SIMPLE EXTENSIONS OF MEASURES AND THE PRESERVATION OF REGULARITY OF CONDITIONAL PROBABILITIES

LOUIS HARVEY BLAKE
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Throughout this paper, the following notation will be
adopted. \((\Omega, \mathcal{A}, P)\) will be a probability space with \(\mathcal{B}\) a sub
\(\sigma\)-field of \(\mathcal{A}\). \(H\) will denote a subset of \(\Omega\) not in \(\mathcal{A}\) and \(\mathcal{A}'\) will be the \(\sigma\)-field generated by \(\mathcal{A}\) and \(H\). \(P_e\) will be a
simple extension of \(P\) to \(\mathcal{A}'\) if \(P_e\) is a probability measure on
\(\mathcal{A}'\) with \(P_e|_{\mathcal{A}} = P\).

The ability to extend the regularity of the conditional
probability \(P^8\) to regularity of \(P_e^8\) has been explored earlier
for canonical extensions of measures. The main results of
this paper are:

(a) If \(P^8\) is regular for some canonical extension \(P_e\) of
\(P\) to \(\mathcal{A}'\), then \(P^8\) is regular for any simple extension \(P_e\) of
\(P\) to \(\mathcal{A}'\).

(b) For some choice of \((\Omega, \mathcal{A}, P), \mathcal{B}\) and \(H\), \(P^8\) is regu-
lar but for no \(P_e\) is \(P^8_e\) regular. This will essentially extend
the Dicudonné example.

Our notation regarding (regular) conditional probabilities will be
consistent with [1].

For extendability see [4]. The example for (b) occurs in [2].

**Proposition 1.** Any simple extension \(P_e\) can be expressed as the
sum of a canonical extension of \(P\) plus a finite signed measure on
\(\mathcal{A}\). (Since the construction is carried out in a unique manner, this
decomposition of \(P_e\) will be called the canonical decomposition of \(P_e\).)

**Proof.** As in [1], let \(K\) be a set which extends \(P\) canonically to
\(\mathcal{A}'\). For any \(A' \in \mathcal{A}'\) with \(A' = A_1H + A_2H^c\) for some \(A_1\) and \(A_2\) in \(\mathcal{A}\)
write

\[ P_e(A') = P(A'K') + P_e(A_1HK) + P_e(A_2H^cK) \]

It may be supposed that \(P(K) \neq 0\). Thus, let \(\alpha_\varphi \equiv P_e(HK)/P(K)\)
and define a set function \(\varepsilon\) on \(\mathcal{A}\) such that for every \(A \in \mathcal{A}\)
write

\[ \varepsilon(A) = P_e(AHK) - \alpha_\varphi P(AK) \]

It is immediate that \(\varepsilon\) is a finite signed measure. It also follows
that for any \(A \in \mathcal{A}\)

\[ P_e(AH^cK) = \beta_\varphi P(AK) - \varepsilon(A) \]
where $\beta_\alpha \equiv 1 - \alpha_\alpha$ inasmuch as it can be written that
\[
P(A) = P_\epsilon(A) = P_\epsilon(AH + AH^c) = P(AK^c) + \alpha_\alpha P(AK) + \varepsilon(A) + P_\epsilon(AH^cK).
\]
Thus, for $A' \in \mathcal{A}'$
\[
P_\epsilon(A') = P(A'K) + \alpha_\alpha P(A_c) + \varepsilon(A_c) + \beta_\alpha P(A_cK) - \varepsilon(A_c).
\]
(Let the sum of the underlined measures be called the canonical part of $P_\epsilon$.)

It is clear that the extension, $P_\epsilon$, of Proposition 1 is canonical if and only if the signed measure $\varepsilon$ is identically zero.

**Lemma 2.** The signed measure $\varepsilon$ is absolutely continuous with respect to $P$.

**Proof.** Let $B \in \mathcal{B}$ be a positive set for $\varepsilon$ according to its Jordan decomposition and let $A \in \mathcal{A}$ with $P(A) = 0$. Then,
\[P_\epsilon(ABHK) \leq P(ABK) \leq P(A) = 0\]
and so $\varepsilon(AB) = 0$. If $C(=B^c)$ is a negative set for $\varepsilon$ then it follows that $\varepsilon(AC) = 0$ where one merely inserts $C$ for $B$ in (2.1). Hence $\varepsilon \ll P$.

