MATRIX REPRESENTATIONS FOR LINEAR
TRANSFORMATIONS ON ANALYTIC SEQUENCES

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Let \( \mathcal{A} \) be the space of all analytic sequences, those complex sequences \( \alpha \) for which there is a positive number \( r \) such that \( \sum \alpha_n r^n \) converges. Those linear transformations from \( \mathcal{A} \) to \( \mathcal{A} \) which have matrix representations are characterized in terms of various spaces and topologies associated with \( \mathcal{A} \). An example is given of a linear transformation from \( \mathcal{A} \) to \( \mathcal{A} \) which has no matrix representation.

Louise Raphael [8] characterizes the matrix transformations from \( \mathcal{A} \) to \( \mathcal{A} \). She makes use of the following: if \( q > 0 \), \( A_q \) is the subspace of \( \mathcal{A} \) to which \( \alpha \) belongs only in case \( \{ \| \alpha \| \} < \infty \) is a bounded sequence, and \( \| \alpha \|_q \) denotes the least number less than no term of that bounded sequence. If \( q > 0 \), \( \{ A_q, \| \|_q \} \) is a complete normed linear space. (See also, I. Heller [5], I. M. Sheffer [10, Th. 6, p. 177], and the more fundamental work of Karl Zeller [12].)

Following M. G. Haplanov [4] and V. Ganapathy Iyer [3], \( S_q \) denotes the subset of \( \mathcal{A} \) to which \( \alpha \) belongs provided that \( \sum \alpha_n z^n \) converges for \( |z| < q \), and, if \( 0 < p < q \), \( N_p(\alpha) \) denotes \( \sum_{k=0}^{\infty} \alpha_k |p|^k \) for each \( \alpha \) in \( S_q \). If \( q > p > 0 \), \( \{ S_q, N_p \} \) is a normed linear space (not complete).

In [11] the author characterizes those linear transformations from \( S_q \) to \( S_p \) which have matrix representations. We continue here in much the same spirit. If \( q > 0 \) and \( \alpha = \{ \alpha_n \}_{n=0}^{\infty} \) is a sequence of sequences in \( \mathcal{A} \) and \( f \) is a sequence of analytic functions such that if \( n \) is a nonnegative integer and \( |z| < q \) then

\[
f_n(z) = \sum_{k=0}^{\infty} \alpha_n z^k
\]

and \( f \) converges uniformly with limit 0 on each closed subset of the (open) disc with center 0 and radius \( q \), then \( \alpha \) is said to have limit 0 analytically relative to \( q \). A sequence has limit 0 analytically if it has limit 0 analytically relative to some positive number.

We recall some fundamental notions from G. Köthe and O. Toeplitz [7] about sequence spaces:

Suppose that \( \lambda \) is a linear sequence space. \( \lambda^* \) (sometimes called the dual or \( \alpha \)-dual of \( \lambda \)) is the collection of all complex sequences \( y \) such that \( \sum |y_n x_n| \) converges for each \( x \) in \( \lambda \). If \( x \) is in \( \lambda \) and \( y \) is in \( \lambda^* \),
A sequence \( x = \{x_p\}_{p=0}^{\infty} \) of sequences in \( \lambda \) is said to converge in \( \lambda \) provided that, for each \( y \) in \( \lambda^* \), the complex sequence \( \{Q(x_p, y)\}_{p=0}^{\infty} \) converges. The transformation \( F \) is sequentially continuous from \( \lambda \) to \( \lambda \) provided that \( \{F(x_p)\}_{p=0}^{\infty} \) converges in \( \lambda \) if \( \{x_p\}_{0}^{\infty} \) converges in \( \lambda \).

Theorems A and B are due to Köthe and Toeplitz.

**Theorem A.** If \( \lambda = \lambda^{**} \) and the matrix \( M \) transforms \( \lambda \) to \( \lambda \) (if \( x \) is in \( \lambda \) and \( y_n = \sum_{k=0}^{\infty} M_{nk} x_k \), \( n = 0, 1, \ldots \), then \( y \) is in \( \lambda \)), then the transformation is sequentially continuous from \( \lambda \) to \( \lambda \) [7, Satz 6, p. 206].

**Theorem B.** Each linear sequentially-continuous transformation from \( \lambda \) to \( \lambda \) has a matrix representation. [7, Satz 7, p. 207].

