

# Pacific Journal of Mathematics

## **MATRIX REPRESENTATIONS FOR LINEAR TRANSFORMATIONS ON ANALYTIC SEQUENCES**

PHILIP C. TONNE

## MATRIX REPRESENTATIONS FOR LINEAR TRANSFORMATIONS ON ANALYTIC SEQUENCES

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**Let  $\mathcal{A}$  be the space of all analytic sequences, those complex sequences  $\alpha$  for which there is a positive number  $r$  such that  $\sum \alpha_n r^n$  converges. Those linear transformations from  $\mathcal{A}$  to  $\mathcal{A}$  which have matrix representations are characterized in terms of various spaces and topologies associated with  $\mathcal{A}$ . An example is given of a linear transformation from  $\mathcal{A}$  to  $\mathcal{A}$  which has no matrix representation.**

Louise Raphael [8] characterizes the matrix transformations from  $\mathcal{A}$  to  $\mathcal{A}$ . She makes use of the following: if  $q > 0$ ,  $A_q$  is the subspace of  $\mathcal{A}$  to which  $\alpha$  belongs only in case  $\{\|\alpha_n\| q^n\}_{n=0}^\infty$  is a bounded sequence, and  $\|\alpha\|_q$  denotes the least number less than no term of that bounded sequence. If  $q > 0$ ,  $\{A_q, \|\cdot\|_q\}$  is a complete normed linear space. (See also, I. Heller [5], I. M. Sheffer [10, Th. 6, p. 177], and the more fundamental work of Karl Zeller [12].)

Following M. G. Haplanov [4] and V. Ganapathy Iyer [3],  $S_q$  denotes the subset of  $\mathcal{A}$  to which  $\alpha$  belongs provided that  $\sum \alpha_n z^n$  converges for  $|z| < q$ , and, if  $0 < p < q$ ,  $N_p(\alpha)$  denotes  $\sum_{k=0}^\infty |\alpha_k| p^k$  for each  $\alpha$  in  $S_q$ . If  $q > p > 0$ ,  $\{S_q, N_p\}$  is a normed linear space (not complete).

In [11] the author characterizes those linear transformations from  $S_i$  to  $S_i$  which have matrix representations. We continue here in much the same spirit. If  $q > 0$  and  $\alpha = \{\alpha_n\}_{n=0}^\infty$  is a sequence of sequences in  $\mathcal{A}$  and  $f$  is a sequence of analytic functions such that if  $n$  is a nonnegative integer and  $|z| < q$  then

$$f_n(z) = \sum_{k=0}^\infty \alpha_{nk} z^k$$

and  $f$  converges uniformly with limit 0 on each closed subset of the (open) disc with center 0 and radius  $q$ , then  $\alpha$  is said to *have limit 0 analytically relative to  $q$* . A sequence *has limit 0 analytically* if it has limit 0 analytically relative to some positive number.

We recall some fundamental notions from G. Köthe and O. Toeplitz [7] about sequence spaces:

Suppose that  $\lambda$  is a linear sequence space.  $\lambda^*$  (sometimes called the dual or  $\alpha$ -dual of  $\lambda$ ) is the collection of all complex sequences  $y$  such that  $\sum |y_n x_n|$  converges for each  $x$  in  $\lambda$ . If  $x$  is in  $\lambda$  and  $y$  is in  $\lambda^*$ ,

$$Q(x, y) = \sum_{n=0}^{\infty} x_n y_n .$$

A sequence  $x = \{x_p\}_{p=0}^{\infty}$  of sequences in  $\lambda$  is said to *converge in*  $\lambda$  provided that, for each  $y$  in  $\lambda^*$ , the complex sequence  $\{Q(x_p, y)\}_{p=0}^{\infty}$  converges. The transformation  $F$  is *sequentially continuous* from  $\lambda$  to  $\lambda$  provided that  $\{F(x_p)\}_{p=0}^{\infty}$  converges in  $\lambda$  if  $\{x_p\}_0^{\infty}$  converges in  $\lambda$ .

Theorems A and B are due to Köthe and Toeplitz.

