0) - \tilde{f}(0)| \leqq \|f\| \frac{1}{2\pi} \Big[2\pi\frac{M}{\delta} + \int_{-\pi}^\pi |a(t)| dt$$

$$+ \int_{-2\pi+\gamma\pi}^{2\pi-\gamma\pi} |a(t)| dt \Big] \ .$$

Put $C/2$ equal to the coefficient of $\|f\|$ in (8). Note that C depends only on M, δ and γ. That inequality applies to smooth functions in

$(\operatorname{Re} A)^{\dagger}$ vanishing at $x = 0$. Now, let f be any smooth function in $(\operatorname{Re} A)^{\dagger}$. We can apply (8) to $f - f(0)$. Since $\tilde{f}(0) = \widetilde{(f - f(0))}(0)$ and $\widetilde{f \circ \phi}(0) = \widetilde{[(f - f(0)) \circ \phi]}(0)$, we conclude that

$$|\widetilde{f \circ \phi}(0) - \tilde{f}(0)| \leqq \|f - f(0)\|\frac{C}{2}$$

$$\Longrightarrow |\widetilde{f \circ \phi}(0) - \tilde{f}(0)| \leqq C\|f\|$$

$$\Longrightarrow |\widetilde{f \circ \phi}(0)| \leqq C\|f\| + |\tilde{f}(0)|$$

$$\leqq C\|f\| + \|\tilde{f}\|$$

$$\leqq (C + 1)\|\|f\|\|.$$

DEFINITION. For $\beta > 0$ define C_β by
$C_\beta = \{\Phi: \Phi$ is a homeomorphism of Γ onto Γ of class C^2 satisfying:

$$\beta \leqq \left|\frac{d}{dt}\Phi(e^{it})\right| \leqq \frac{1}{\beta}$$

$$\beta \leqq \left|\frac{d}{dt}\Phi^{-1}(e^{it})\right| \leqq \frac{1}{\beta}$$

$$\left|\frac{d^2}{dt^2}\Phi(e^{it})\right| \leqq \frac{1}{\beta}$$

$$\left|\frac{d^2}{dt^2}\Phi^{-1}(e^{it})\right| \leqq \frac{1}{\beta}\}.$$

LEMMA 2. *There exists $K_\beta > 0$, depending only on β such that for all Φ belonging to C_β with $\Phi(1) = 1$ and for all smooth functions F in $\operatorname{Re} A$, we have*

$$|\widetilde{F(\Phi)}(1)| \leqq K_\beta \|\|F\|\|.$$

Proof. Define ϕ on $[-\pi, \pi]$ by $\Phi(e^{it}) = e^{i\phi(t)}$. Then ϕ is of class C^2 and $\phi(0) = 0$. If $\Psi = \Phi^{-1}$ and $\Psi(e^{iu}) = e^{i\psi(u)}$ for $u \in \phi([-\pi, \pi])$, then ψ is the inverse of ϕ. Furthermore

$$|\phi'(t)| = \left|\frac{d}{dt}\Phi(e^{it})\right|,$$

$$|\psi'(t)| = \left|\frac{d}{dt}\Psi(e^{it})\right|,$$

and

$$|\psi''(t)| = \left|\frac{d^2}{dt^2}\Psi(e^{it})\right| + \left|\frac{d}{dt}\Psi(e^{it})\right|^2.$$

If we set $\gamma = \beta, \delta = \beta$ and $M = 1/\beta + 1/\beta^2$, then ϕ satisfies the

hypotheses of Lemma 1, and if we apply the lemma to ϕ, the constant C obtained in (8) depends only on β.

If F is a smooth function in Re A and if f is defined on $\mathbf{R}$ by $f(\theta) = F(e^{i\theta})$, then f is a smooth function in $(\text{Re } A)^\dagger$ with $|||F||| = |||f|||$ and $\widetilde{f \circ \phi}(0) = \widetilde{F \circ \Phi}(1)$. Hence (9) yields that

$$|\widetilde{F(\Phi)}(1)| \leqq (C + 1)|||F||| .$$

Therefore the lemma is proved by defining $K_\beta = C + 1$.

LEMMA 3. *If Φ belongs to C_β, then*

$$|\widetilde{F(\Phi)}(1)| \leqq K_\beta |||F|||$$

for all smooth functions F contained in Re A.