A subset \( X \) of the sequence space \( \lambda \) is bounded in \( \lambda \) if for each \( u \) in \( \lambda^* \) there is a number \( m \) such that if \( x \) is in \( X \) then \( |Q(x, u)| \leq m \). If \( F \) is a transformation from \( \lambda \) to \( \lambda \), the adjoint \( F^* \) of \( F \) is the relation to which the ordered pair \( \{x, y\} \) belongs only in the case that

\[ Q(x, F(z)) = Q(y, z) \]

for each \( z \) in \( \lambda \).

Let \( \mathcal{E} \) be the space of all entire sequences, those complex sequences which are coefficient sequences for power-series expansions of entire functions. \( \mathcal{E} = \mathcal{A}^* \) and \( \mathcal{A}^* = \mathcal{A} \). The matrix transformations from \( \mathcal{E} \) to \( \mathcal{E} \) have been characterized by H. I. Brown [1] and, in another manner, by K. Chandrasekhara Rao [2].

**Theorem.** Let \( L \) be a linear transformation from \( \mathcal{A} \) to \( \mathcal{A} \). These statements are equivalent:

1. \( L \) has a matrix representation.
2. \( L \) is sequentially continuous from \( \mathcal{A} \) to \( \mathcal{A} \).
3. If \( p > 0 \) there is a positive number \( q \) such that \( L \) maps \( \{A_p, || \|_p\} \) continuously into \( \{A_q, || \|_q\} \) (with respect to the norms).
3'. If \( p > 0 \) there is a positive number \( q \) such that \( L \) maps \( A_p \) into \( A_q \).
4. If \( X \) is a set bounded in \( \mathcal{A} \) then \( L(X) \) is also.
5. If \( 0 < p < r \) there is a positive number \( R \) such that, if \( 0 < P < R \), then \( L \) maps \( \{S_r, N_r\} \) into \( \{S_p, N_p\} \) continuously.
6. \( L^* \) is a sequentially continuous transformations from \( \mathcal{E} \) to \( \mathcal{E} \).
7. If \( \alpha \) has limit 0 analytically, so does \( \{L(\alpha_n)\}_{n=0}^{\infty} \).
and \( i = \mathcal{A} \) and \( \mathcal{C}^* = \mathcal{C} \). This and the following lemmas are useful in the proof of our theorem.

**Lemma 0.** Suppose that \( \mathcal{A} \) is a sequence space and \( \mathcal{A}^* = \mathcal{A} \) and \( T \) is a linear sequentially continuous transformation from \( \mathcal{A} \) to \( \mathcal{A} \). Then \( T^* \) is a sequentially continuous transformation from \( \mathcal{A}^* \) into \( \mathcal{A}^* \).

Via [7, Satz 6, p. 200], a characterization of linear functionals, Lemma 0 is easy to prove. (See also [9, p. 158].)

**Lemma 1.** If \( B \) is a set bounded in \( \mathcal{A} \), then there is a member \( \alpha \) of \( \mathcal{A} \) such that if \( \beta \) is in \( B \) then \( | \beta_k | \leq \alpha_k, k = 0, 1, \ldots \).

**Proof.** Otherwise, there is a sequence \( \beta \) of sequences in \( B \) and an increasing sequence \( n \) of nonnegative integers such that, if \( k \) is a positive integer, \( | \beta_{k,n_k} | > k^{1+n_k} \). Let us indicate how to define such a sequence. Let \( \beta_1 \) be a member of \( B \) and \( n_1 \) be a positive integer such that \( | \beta_{1,n_1} | > 1^{1+n_1} \). Let \( t \) be a number such that if \( b \) is in \( B \) then \( | b_k | \leq t \), \( k = 0, 1, \ldots, n_1 \). Let \( \beta_2 \) be a member of \( B \) and \( n_2 \) be a positive integer such that \( | \beta_{2,n_2} | > t \cdot 2^{1+n_2} \). \( n_2 > n_1 \). Please continue.

Let \( e \) be a sequence such that if \( k \) is a nonnegative integer then \( e_{n_k} = k^{-n_k} \) and \( e_k = 0 \) if there is no positive integer \( j \) such that \( \beta_j \Lambda 2^0 \cdot \beta \). \( e \) is in \( \mathcal{C} \).