**THEOREM A.** *If  $\lambda = \lambda^{**}$  and the matrix  $M$  transforms  $\lambda$  to  $\lambda$  (if  $x$  is in  $\lambda$  and  $y_n = \sum_{k=0}^{\infty} M_{nk} x_k$ ,  $n = 0, 1, \dots$ , then  $y$  is in  $\lambda$ ), then the transformation is sequentially continuous from  $\lambda$  to  $\lambda$  [7, Satz 6, p. 206].*

**THEOREM B.** *Each linear sequentially-continuous transformation from  $\lambda$  to  $\lambda$  has a matrix representation. [7, Satz 7, p. 207].*

A subset  $X$  of the sequence space  $\lambda$  is *bounded in*  $\lambda$  if for each  $u$  in  $\lambda^*$  there is a number  $m$  such that if  $x$  is in  $X$  then  $|Q(x, u)| \leq m$ . If  $F$  is a transformation from  $\lambda$  to  $\lambda$ , the *adjoint*  $F^*$  of  $F$  is the relation to which the ordered pair  $\{x, y\}$  belongs only in the case that

$$Q(x, F(z)) = Q(y, z)$$

for each  $z$  in  $\lambda$ .

Let  $\mathcal{E}$  be the space of all *entire* sequences, those complex sequences which are coefficient sequences for power-series expansions of entire functions.  $\mathcal{E} = \mathcal{A}^*$  and  $\mathcal{E}^* = \mathcal{A}$ . The matrix transformations from  $\mathcal{E}$  to  $\mathcal{E}$  have been characterized by H. I. Brown [1] and, in another manner, by K. Chandrasekhara Rao [2].

**THEOREM.** *Let  $L$  be a linear transformation from  $\mathcal{A}$  to  $\mathcal{A}$ . These statements are equivalent:*

- (1)  $L$  has a matrix representation.
- (2)  $L$  is sequentially continuous from  $\mathcal{A}$  to  $\mathcal{A}$ .
- (3) If  $p > 0$  there is a positive number  $q$  such that  $L$  maps  $\{A_p, \| \cdot \|_p\}$  continuously into  $\{A_q, \| \cdot \|_q\}$  (with respect to the norms).
- (3') If  $p > 0$  there is a positive number  $q$  such that  $L$  maps  $A_p$  into  $A_q$ .
- (4) If  $X$  is a set bounded in  $\mathcal{A}$  then  $L(X)$  is also.
- (5) If  $0 < p < r$  there is a positive number  $R$  such that, if  $0 < P < R$ , then  $L$  maps  $\{S_r, N_p\}$  into  $\{S_R, N_P\}$  continuously.
- (6)  $L^*$  is a sequentially continuous transformations from  $\mathcal{E}$  to  $\mathcal{E}$ .
- (7) If  $\alpha$  has limit 0 analytically, so does  $\{L(\alpha_n)\}_{n=0}^{\infty}$ .

$\mathcal{A}^{**} = \mathcal{A}$  and  $\mathcal{E}^{**} = \mathcal{E}$ . This and the following lemmas are useful in the proof of our theorem.

**LEMMA 0.** *Suppose that  $\lambda$  is a sequence space and  $\lambda^{**} = \lambda$  and  $T$  is a linear sequentially continuous transformation from  $\lambda$  to  $\lambda$ . Then  $T^*$  is a sequentially continuous transformation from  $\lambda^*$  into  $\lambda^*$ .*

Via [7, Satz 6, p. 200], a characterization of linear functionals, Lemma 0 is easy to prove. (See also [9, p. 158].)

**LEMMA 1.** *If  $B$  is a set bounded in  $\mathcal{A}$ , then there is a member  $\alpha$  of  $\mathcal{A}$  such that if  $\beta$  is in  $B$  then  $|\beta_k| \leq \alpha_k, k = 0, 1, \dots$ .*

*Proof.* Otherwise, there is a sequence  $\beta$  of sequences in  $B$  and an increasing sequence  $n$  of nonnegative integers such that, if  $k$  is a positive integer,  $|\beta_{k, n_k}| > k^{1+n_k}$ . Let us indicate how to define such a sequence. Let  $\beta_1$  be a member of  $B$  and  $n_1$  be a positive integer such that  $|\beta_{1, n_1}| > 1^{1+n_1}$ . Let  $t$  be a number such that if  $b$  is in  $B$  then  $|b_k| \leq t, k = 0, 1, \dots, n_1$ . Let  $\beta_2$  be a member of  $B$  and  $n_2$  be a positive integer such that  $|\beta_{2, n_2}| > t \cdot 2^{1+n_2}$ .  $n_2 > n_1$ . Please continue.

