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**MATRIX REPRESENTATIONS FOR LINEAR
TRANSFORMATIONS ON ANALYTIC SEQUENCES**

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Let \mathcal{A} be the space of all analytic sequences, those complex sequences α for which there is a positive number r such that $\sum \alpha_n r^n$ converges. Those linear transformations from \mathcal{A} to \mathcal{A} which have matrix representations are characterized in terms of various spaces and topologies associated with \mathcal{A} . An example is given of a linear transformation from \mathcal{A} to \mathcal{A} which has no matrix representation.

Louise Raphael [8] characterizes the matrix transformations from \mathcal{A} to \mathcal{A} . She makes use of the following: if $q > 0$, A_q is the subspace of \mathcal{A} to which α belongs only in case $\{\|\alpha_n\| q^n\}_{n=0}^{\infty}$ is a bounded sequence, and $\|\alpha\|_q$ denotes the least number less than no term of that bounded sequence. If $q > 0$, $\{A_q, \|\cdot\|_q\}$ is a complete normed linear space. (See also, I. Heller [5], I. M. Sheffer [10, Th. 6, p. 177], and the more fundamental work of Karl Zeller [12].)

Following M. G. Haplanov [4] and V. Ganapathy Iyer [3], S_q denotes the subset of \mathcal{A} to which α belongs provided that $\sum \alpha_n z^n$ converges for $|z| < q$, and, if $0 < p < q$, $N_p(\alpha)$ denotes $\sum_{k=0}^{\infty} |\alpha_k| p^k$ for each α in S_q . If $q > p > 0$, $\{S_q, N_p\}$ is a normed linear space (not complete).

In [11] the author characterizes those linear transformations from S_1 to S_1 which have matrix representations. We continue here in much the same spirit. If $q > 0$ and $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is a sequence of sequences in \mathcal{A} and f is a sequence of analytic functions such that if n is a nonnegative integer and $|z| < q$ then

$$f_n(z) = \sum_{k=0}^{\infty} \alpha_{nk} z^k$$

and f converges uniformly with limit 0 on each closed subset of the (open) disc with center 0 and radius q , then α is said to *have limit 0 analytically relative to q* . A sequence *has limit 0 analytically* if it has limit 0 analytically relative to some positive number.

We recall some fundamental notions from G. Köthe and O. Toeplitz [7] about sequence spaces:

Suppose that λ is a linear sequence space. λ^* (sometimes called the dual or α -dual of λ) is the collection of all complex sequences y such that $\sum |y_n x_n|$ converges for each x in λ . If x is in λ and y is in λ^* ,

$$Q(x, y) = \sum_{n=0}^{\infty} x_n y_n .$$

A sequence $x = \{x_p\}_{p=0}^{\infty}$ of sequences in λ is said to *converge in λ* provided that, for each y in λ^* , the complex sequence $\{Q(x_p, y)\}_{p=0}^{\infty}$ converges. The transformation F is *sequentially continuous* from λ to λ provided that $\{F(x_p)\}_{p=0}^{\infty}$ converges in λ if $\{x_p\}_0^{\infty}$ converges in λ .

Theorems A and B are due to Köthe and Toeplitz.

THEOREM A. *If $\lambda = \lambda^{**}$ and the matrix M transforms λ to λ (if x is in λ and $y_n = \sum_{k=0}^{\infty} M_{nk} x_k$, $n = 0, 1, \dots$, then y is in λ), then the transformation is sequentially continuous from λ to λ [7, Satz 6, p. 206].*

THEOREM B. *Each linear sequentially-continuous transformation from λ to λ has a matrix representation. [7, Satz 7, p. 207].*

A subset X of the sequence space λ is *bounded in λ* if for each u in λ^* there is a number m such that if x is in X then $|Q(x, u)| \leq m$. If F is a transformation from λ to λ , the *adjoint F^** of F is the relation to which the ordered pair $\{x, y\}$ belongs only in the case that

$$Q(x, F(z)) = Q(y, z)$$

for each z in λ .

Let \mathcal{E} be the space of all *entire* sequences, those complex sequences which are coefficient sequences for power-series expansions of entire functions. $\mathcal{E} = \mathcal{A}^*$ and $\mathcal{E}^* = \mathcal{A}$. The matrix transformations from \mathcal{E} to \mathcal{E} have been characterized by H. I. Brown [1] and, in another manner, by K. Chandrasekhara Rao [2].

THEOREM. *Let L be a linear transformation from \mathcal{A} to \mathcal{A} . These statements are equivalent:*

- (1) *L has a matrix representation.*
- (2) *L is sequentially continuous from \mathcal{A} to \mathcal{A} .*
- (3) *If $p > 0$ there is a positive number q such that L maps $\{A_p, \| \cdot \|_p\}$ continuously into $\{A_q, \| \cdot \|_q\}$ (with respect to the norms).*
- (3') *If $p > 0$ there is a positive number q such that L maps A_p into A_q .*
- (4) *If X is a set bounded in \mathcal{A} then $L(X)$ is also.*
- (5) *If $0 < p < r$ there is a positive number R such that, if $0 < P < R$, then L maps $\{S_r, N_p\}$ into $\{S_R, N_P\}$ continuously.*
- (6) *L^* is a sequentially continuous transformations from \mathcal{E} to \mathcal{E} .*
- (7) *If α has limit 0 analytically, so does $\{L(\alpha_n)\}_{n=0}^{\infty}$.*

$\mathcal{A}^{**} = \mathcal{A}$ and $\mathcal{E}^{**} = \mathcal{E}$. This and the following lemmas are useful in the proof of our theorem.