The set \( D \) to which \( d \) belongs only in case \( | d_k | \leq | e_k |, k = 0, 1, \ldots \), is bounded in \( \mathcal{C} \). Since \( B \) is bounded, it is strongly bounded (see [7, Satz 1, p. 201] or [6, p. 413 (5)], so that there is a number \( c \) such that if \( b \) is in \( B \) and \( d \) is in \( D \) then \( | Q(b, d) | \leq c \). Let \( k \) be a positive integer. Let \( u \) be a complex sequence such that if \( j \) is a nonnegative integer then \( | u_j | = 1 \) and \( \beta_{k,j} u_j \geq 0 \). \( u \cdot e \) is in \( D \).

\[
\begin{align*}
\sum_{j=0}^\infty | Q(\beta_{k,j}, u \cdot e) | &= \left| \sum_{j=0}^\infty \beta_{k,j} u_j e_j \right| \\
&\geq \beta_{k,n_k} u_{n_k} e_{n_k} = | \beta_{k,n_k} | e_{n_k} > k^{1+n_k} k^{-n_k} = k.
\end{align*}
\]

So there is a member \( \alpha \) of \( \mathcal{A} \) such that if \( b \) is in \( B \) then \( | b_k | \leq \alpha_k, k = 0, 1, \ldots \).

**Lemma 2.** If \( \alpha \) is a sequence of sequences in \( \mathcal{A} \), then these are equivalent:

1. \( \alpha \) has limit 0 analytically.
2. \( \alpha \) has limit 0 in \( \mathcal{A} \).

**Proof.** Suppose that \( \alpha \) has limit 0 analytically (relative to \( q \)). Then \( \alpha \) has limit 0 in \( S_q \) (see [11, Lemma 1]). \( \alpha \) is a sequence
bounded in $S_\cdot$. $\mathcal{A}^\ast$ is a subset of $S_r^\ast$, so $\alpha$ is a sequence bounded in $\mathcal{A}$, and there is a member $\beta$ of $\mathcal{A}$ such that if each of $j$ and $k$ is a nonnegative integer then $|\alpha_{jk}| \leq \beta_k$. Let $t$ be a positive number such that $\beta_k \leq t^{k+1}$, $k = 0, 1, \ldots$. Let $\varepsilon$ be in $\mathcal{E}$. Let $m$ be a positive integer such that $2 \sum_{k=m}^{m-1} |a_{jk}| |e_k| < \varepsilon$. Let $J$ be a positive integer such that if $j$ is a nonnegative integer then $2 \sum_{k=m}^{m-1} |a_{jk}| |e_k| < \varepsilon$. Then, if $j > J$,

$$\sum_{k=0}^{m-1} |\alpha_{jk}| |e_k| \leq \sum_{k=m}^{m-1} |a_{jk}| |e_k| < \varepsilon.$$ 

So $\alpha$ has limit 0 in $\mathcal{A}$.

Now, suppose that $\alpha$ has limit 0 in $\mathcal{A}$. $\alpha$ is a sequence bounded in $\mathcal{A}$. There is a positive number $t$ such that $|\alpha_{jk}| \leq t^{k+1}$, $j, k = 0, 1, \ldots$. Let $q$ be a number between 0 and $1/t$. Let $\varepsilon$ be a positive number. Let $m$ be a positive integer such that $2 \sum_{k=m}^{m-1} q^k t^{k+1} < \varepsilon$. Let $J$ be a positive integer such that if $j$ is an integer exceeding $J$ then $2 \sum_{k=m}^{m-1} |a_{jk}| |e_k| < \varepsilon$. Now, if $j > J$ and $|z| \leq q$,

$$\sum_{k=0}^{m-1} \alpha_{jk} z^k \leq \sum_{k=m}^{m-1} |\alpha_{jk}| t^{k+1} < \varepsilon.$$ 

So $\alpha$ has limit 0 analytically relative to $1/t$.

**Lemma 3.** Suppose that $0 < p < r$ and $R > P > 0$ and $L$ is a continuous linear transformation from $\{S_r, N_p\}$ to $\{S_r, N_P\}$. Then $L$ has a matrix representation.

**Proof.** By [11, Theorem 1] this is true if $r = R = 1$.

Suppose that, for each positive number $\rho$, $t(\rho)$ is the function from $\mathcal{A}$ to $\mathcal{A}$ such that if $\alpha$ is in $\mathcal{A}$ and $n$ is a nonnegative integer then $t(\rho)(\alpha)_n = \alpha_n \rho^n$, so that, if $0 < q < \rho$, $t(\rho)$ maps $\{S_r, N_q\}$ continuously onto $\{S_r, N_{q/r}\}$.