Let  $e$  be a sequence such that if  $k$  is a nonnegative integer then  $e_{n_k} = k^{-n_k}$  and  $e_k = 0$  if there is no positive integer  $j$  such that  $n_j = k$ .  $e$  is in  $\mathcal{E}$ .

The set  $D$  to which  $d$  belongs only in case  $|d_k| \leq |e_k|, k = 0, 1, \dots$ , is bounded in  $\mathcal{E}$ . Since  $B$  is bounded, it is strongly bounded (see [7, Satz 1, p. 201] or [6, p. 413 (5)], so that there is a number  $c$  such that if  $b$  is in  $B$  and  $d$  is in  $D$  then  $|Q(b, d)| \leq c$ . Let  $k$  be a positive integer. Let  $u$  be a complex sequence such that if  $j$  is a nonnegative integer then  $|u_j| = 1$  and  $\beta_{k_j} u_j \geq 0$ .  $u \cdot e$  is in  $D$ .

$$\begin{aligned} c &\geq |Q(\beta_k, u \cdot e)| = \left| \sum_{j=0}^{\infty} \beta_{k_j} u_j e_j \right| = \sum_{j=0}^{\infty} \beta_{k_j} u_j e_j \\ &\geq \beta_{k, n_k} u_{n_k} e_{n_k} = |\beta_{k, n_k}| e_{n_k} > k^{1+n_k} k^{-n_k} = k. \end{aligned}$$

So there is a member  $\alpha$  of  $\mathcal{A}$  such that if  $b$  is in  $B$  then  $|b_k| \leq \alpha_k, k = 0, 1, \dots$ .

**LEMMA 2.** *If  $\alpha$  is a sequence of sequences in  $\mathcal{A}$ , then these are equivalent:*

- (1)  $\alpha$  has limit 0 analytically.
- (2)  $\alpha$  has limit 0 in  $\mathcal{A}$ .

*Proof.* Suppose that  $\alpha$  has limit 0 analytically (relative to  $g$ ). Then  $\alpha$  has limit 0 in  $S_g$  (see [11, Lemma 1]).  $\alpha$  is a sequence

bounded in  $S_q$ .  $\mathcal{A}^*$  is a subset of  $S_q^*$ , so  $\alpha$  is a sequence bounded in  $\mathcal{A}$ , and there is a member  $\beta$  of  $\mathcal{A}$  such that if each of  $j$  and  $k$  is a nonnegative integer then  $|\alpha_{jk}| \leq \beta_k$ . Let  $t$  be a positive number such that  $\beta_k \leq t^{k+1}$ ,  $k = 0, 1, \dots$ . Let  $e$  be in  $\mathcal{E}$ . ( $\mathcal{E} = \mathcal{A}^*$ .) Let  $\varepsilon$  be a positive number. Let  $m$  be a positive integer such that  $2 \sum_{k=m}^{\infty} |e_k| t^{k+1} < \varepsilon$ . Let  $J$  be a positive integer such that if  $j$  is an integer exceeding  $J$  then  $2 \sum_{k=0}^{m-1} |\alpha_{jk}| |e_k| < \varepsilon$ . Then, if  $j > J$ ,

$$\begin{aligned} |Q(\alpha_j, e)| &= \left| \sum_{k=0}^{\infty} \alpha_{jk} e_k \right| \leq \sum_{k=0}^{\infty} |\alpha_{jk}| |e_k| \\ &\leq \sum_{k=0}^{m-1} |\alpha_{jk}| |e_k| + \sum_{k=m}^{\infty} |e_k| t^{k+1} < \varepsilon. \end{aligned}$$

So  $\alpha$  has limit 0 in  $\mathcal{A}$ .

Now, suppose that  $\alpha$  has limit 0 in  $\mathcal{A}$ .  $\alpha$  is a sequence bounded in  $\mathcal{A}$ . There is a positive number  $t$  such that  $|\alpha_{jk}| \leq t^{k+1}$ ,  $j, k = 0, 1, \dots$ . Let  $q$  be a number between 0 and  $1/t$ . Let  $\varepsilon$  be a positive number. Let  $m$  be a positive integer such that  $2 \sum_{k=m}^{\infty} q^k t^{k+1} < \varepsilon$ . Let  $J$  be a positive integer such that if  $j$  is an integer exceeding  $J$  then  $2 \sum_{k=0}^{m-1} |\alpha_{jk}| q^k < \varepsilon$ . Now, if  $j > J$  and  $|z| \leq q$ ,

$$\left| \sum_{k=0}^{\infty} \alpha_{jk} z^k \right| \leq \sum_{k=0}^{\infty} |\alpha_{jk}| q^k \leq \varepsilon.$$

So  $\alpha$  has limit 0 analytically relative to  $1/t$ .