Proof. Define λ by $\Phi(1) = e^{i\lambda}$ and set $\Phi_1 = e^{-i\lambda}\Phi$. Note that $\Phi_1(1) = 1$. If $\Psi = \Phi^{-1}$ and $\Psi_1 = \Phi_1^{-1}$ then $\Psi_1(e^{it}) = \Psi(e^{i(t+\lambda)})$. We check that $\Phi_1 \in C_\beta$ by noting that

$$\left| \frac{d}{dt} \Phi_1(e^{it}) \right| = \left| \frac{d}{dt} \Phi(e^{it}) \right|$$

$$\left| \frac{d^2}{dt^2} \Phi_1(e^{it}) \right| = \left| \frac{d^2}{dt^2} \Phi(e^{it}) \right|$$

$$\left| \frac{d}{du} \Psi_1(e^{iu}) \right| = \left| \frac{d}{du} \Psi(e^{i(u+\lambda)}) \right|$$

$$\left| \frac{d^2}{du^2} \Psi_1(e^{iu}) \right| = \left| \frac{d^2}{du^2} \Psi(e^{i(u+\lambda)}) \right| .$$

Thus, by Lemma 2, $|\widetilde{G(\Phi_1)}(1)| \leqq K_\beta |||G|||$ for all smooth functions G belonging to Re A. Now, given F smooth in Re A, put $G(e^{i\theta}) = F(e^{i(\theta+\lambda)})$. Then $G(\Phi_1) = F(\Phi)$, $\widetilde{G(\Phi_1)} = \widetilde{F(\Phi)}$ and $|||F||| = |||G|||$. Hence

$$|\widetilde{F(\Phi)}(1)| \leqq K_\beta |||F||| .$$

LEMMA 4. *For $\Phi \in C_\beta$ and ζ with $|\zeta| = 1$,*

$$|\widetilde{F(\Phi)}(\zeta)| \leqq K_\beta |||F|||$$

for all smooth functions F in Re A.

Proof. Fix $\zeta = e^{i\lambda}$ on Γ. Define Φ^* by $\Phi^*(e^{i\theta}) = \Phi(e^{i(\theta+\lambda)})$. By comparing the derivatives of Φ, Φ^* and their respective inverses in a manner similar to that in Lemma 3, we can show that $\Phi^* \in C_\beta$. Therefore, for a smooth function F in Re A,

$$|\widetilde{F(\varPhi^*)}(1)| \leqq K_\beta |||F|||\ .$$

But $\widetilde{F(\varPhi)}(\zeta) = \widetilde{F(\varPhi^*)}(1)$.   Hence

$$|\widetilde{F(\varPhi)}(\zeta)| \leqq K_\beta |||F|||\ .$$

LEMMA 5.   *If $\varPhi$ belongs to C_β for some $\beta > 0$, then* $\operatorname{Re} A(\varPhi) \subseteqq \operatorname{Re} A$.

Proof.   From Lemma 4, it follows that if F is a smooth function in $\operatorname{Re} A$,

$$\|\widetilde{F(\varPhi)}\| \leqq K_\beta |||F|||\ .$$

Hence

$$|||F(\varPhi)||| = \|F(\varPhi)\| + \|\widetilde{F(\varPhi)}\|$$
$$\leqq \|F\| + K_\beta |||F||| \leqq (K_\beta + 1)|||F|||\ .$$

Defining T by $T(F) = F(\varPhi)$, we see that T is a bounded linear transformation defined on smooth functions in $\operatorname{Re} A$ and mapping into $\operatorname{Re} A$. The domain of T is dense in $\operatorname{Re} A$, so that T has a norm preserving extension which maps $\operatorname{Re} A \to \operatorname{Re} A$; we also denote this extension by T. If u belongs to $\operatorname{Re} A$, there are smooth function F_n in $\operatorname{Re} A$ with $|||F_n - u||| \to 0$, and therefore $|||F_n(\varPhi) - Tu||| \to 0$. These imply that $\|F_n - u\| \to 0$ and $\|F_n(\varPhi) - Tu\| \to 0$. Hence $Tu = u(\varPhi)$. That is $u(\varPhi)$ belongs to $\operatorname{Re} A$ for every u in $\operatorname{Re} A$.

THEOREM 3.   *Let $\varPhi$ belong to C_β for some $\beta > 0$. Then* $\operatorname{Re} A(\varPhi) = \operatorname{Re} A$.

Proof.   We apply Lemma 5 to $\varPhi$ and $\varPhi^{-1}$, both of which are contained in C_β, to conclude $\operatorname{Re} A(\varPhi) \subseteqq \operatorname{Re} A$ and $\operatorname{Re} A(\varPhi^{-1}) \subseteqq \operatorname{Re} A$. Therefore $\operatorname{Re} A(\varPhi) = \operatorname{Re} A$.

COROLLARY.   *Let $\varPhi$ be a homeomorphism of $\varGamma$ onto $\varGamma$ of class C^2 with $d/dt(\varPhi(e^{it})) \neq 0$ for all t. Then* $\operatorname{Re} A(\varPhi) = \operatorname{Re} A$.

REMARK.   The assumption that $d/dt(\varPhi(e^{it})) \neq 0$ cannot be eliminated, as there do exist examples of homeomorphisms $\varPhi$ of $\varGamma$ onto $\varGamma$ which are of class C^∞, but for which $\operatorname{Re} A(\varPhi) \neq \operatorname{Re} A$.

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