Let $L'$ be the continuous linear transformation from $\{S_{1}, N_{1/\rho}\}$ into $\{S_{1}, N_{1/r}\}$ such that if $x$ is in $S_{1}$ then

$$L'(x) = t(R)Lt(1/r)(x).$$

$L'$ has a matrix representation, so $L$ has a matrix representation.

**Lemma 4.** Suppose that $0 < p < r$. If $\alpha$ is in $A_r$, then $\alpha$ is in $S_r$ and

$$N_p(\alpha) \leq \|\alpha\|/\left(1 - p/r\right).$$
If $\alpha$ is in $S_r$, then $\alpha$ is in $A_p$ and

$$\| \alpha \|_p \leq N_p(\alpha).$$

The proof is straight-forward and omitted.

**Proof of Theorem.** 1 $\iff$ 2. That statements (1) and (2) are equivalent is seen from Theorems A and B.

1 $\rightarrow$ 3. Mrs. Raphael has shown that statement (3) follows from (1) [8, Theorem 4, p. 124].

2 $\rightarrow$ 4. That statement (4) follows from (2) is a consequence of [7, Satz 5, p. 207].

4 $\rightarrow$ 3'. Suppose that if $X$ is a set bounded in $\mathcal{X}$ then $L(X)$ is too. Let $p$ be a positive number. Let $X$ be the set of all points $x$ of $A_p$ such that $\| x \|_p \leq 1$. Let $e$ be in $E$. Let $x$ be in $X$.

$$|Q(x, e)| = \left| \sum_{k=0}^{\infty} e_k x_k \right| \leq \sum_{k=0}^{\infty} |e_k| \| x_k \| \leq \sum_{k=0}^{\infty} |e_k| p^{-k},$$

so $X$ is bounded in $A$.

$L(X)$ is bounded in $A$. By Lemma 1 there is a positive number $q$ such that if $y$ is in $L(X)$ then $|y_n| \leq q^{n+1}$, $n = 0, 1, \ldots$. So, if $x$ is in $A_p$, $L(x)$ is in $A_r$. Therefore statement (3') follows from statement (4).

3' $\rightarrow$ 1. That statement 3' implies that statement (1) is true is evident from part 4 of Karl Zeller's theorem in [12].

2 $\iff$ 6. That statements (2) and (6) are equivalent is a consequence of Lemma 0. One might also use Theorems A and B (of [7]) and [7, Satz 4, p. 206].

2 $\iff$ 7. That statements (2) and (7) are equivalent is evident from Lemma 2.

3 $\rightarrow$ 5. Suppose that $0 < p < r$. Let $q$ be a positive number such that $L$ maps $\{ A_p, || \|_p \}$ continuously into $\{ A_q, || \|_q \}$. Let $K$ be a positive number such that if $x$ is in $A_p$ then $|| L(x) ||_q \leq K || x ||_p$. Let $P$ be a number between 0 and $q$. Then, by Lemma 4, if $x$ is in $S_r$, $x$ is in $A_q$, $L(x)$ in $A_q$, $L(x)$ is in $S_q$, and

$$N_p(L(x)) \leq \frac{|| L(x) ||_q}{1 - P/q} \leq \frac{K}{1 - P/q} \frac{K}{1 - P/q} N_p(x).$$

So statement (5) follows from statement (3).

5 $\rightarrow$ 1. Since each point of $A$ belongs to $S_r$ for some positive number $r$, it follows from Lemma 3 that $L$ has a matrix representation (statement (1)) if statement (5) is true.

One can add to the seven statements in the theorem by taking other combinations of these spaces and notions. I have presented those which seem most interesting.
EXAMPLE. Let \( S \) be a maximal linearly independent subset of \( A \) which contains the unit vectors \((1, 0, 0, \ldots)\), etc., and the constant sequence \( k = (1, 1, \ldots) \). We define a function \( l \) from \( S \) to the plane such that if \( s \) is in \( S \) and \( s \neq k \) then \( l(s) = 0 \) and \( l(k) = 1 \). Let \( l' \) be the linear extension of \( l \) to \( A \). Let \( L \) be the linear transformation from \( A \) to \( A \) such that if \( x \) is in \( A \) and \( n \) is a nonnegative integer then

\[
L(x)_n = l'(x).
\]

\( L \) is a linear transformation from \( A \) to \( A \) (indeed to the constant sequences) and, since \( l' \) cannot be represented by a sequence, \( L \) has no matrix representation.

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