

Pacific Journal of Mathematics

**SIMPLE EXTENSIONS OF MEASURES AND THE
PRESERVATION OF REGULARITY OF CONDITIONAL
PROBABILITIES**

LOUIS HARVEY BLAKE

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Throughout this paper, the following notation will be adopted. $(\Omega, \mathfrak{A}, P)$ will be a probability space with \mathfrak{B} a sub σ -field of \mathfrak{A} . H will denote a subset of Ω not in \mathfrak{A} and \mathfrak{A}' will be the σ -field generated by \mathfrak{A} and H . P_e will be a simple extension of P to \mathfrak{A}' if P_e is a probability measure on \mathfrak{A}' with $P_e|_{\mathfrak{A}} = P$.

The ability to extend the regularity of the conditional probability $P^{\mathfrak{B}}$ to regularity of $P_e^{\mathfrak{B}}$ has been explored earlier for canonical extensions of measures. The main results of this paper are:

(a) If $P_e^{\mathfrak{B}}$ is regular for some canonical extension P_e of P to \mathfrak{A}' , then $P_e^{\mathfrak{B}}$ is regular for any simple extension P_e of P to \mathfrak{A}' .

(b) For some choice of $(\Omega, \mathfrak{A}, P)$, \mathfrak{B} and H , $P^{\mathfrak{B}}$ is regular but for no P_e is $P_e^{\mathfrak{B}}$ regular. This will essentially extend the Dicuodonné example,

Our notation regarding (regular) conditional probabilities will be consistent with [1].

For extendability see [4]. The example for (b) occurs in [2].

PROPOSITION 1. *Any simple extension P_e can be expressed as the sum of a canonical extension of P plus a finite signed measure on \mathfrak{A} . (Since the construction is carried out in a unique manner, this decomposition of P_e will be called the canonical decomposition of P_e .)*

Proof. As in [1], let K be a set which extends P canonically to \mathfrak{A}' . For any $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some A_1 and A_2 in \mathfrak{A} write

$$P_e(A') = P(A'K^c) + P_e(A_1HK) + P_e(A_2H^cK).$$

It may be supposed that $P(K) \neq 0$. Thus, let $\alpha_0 \equiv P_e(HK)/P(K)$ and define a set function ε on \mathfrak{A} such that for every $A \in \mathfrak{A}$

$$\varepsilon(A) = P_e(AHK) - \alpha_0 P(AK).$$

It is immediate that ε is a finite signed measure. It also follows that for any $A \in \mathfrak{A}$

$$P_e(AH^cK) = \beta_0 P(AK) - \varepsilon(A)$$

where $\beta_2 \equiv 1 - \alpha_2$ inasmuch as it can be written that

$$P(A) = P_e(A) = P_e(AH + AH^c) = P(AK^c) + \alpha_2 P(AK) + \varepsilon(A) + P_e(AH^c K) .$$

Thus, for $A' \in \mathfrak{A}'$

$$P_e(A') = \underbrace{P(A'K^c)} + \underbrace{\alpha_2 P(A_1 K)} + \varepsilon(A_1) + \underbrace{\beta_2 P(A_2 K)} - \varepsilon(A_2) .$$

(Let the sum of the underlined measures be called the *canonical part* of P_e .)

It is clear that the extension, P_e , of Proposition 1 is canonical if and only if the signed measure ε is identically zero.

LEMMA 2. *The signed measure ε is absolutely continuous with respect to P .*

Proof. Let $B \in \mathfrak{A}$ be a positive set for ε according to its Jordan decomposition and let $A \in \mathfrak{A}$ with $P(A) = 0$. Then,

$$(2.1) \quad P_e(ABHK) \leq P(ABK) \leq P(A) = 0$$

and so $\varepsilon(AB) = 0$. If $C (= B^c)$ is a negative set for ε then it follows that $\varepsilon(AC) = 0$ where one merely inserts C for B in (2.1). Hence $\varepsilon \ll P$.

LEMMA 3. *If $\Omega_0 \in \mathfrak{A}$ with $P(\Omega_0) = 1$ then $\varepsilon(\Omega_0) = 0$.*

Proof. Immediate.

The following lemma is needed before the main result can be presented.