LEMMA 0. Suppose that λ is a sequence space and $\lambda^{**} = \lambda$ and T is a linear sequentially continuous transformation from λ to λ . Then T^* is a sequentially continuous transformation from λ^* into λ^* .

Via [7, Satz 6, p. 200], a characterization of linear functionals, Lemma 0 is easy to prove. (See also [9, p. 158].)

LEMMA 1. If B is a set bounded in \mathcal{A} , then there is a member α of \mathcal{A} such that if β is in B then $|\beta_k| \leq \alpha_k, k = 0, 1, \dots$.

Proof. Otherwise, there is a sequence β of sequences in B and an increasing sequence n of nonnegative integers such that, if k is a positive integer, $|\beta_{k, n_k}| > k^{1+n_k}$. Let us indicate how to define such a sequence. Let β_1 be a member of B and n_1 be a positive integer such that $|\beta_{1, n_1}| > 1^{1+n_1}$. Let t be a number such that if b is in B then $|b_k| \leq t, k = 0, 1, \dots, n_1$. Let β_2 be a member of B and n_2 be a positive integer such that $|\beta_{2, n_2}| > t \cdot 2^{1+n_2}$. $n_2 > n_1$. Please continue.

Let e be a sequence such that if k is a nonnegative integer then $e_{n_k} = k^{-n_k}$ and $e_k = 0$ if there is no positive integer j such that $n_j = k$. e is in \mathcal{E} .

The set D to which d belongs only in case $|d_k| \leq |e_k|, k = 0, 1, \dots$, is bounded in \mathcal{E} . Since B is bounded, it is strongly bounded (see [7, Satz 1, p. 201] or [6, p. 413 (5)]), so that there is a number c such that if b is in B and d is in D then $|Q(b, d)| \leq c$. Let k be a positive integer. Let u be a complex sequence such that if j is a nonnegative integer then $|u_j| = 1$ and $\beta_{k_j} u_j \geq 0$. $u \cdot e$ is in D .

$$\begin{aligned} c &\geq |Q(\beta_k, u \cdot e)| = \left| \sum_{j=0}^{\infty} \beta_{k_j} u_j e_j \right| = \sum_{j=0}^{\infty} \beta_{k_j} u_j e_j \\ &\geq \beta_{k, n_k} u_{n_k} e_{n_k} = |\beta_{k, n_k}| e_{n_k} > k^{1+n_k} k^{-n_k} = k. \end{aligned}$$

So there is a member α of \mathcal{A} such that if b is in B then $|b_k| \leq \alpha_k, k = 0, 1, \dots$.

LEMMA 2. If α is a sequence of sequences in \mathcal{A} , then these are equivalent:

- (1) α has limit 0 analytically.
- (2) α has limit 0 in \mathcal{A} .

Proof. Suppose that α has limit 0 analytically (relative to q). Then α has limit 0 in S_q (see [11, Lemma 1]). α is a sequence

bounded in S_q . \mathcal{A}^* is a subset of S_q^* , so α is a sequence bounded in \mathcal{A} , and there is a member β of \mathcal{A} such that if each of j and k is a nonnegative integer then $|\alpha_{jk}| \leq \beta_k$. Let t be a positive number such that $\beta_k \leq t^{k+1}$, $k = 0, 1, \dots$. Let e be in \mathcal{E} . ($\mathcal{E} = \mathcal{A}^*$.) Let ε be a positive number. Let m be a positive integer such that $2 \sum_{k=m}^{\infty} |e_k| t^{k+1} < \varepsilon$. Let J be a positive integer such that if j is an integer exceeding J then $2 \sum_{k=0}^{m-1} |\alpha_{jk}| |e_k| < \varepsilon$. Then, if $j > J$,

$$\begin{aligned} |Q(\alpha_j, e)| &= \left| \sum_{k=0}^{\infty} \alpha_{jk} e_k \right| \leq \sum_{k=0}^{\infty} |\alpha_{jk}| |e_k| \\ &\leq \sum_{k=0}^{m-1} |\alpha_{jk}| |e_k| + \sum_{k=m}^{\infty} |e_k| t^{k+1} < \varepsilon. \end{aligned}$$

So α has limit 0 in \mathcal{A} .