**Lemma 3.** If $\Omega_0 \in \mathcal{A}$ with $P(\Omega_0) = 1$ then $\varepsilon(\Omega_0) = 0$.

**Proof.** Immediate.

The following lemma is needed before the main result can be presented.

**Lemma 4.** Let $(\Omega, \mathcal{A}, P)$ be a probability space with $\mathcal{B} \subset \mathcal{A}$. Let $P_\circ$ be another measure on $\mathcal{A}$ with $P = P_\circ$ on $\mathcal{B}$ and $P \ll P_\circ$. Suppose $P_\circ$ is regular. Then, $P_\circ$ is regular.

**Proof.** Let $p_\circ(\cdot | \mathcal{B})$ be a version of $P_\circ$ such that $p_\circ(\omega, \cdot | \mathcal{B})$ is a measure ($P_\circ|_\mathcal{B}$ a.e.). Also, let $X = dP/dP_\circ$ where for all $A \in \mathcal{A}$
\[P(A) = \int_A X dP_\circ.
\]
Hence, define
\[(4.1) h(\omega, A) = \int_A X(\omega')p_\circ(\omega, d\omega' | \mathcal{B}).\]
From (4.1) it is immediate that $h(\cdot, A)$ is $\mathcal{B}$-measurable for every $A \in \mathcal{A}$ and for fixed $\omega \in \Omega$, $h(\omega, \cdot)$ is a measure on $\mathcal{A}$. It remains to show that for any $B \in \mathcal{B}$

$$\int_B h(\omega, A) P(d\omega) = P(AB).$$

To show this, begin by establishing that

$$X \in L_1(\Omega, \mathcal{A}, p_0(\omega, \cdot | \mathcal{B})) P_0 | \mathcal{B} \text{ a.e.}$$

This follows at once by observing that

$$\int_\Omega X(\omega') P_0(\omega', d\omega') = (E^o X)(\omega)$$

and

$$\int_\Omega (E^o X)(\omega) P_0(d\omega) = \int_\Omega X(\omega) P_0(d\omega) = 1.$$

Next, write

$$X = \lim_{n \to \infty} X_n$$

where $X_n = \sum_{k=1}^n \zeta_{k,n} \mathcal{F} A_k, n)$ where

$\zeta_{k,n}$ is a real constant, $\mathcal{F} A_k, n)$ is the characteristic function of $A_k, n) \in \mathcal{A}$ and $\{X_n\}_{n \geq 1}$ is an increasing sequence.

Finally, since $X \in L_1(\Omega, \mathcal{A}, p_0(\omega, \cdot | \mathcal{B})) P_0 | \mathcal{B}$ a.e., the monotone convergence theorem can be used on the following chain of equalities to give the desired result:

$$\int_B h(\omega', A) P(d\omega') = \int_B \left\{ \int_A X(\omega) p_0(\omega', d\omega | \mathcal{B}) \right\} P(d\omega')$$

$$= \int_B \left\{ \lim_{n \to \infty} \sum_{k=1}^n \zeta_{k,n} p_0(\omega', A_k, n A | \mathcal{B}) \right\} P(d\omega')$$

$$= \lim_{n \to \infty} \left\{ \int_B \sum_{k=1}^n \zeta_{k,n} p_0(\omega', AA_k, n | \mathcal{B}) \right\} P(d\omega')$$

(since $P = P_0$ on $\mathcal{B}$)

$$= \lim_{n \to \infty} \left\{ \int_B \sum_{k=1}^n \zeta_{k,n} p_0(\omega', AA_k, n B) \right\}$$

$$= \lim_{n \to \infty} \left\{ \int_{AB} \sum_{k=1}^n \zeta_{k,n} (\mathcal{F} A_k, n)(\omega) P_0(d\omega) \right\}$$

$$= \int_{AB} X(\omega) P_0(d\omega) = P(AB).$$

Lemma 4 gives immediately
THEOREM 5. Let \((\Omega, \mathcal{A}, P), \mathcal{B} \subset \mathcal{A}, \) and \(\mathcal{A}' \) be given. Let \(P_e\) be any simple extension of \(P\) to \(\mathcal{A}'\). Let \(P^*\) be regular. A sufficient condition that \(P^*_e\) be regular is that \(P^*_e\) be regular where \(P_e\) is the canonical part of \(P_e\). (Let \(K\) be the set which extends \(P\) canonically to \(\mathcal{A}'\) as in [1].)