LEMMA 3. *Suppose that  $r > p > 0$  and  $R > P > 0$  and  $L$  is a continuous linear transformation from  $\{S_r, N_p\}$  to  $\{S_R, N_P\}$ . Then  $L$  has a matrix representation.*

*Proof.* By [11, Theorem 1] this is true if  $r = R = 1$ .

Suppose that, for each positive number  $\rho$ ,  $t(\rho)$  is the function from  $\mathcal{A}$  to  $\mathcal{A}$  such that if  $\alpha$  is in  $\mathcal{A}$  and  $n$  is a nonnegative integer then  $t(\rho)(\alpha)_n = \alpha_n \rho^n$ , so that, if  $0 < q < \rho$ ,  $t(\rho)$  maps  $\{S_\rho, N_q\}$  continuously onto  $\{S_1, N_{q/\rho}\}$ .

Let  $L'$  be the continuous linear transformation from  $\{S_1, N_{p/r}\}$  into  $\{S_1, N_{P/R}\}$  such that if  $x$  is in  $S_1$  then

$$L'(x) = t(R)Lt(1/r)(x).$$

$L'$  has a matrix representation, so  $L$  has a matrix representation.

LEMMA 4. *Suppose that  $0 < p < r$ . If  $\alpha$  is in  $A_r$ , then  $\alpha$  is in  $S_r$  and*

$$N_p(\alpha) \leq \|\alpha\|_r / (1 - p/r).$$

If  $\alpha$  is in  $S_r$ , then  $\alpha$  is in  $A_p$  and

$$\|\alpha\|_p \leq N_p(\alpha).$$

The proof is straight-forward and omitted.

*Proof of Theorem.*  $1 \leftrightarrow 2$ . That statements (1) and (2) are equivalent is seen from Theorems A and B.

$1 \rightarrow 3$ . Mrs. Raphael has shown that statement (3) follows from (1) [8, Theorem 4, p. 124].

$2 \rightarrow 4$ . That statement (4) follows from (2) is a consequence of [7, Satz 5, p. 207].

$4 \rightarrow 3'$ . Suppose that if  $X$  is a set bounded in  $\mathcal{A}$  then  $L(X)$  is too. Let  $p$  be a positive number. Let  $X$  be the set of all points  $x$  of  $A_p$  such that  $\|x\|_p \leq 1$ . Let  $e$  be in  $E$ . Let  $x$  be in  $X$ .

$$|Q(x, e)| = \left| \sum_{k=0}^{\infty} e_k x_k \right| \leq \sum_{k=0}^{\infty} |e_k| |x_k| \leq \sum_{k=0}^{\infty} |e_k| p^{-k},$$

so  $X$  is bounded in  $A$ .

$L(X)$  is bounded in  $A$ . By Lemma 1 there is a positive number  $q$  such that if  $y$  is in  $L(X)$  then  $|y_n| \leq q^{n+1}$ ,  $n = 0, 1, \dots$ . So, if  $x$  is in  $A_p$ ,  $L(x)$  is in  $A_q$ . Therefore statement (3') follows from statement (4).

$3' \rightarrow 1$ . That statement 3' implies that statement (1) is true is evident from part 4 of Karl Zeller's theorem in [12].

$2 \leftrightarrow 6$ . That statements (2) and (6) are equivalent is a consequence of Lemma 0. One might also use Theorems A and B (of [7]) and [7, Satz 4, p. 206].

$2 \leftrightarrow 7$ . That statements (2) and (7) are equivalent is evident from Lemma 2.

$3 \rightarrow 5$ . Suppose that  $0 < p < r$ . Let  $q$  be a positive number such that  $L$  maps  $\{A_p, \|\cdot\|_p\}$  continuously into  $\{A_q, \|\cdot\|_q\}$ . Let  $K$  be a positive number such that if  $x$  is in  $A_p$  then  $\|L(x)\|_q \leq K \|x\|_p$ . Let  $P$  be a number between 0 and  $q$ . Then, by Lemma 4, if  $x$  is in  $S_r$ ,  $x$  is in  $A_p$ ,  $L(x)$  in  $A_q$ ,  $L(x)$  is in  $S_q$ , and

$$N_p(L(x)) \leq \frac{\|L(x)\|_q}{1 - P/q} \leq \frac{K}{1 - P/q} \|x\|_p \leq \frac{K}{1 - P/q} N_p(x).$$

So statement (5) follows from statement (3).