Now, suppose that α has limit 0 in \mathcal{A} . α is a sequence bounded in \mathcal{A} . There is a positive number t such that $|\alpha_{jk}| \leq t^{k+1}$, $j, k = 0, 1, \dots$. Let q be a number between 0 and $1/t$. Let ε be a positive number. Let m be a positive integer such that $2 \sum_{k=m}^{\infty} q^k t^{k+1} < \varepsilon$. Let J be a positive integer such that if j is an integer exceeding J then $2 \sum_{k=0}^{m-1} |\alpha_{jk}| q^k < \varepsilon$. Now, if $j > J$ and $|z| \leq q$,

$$\left| \sum_{k=0}^{\infty} \alpha_{jk} z^k \right| \leq \sum_{k=0}^{\infty} |\alpha_{jk}| q^k \leq \varepsilon.$$

So α has limit 0 analytically relative to $1/t$.

LEMMA 3. *Suppose that $r > p > 0$ and $R > P > 0$ and L is a continuous linear transformation from $\{S_r, N_p\}$ to $\{S_R, N_P\}$. Then L has a matrix representation.*

Proof. By [11, Theorem 1] this is true if $r = R = 1$.

Suppose that, for each positive number ρ , $t(\rho)$ is the function from \mathcal{A} to \mathcal{A} such that if α is in \mathcal{A} and n is a nonnegative integer then $t(\rho)(\alpha)_n = \alpha_n \rho^n$, so that, if $0 < q < \rho$, $t(\rho)$ maps $\{S_\rho, N_q\}$ continuously onto $\{S_1, N_{q/\rho}\}$.

Let L' be the continuous linear transformation from $\{S_1, N_{p/r}\}$ into $\{S_1, N_{P/R}\}$ such that if x is in S_1 then

$$L'(x) = t(R)Lt(1/r)(x).$$

L' has a matrix representation, so L has a matrix representation.

LEMMA 4. *Suppose that $0 < p < r$. If α is in A_r , then α is in S_r and*

$$N_p(\alpha) \leq \|\alpha\|_r / (1 - p/r).$$

If α is in S_r , then α is in A_p and

$$\|\alpha\|_p \leq N_p(\alpha).$$

The proof is straight-forward and omitted.

Proof of Theorem. $1 \leftrightarrow 2$. That statements (1) and (2) are equivalent is seen from Theorems A and B.

$1 \rightarrow 3$. Mrs. Raphael has shown that statement (3) follows from (1) [8, Theorem 4, p. 124].

$2 \rightarrow 4$. That statement (4) follows from (2) is a consequence of [7, Satz 5, p. 207].

$4 \rightarrow 3'$. Suppose that if X is a set bounded in \mathcal{A} then $L(X)$ is too. Let p be a positive number. Let X be the set of all points x of A_p such that $\|x\|_p \leq 1$. Let e be in E . Let x be in X .

$$|Q(x, e)| = \left| \sum_{k=0}^{\infty} e_k x_k \right| \leq \sum_{k=0}^{\infty} |e_k| |x_k| \leq \sum_{k=0}^{\infty} |e_k| p^{-k},$$

so X is bounded in A .

$L(X)$ is bounded in A . By Lemma 1 there is a positive number q such that if y is in $L(X)$ then $|y_n| \leq q^{n+1}$, $n = 0, 1, \dots$. So, if x is in A_p , $L(x)$ is in A_q . Therefore statement (3') follows from statement (4).

$3' \rightarrow 1$. That statement 3' implies that statement (1) is true is evident from part 4 of Karl Zeller's theorem in [12].

$2 \leftrightarrow 6$. That statements (2) and (6) are equivalent is a consequence of Lemma 0. One might also use Theorems A and B (of [7]) and [7, Satz 4, p. 206].

$2 \leftrightarrow 7$. That statements (2) and (7) are equivalent is evident from Lemma 2.

$3 \rightarrow 5$. Suppose that $0 < p < r$. Let q be a positive number such that L maps $\{A_p, \|\cdot\|_p\}$ continuously into $\{A_q, \|\cdot\|_q\}$. Let K be a positive number such that if x is in A_p then $\|L(x)\|_q \leq K \|x\|_p$. Let P be a number between 0 and q . Then, by Lemma 4, if x is in S_r , x is in A_p , $L(x)$ in A_q , $L(x)$ is in S_q , and

$$N_p(L(x)) \leq \frac{\|L(x)\|_q}{1 - P/q} \leq \frac{K}{1 - P/q} \|x\|_p \leq \frac{K}{1 - P/q} N_p(x).$$

So statement (5) follows from statement (3).

$5 \rightarrow 1$. Since each point of A belongs to S_r for some positive number r , it follows from Lemma 3 that L has a matrix representation (statement (1)) if statement (5) is true.

One can add to the seven statements in the theorem by taking other combinations of these spaces and notions. I have presented those which seem most interesting.

EXAMPLE. Let S be a maximal linearly independent subset of A which contains the unit vectors $(1, 0, 0, \dots)$, etc., and the constant sequence $k = (1, 1, \dots)$. We define a function l from S to the plane such that if s is in S and $s \neq k$ then $l(s) = 0$ and $l(k) = 1$. Let l' be the linear extension of l to A . Let L be the linear transformation from A to A such that if x is in A and n is a nonnegative integer then

$$L(x)_n = l'(x) .$$

L is a linear transformation from A to A (indeed to the constant sequences) and, since l' cannot be represented by a sequence, L has no matrix representation.

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