LEMMA 4. *Let $(\Omega, \mathfrak{A}, P)$ be a probability space with $\mathfrak{B} \subset \mathfrak{A}$. Let P_0 be another measure on \mathfrak{A} with $P = P_0$ on \mathfrak{B} and $P \ll P_0$. Suppose $P_0^{\mathfrak{B}}$ is regular. Then, $P^{\mathfrak{B}}$ is regular.*

Proof. Let $p_0(\cdot, \cdot | \mathfrak{B})$ be a version of $P_0^{\mathfrak{B}}$ such that $p_0(\omega, \cdot | \mathfrak{B})$ is a measure ($P_0 |_{\mathfrak{B}}$ a.e.). Also, let $X = dP/dP_0$ where for all $A \in \mathfrak{A}$

$$P(A) = \int_A X dP_0 .$$

Hence, define

$$(4.1) \quad h(\omega, A) = \int_A X(\omega') p_0(\omega, d\omega' | \mathfrak{B}) .$$

From (4.1) it is immediate that $h(\cdot, A)$ is \mathfrak{B} -measurable for every $A \in \mathfrak{A}$ and for fixed $\omega \in \Omega$, $h(\omega, \cdot)$ is a measure on \mathfrak{A} . It remains to show that for any $B \in \mathfrak{B}$

$$\int_B h(\omega, A)P(d\omega) = P(AB) .$$

To show this, begin by establishing that

$$X \in L_1(\Omega, \mathfrak{A}, p_0(\omega, \cdot | \mathfrak{B})) P_0 |_{\mathfrak{B}} \text{ a.e.}$$

This follows at once by observing that

$$\int_{\Omega} X(\omega')p(\omega, d\omega' | \mathfrak{B}) = (E^{\mathfrak{B}}X)(\omega)$$

and

$$\int_{\Omega} (E^{\mathfrak{B}}X)(\omega)P_0(d\omega) = \int_{\Omega} X(\omega)P_0(d\omega) = 1 .$$

Next, write

$$X = \lim_{n \rightarrow \infty} X_n \quad \text{where} \quad X_n = \sum_{k=1}^{m_n} \zeta_{k,n}(\Psi A_{k,n}) \quad \text{where}$$

$\zeta_{k,n}$ is a real constant, $(\Psi A_{k,n})$ is the characteristic function of $A_{k,n} \in \mathfrak{A}$ and $\{X_n\}_{n \geq 1}$ is an increasing sequence.

Finally, since $X \in L_1(\Omega, \mathfrak{A}, p_0(\omega, \cdot | \mathfrak{B})) P_0 |_{\mathfrak{B}}$ a.e., the monotone convergence theorem can be used on the following chain of equalities to give the desired result:

$$\begin{aligned} \int_B h(\omega', A)P(d\omega') &= \int_B \left\{ \int_A X(\omega) p_0(\omega', d\omega | \mathfrak{B}) \right\} P(d\omega') \\ &= \int_B \left\{ \lim_{n \rightarrow \infty} \sum_{k=1}^{m_n} \zeta_{k,n} p_0(\omega', A_{k,n} A | \mathfrak{B}) \right\} P(d\omega') \\ &= \lim_{n \rightarrow \infty} \left\{ \int_B \sum_{k=1}^{m_n} \zeta_{k,n} p_0(\omega', A A_{k,n} | \mathfrak{B}) \right\} P(d\omega') \\ &\hspace{20em} (\text{since } P = P_0 \text{ on } \mathfrak{B}) \\ &= \lim_{n \rightarrow \infty} \left\{ \int_B \sum_{k=1}^{m_n} \zeta_{k,n} p_0(\omega', A A_{k,n} | \mathfrak{B}) \right\} P_0(d\omega') \\ &= \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^{m_n} \zeta_{k,n} P_0(A A_{k,n} B) \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \int_{AB} \sum_{k=1}^{m_n} \zeta_{k,n} (\Psi A_{k,n})(\omega) P_0(d\omega) \right\} \\ &= \int_{AB} X(\omega) P_0(d\omega) = P(AB) . \end{aligned}$$

Lemma 4 gives immediately

THEOREM 5. *Let $(\Omega, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$, and \mathfrak{A}' be given. Let P_e be any simple extension of P to \mathfrak{A}' . Let $P^{\mathfrak{B}}$ be regular. A sufficient condition that $P_e^{\mathfrak{B}}$ be regular is that $P_e^{\mathfrak{B}}$ be regular where P_e is the canonical part of P_e . (Let K be the set which extends P canonically to \mathfrak{A}' as in [1].)*