Proof. It is immediate that \(P_e|_\mathcal{B} = P = P_e|_\mathcal{B}\). Thus the proof will be complete by Lemma 4 if it can be shown that \(P_e \ll P\). To do so, suppose \(A' \in \mathcal{A}'\) with \(A' = A_1H + A_2H'\) and \(A_i \in \mathcal{A}, i = 1, 2\). If \(P_e(A') = 0\), it follows that \(P(A_1K) = P(A_2K) = 0\). Thus

\[
\varepsilon(A_iK) = \varepsilon(A_2K) = 0
\]

by Lemma 2. But, by Proposition 1 it follows that \(\varepsilon(A) = \varepsilon(AK)\) for all \(A \in \mathcal{A}\); hence \(\varepsilon(A_i) = \varepsilon(A_2) = 0\) and thus \(P_e(A') = 0\).

COROLLARY 6. With the notation of Theorem 5, assume \(P^*_e\) is regular with \(0 < \alpha_0 < 1\). Let \(P_e\) be any other canonical extension of \(P\) to \(\mathcal{A}'\), then \(P^*_e\) is regular.

Proof. \(P_e \ll P\) and the proof is complete by Lemma 4.

The representation of an arbitrary simple extension as constructed in Proposition 1 helps establish the following interesting

PROPOSITION 7. Let \((\Omega, \mathcal{A}, P)\) be given with \(\mathcal{A}\) countably generated and \(\{\omega\} \in \mathcal{A}\) for all \(\omega \in \Omega\). Suppose \(H \in \mathcal{A}\) with \(P^*(H) = 0\) and \(P^*(H) = 1\). Then there exists no simple extension \(P_e\) of \(P\) to \(\mathcal{A}' \equiv \sigma(\mathcal{A}, H)\) such that \(P^*_e\) is regular.

Proof. With \(H\) so chosen, it follows that the set \(K\) associated with the canonical part of \(P_e\) has \(P\)-measure one.

By Proposition 1 write

\[
P_e(A') = \alpha_2P(A_1K) + \varepsilon(A_1) + \beta_2P(A_2K) - \varepsilon(A_2)
\]

for any \(A' \in \mathcal{A}'\) with \(A' = A_1H + A_2H'\) and \(A_i \in \mathcal{A}, i = 1, 2\). It may be assumed that \(0 < \alpha_0 < 1\); otherwise, \(P_e\) would be canonical (see [1]) and the result would follow directly as in [3], p. 210.

Suppose there exists a version of \(P^*_e\), \(p_e(\cdot, \cdot | \mathcal{A})\), such that \(p_e(\omega, \cdot | \mathcal{A})\) is a measure on \(\mathcal{A}'\). Define

\[
B \equiv \{\omega | p_e(\omega, H | \mathcal{A}) = 0\}
\]

It follows that \(P(B) < 1\), otherwise write
\[ 0 = \int_B p_\omega(\omega, H | 21) P_\omega(d\omega) = P_\omega(BH) = \alpha_0 P(BK) + \varepsilon(B) \]
\[ = \alpha_0 P(B) + \varepsilon(B) = \alpha_0 , \]

where \( P(B) = 1 \) and \( \varepsilon(B) = 0 \) by Lemma 3, and get \( \alpha_0 = 0 \), a contradiction.

Define a set \( E \) where \( E \) is the set of points \( \omega \) for which it is not true that \( p_\omega(\omega, D | 21) = (\psi D)(\omega) \) identically for all \( D \in 21 \) (where \( \psi D \) is the characteristic function of \( D \)). Since \( 21 \) is countably generated, \( P(E) = 0 \) (see [3, p. 210]).

It then follows that \((E \cup B)^c \subset H\). Suppose otherwise; that is, \( \omega \in (E \cup B)^c \) and \( \omega \notin H^c \) and get
\[
0 = (\psi\{\omega}\rangle \cup H | 21) = (\psi\{\omega}\rangle | 21) + p_\omega(\omega, H | 21) = \varepsilon(\{\omega}\rangle | 21) + p_\omega(\omega, H | 21) > 1 ,
\]
a contradiction.

But \( P((E \cup B)^c) > 0 \) and \((E \cup B)^c \subset H\). This contradicts construction of \( H \) and so \( P^*_\omega \) cannot be regular for any simple extension of \( P \) to \( 21' \).

**References**


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