$5 \rightarrow 1$ . Since each point of  $A$  belongs to  $S_r$  for some positive number  $r$ , it follows from Lemma 3 that  $L$  has a matrix representation (statement (1)) if statement (5) is true.

One can add to the seven statements in the theorem by taking other combinations of these spaces and notions. I have presented those which seem most interesting.

EXAMPLE. Let  $S$  be a maximal linearly independent subset of  $A$  which contains the unit vectors  $(1, 0, 0, \dots)$ , etc., and the constant sequence  $k = (1, 1, \dots)$ . We define a function  $l$  from  $S$  to the plane such that if  $s$  is in  $S$  and  $s \neq k$  then  $l(s) = 0$  and  $l(k) = 1$ . Let  $l'$  be the linear extension of  $l$  to  $A$ . Let  $L$  be the linear transformation from  $A$  to  $A$  such that if  $x$  is in  $A$  and  $n$  is a nonnegative integer then

$$L(x)_n = l'(x) .$$

$L$  is a linear transformation from  $A$  to  $A$  (indeed to the constant sequences) and, since  $l'$  cannot be represented by a sequence,  $L$  has no matrix representation.

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Allan Francis Abrahamse, <i>Uniform integrability of derivatives on <math>\sigma</math>-lattices</i> .....	1
Ronald Alter and K. K. Kubota, <i>The diophantine equation <math>x^2 + D = p^n</math></i> .....	11
Grahame Bennett, <i>Some inclusion theorems for sequence spaces</i> .....	17
William Cutler, <i>On extending isotopies</i> .....	31
Robert Jay Daverman, <i>Factored codimension one cells in Euclidean <math>n</math>-space</i> .....	37
Patrick Barry Eberlein and Barrett O'Neill, <i>Visibility manifolds</i> .....	45
M. Edelstein, <i>Concerning dentability</i> .....	111
Edward Graham Evans, Jr., <i>Krull-Schmidt and cancellation over local rings</i> .....	115
C. D. Feustel, <i>A generalization of Kneser's conjecture</i> .....	123
Avner Friedman, <i>Uniqueness for the Cauchy problem for degenerate parabolic equations</i> .....	131
David Golber, <i>The cohomological description of a torus action</i> .....	149
Alain Goulet de Rugy, <i>Un théorème du genre "Andô-Edwards" pour les Fréchet ordonnés normaux</i> .....	155
Louise Hay, <i>The class of recursively enumerable subsets of a recursively enumerable set</i> .....	167
John Paul Helm, Albert Ronald da Silva Meyer and Paul Ruel Young, <i>On orders of translations and enumerations</i> .....	185
Julien O. Hennefeld, <i>A decomposition for <math>B(X)^*</math> and unique Hahn-Banach extensions</i> .....	197
Gordon G. Johnson, <i>Moment sequences in Hilbert space</i> .....	201
Thomas Rollin Kramer, <i>A note on countably subparacompact spaces</i> .....	209
Yves A. Lequain, <i>Differential simplicity and extensions of a derivation</i> .....	215
Peter Lorimer, <i>A property of the groups <math>\text{Aut PU}(3, q^2)</math></i> .....	225
Yasou Matsugu, <i>The Levi problem for a product manifold</i> .....	231
John M.F. O'Connell, <i>Real parts of uniform algebras</i> .....	235
William Lindall Paschke, <i>A factorable Banach algebra without bounded approximate unit</i> .....	249
Ronald Joel Rudman, <i>On the fundamental unit of a purely cubic field</i> .....	253
Tsuan Wu Ting, <i>Torsional rigidities in the elastic-plastic torsion of simply connected cylindrical bars</i> .....	257
Philip C. Tonne, <i>Matrix representations for linear transformations on analytic sequences</i> .....	269
Jung-Hsien Tsai, <i>On <math>E</math>-compact spaces and generalizations of perfect mappings</i> .....	275
Alfons Van Daele, <i>The upper envelope of invariant functionals majorized by an invariant weight</i> .....	283
Giulio Varsi, <i>The multidimensional content of the frustum of the simplex</i> .....	303