Proof. It is immediate that $P_e|_{\mathfrak{B}} = P = P_e|_{\mathfrak{B}}$. Thus the proof will be complete by Lemma 4 if it can be shown that $P_e \ll P_e$. To do so, suppose $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ and $A_i \in \mathfrak{A}$, $i = 1, 2$. If $P_e(A') = 0$, it follows that $P(A_1K) = P(A_2K) = 0$. Thus

$$\varepsilon(A_1K) = \varepsilon(A_2K) = 0$$

by Lemma 2. But, by Proposition 1 it follows that $\varepsilon(A) = \varepsilon(AK)$ for all $A \in \mathfrak{A}$; hence $\varepsilon(A_1) = \varepsilon(A_2) = 0$ and thus $P_e(A') = 0$.

COROLLARY 6. *With the notation of Theorem 5, assume $P_e^{\mathfrak{B}}$ is regular with $0 < \alpha_{\Omega} < 1$. Let $P_{e'}$ be any other canonical extension of P to \mathfrak{A}' , then $P_{e'}^{\mathfrak{B}}$ is regular.*

Proof. $P_{e'} \ll P_e$ and the proof is complete by Lemma 4.

The representation of an arbitrary simple extension as constructed in Proposition 1 helps establish the following interesting

PROPOSITION 7. *Let $(\Omega, \mathfrak{A}, P)$ be given with \mathfrak{A} countably generated and $\{\omega\} \in \mathfrak{A}$ for all $\omega \in \Omega$. Suppose $H \notin \mathfrak{A}$ with $P_*(H) = 0$ and $P^*(H) = 1$. Then there exists no simple extension P_e of P to $\mathfrak{A}' \equiv \sigma(\mathfrak{A}, H)$ such that $P_e^{\mathfrak{A}}$ is regular.*

Proof. With H so chosen, it follows that the set K associated with the canonical part of P_e has P -measure one.

By Proposition 1 write

$$P_e(A') = \alpha_{\Omega}P(A_1K) + \varepsilon(A_1) + \beta_{\Omega}P(A_2K) - \varepsilon(A_2)$$

for any $A' \in \mathfrak{A}'$ with $A' \in A_1H + A_2H^c$ and $A_i \in \mathfrak{A}$, $i = 1, 2$. It may be assumed that $0 < \alpha_{\Omega} < 1$; otherwise, P_e would be canonical (see [1]) and the result would follow directly as in [3], p. 210.

Suppose there exists a version of $P_e^{\mathfrak{A}}$, $p_e(\cdot, \cdot | \mathfrak{A})$, such that $p_e(\omega, \cdot | \mathfrak{A})$ is a measure on \mathfrak{A}' . Define

$$B \equiv \{\omega | p_e(\omega, H | \mathfrak{A}) = 0\}.$$

It follows that $P(B) < 1$, otherwise write

$$\begin{aligned} 0 &= \int_B p_e(\omega, H | \mathfrak{A}) P_e(d\omega) = P_e(BH) = \alpha_\rho P(BK) + \varepsilon(B) \\ &= \alpha_\rho P(B) + \varepsilon(B) = \alpha_\rho, \end{aligned}$$

where $P(B) = 1$ and $\varepsilon(B) = 0$ by Lemma 3, and get $\alpha_\rho = 0$, a contradiction.

Define a set E where E is the set of points ω for which it is not true that $p_e(\omega, D | \mathfrak{A}) = (\psi D)(\omega)$ identically for all $D \in \mathfrak{A}$ (where ψD is the characteristic function of D). Since \mathfrak{A} is countably generated, $P(E) = 0$ (see [3, p. 210]).

It then follows that $(E \cup B)^c \subset H$. Suppose otherwise; that is, $\omega \in (E \cup B)^c$ and $\omega \in H^c$ and get

$$\begin{aligned} p_e(\omega, \{\omega\} \cup H | \mathfrak{A}) &= p_e(\omega, \{\omega\} | \mathfrak{A}) + p_e(\omega, H | \mathfrak{A}) \\ &= (\psi\{\omega\})(\omega) + p_e(\omega, H | \mathfrak{A}) > 1, \end{aligned}$$

a contradiction.

But $P((E \cup B)^c) > 0$ and $(E \cup B)^c \subset H$. This contradicts construction of H and so P_e^* cannot be regular for any simple extension of P to \mathfrak{A}' .

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