Pacific Journal of Mathematics

SIMPLE EXTENSIONS OF MEASURES AND THE PRESERVATION OF REGULARITY OF CONDITIONAL PROBABILITIES

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Vol. 46, No. 2 December 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 2, 1973

SIMPLE EXTENSIONS OF MEASURES AND THE PRESERVATION OF REGULARITY OF CONDITIONAL PROBABILITIES

LOUIS H. BLAKE

Throughout this paper, the following notation will be adopted. $(\Omega, \mathfrak{A}, P)$ will be a probability space with $\mathfrak{B}$ a sub σ-field of $\mathfrak{A}$. H will denote a subset of Ω not in $\mathfrak{A}$ and $\mathfrak{A}'$ will be the σ-field generated by $\mathfrak{A}$ and H. P_e will be a simple extension of P to $\mathfrak{A}'$ if P_e is a probability measure on $\mathfrak{A}'$ with $P_e\,|_{\mathfrak{A}} = P$.

The ability to extend the regularity of the conditional probability $P^{\mathfrak{B}}$ to regularity of $P_e^{\mathfrak{B}}$ has been explored earlier for canonical extensions of measures. The main results of this paper are:

(a) If $P_c^{\mathfrak{B}}$ is regular for some canonical extension P_c of P to $\mathfrak{A}'$, then $P_e^{\mathfrak{B}}$ is regular for any simple extension P_e of P to $\mathfrak{A}'$.

(b) For some choice of $(\Omega, \mathfrak{A}, P)$, $\mathfrak{B}$ and H, $P^{\mathfrak{B}}$ is regular but for *no* P_e is $P_e^{\mathfrak{B}}$ regular. This will essentially extend the Dicudonné example,

Our notation regarding (regular) conditional probabilities will be consistent with [1].

For extendability see [4]. The example for (b) occurs in [2].

PROPOSITION 1. *Any simple extension P_e can be expressed as the sum of a canonical extension of P plus a finite signed measure on $\mathfrak{A}$. (Since the construction is carried out in a unique manner, this decomposition of P_e will be called the canonical decomposition of P_e.)*

Proof. As in [1], let K be a set which extends P canonically to $\mathfrak{A}'$. For any $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ for some A_1 and A_2 in $\mathfrak{A}$ write

$$P_e(A') = P(A'K^c) + P_e(A_1HK) + P_e(A_2H^cK)\ .$$

It may be supposed that $P(K) \neq 0$. Thus, let $\alpha_\Omega \equiv P_e(HK)/P(K)$ and define a set function ε on $\mathfrak{A}$ such that for every $A \in \mathfrak{A}$

$$\varepsilon(A) = P_e(AHK) - \alpha_\Omega P(AK)\ .$$

It is immediate that ε is a finite signed measure. It also follows that for any $A \in \mathfrak{A}$

$$P_e(AH^cK) = \beta_\Omega P(AK) - \varepsilon(A)$$

where $\beta_\Omega \equiv 1 - \alpha_\Omega$ inasmuch as it can be written that

$$P(A) = P_e(A) = P_e(AH + AH^c) =$$
$$P(AK^c) + \alpha_\Omega P(AK) + \varepsilon(A) + P_e(AH^cK) .$$

Thus, for $A' \in \mathfrak{A}'$

$$P_e(A') = \underline{P(A'K^c)} + \underline{\alpha_\Omega P(A_1K)} + \varepsilon(A_1)$$
$$+ \underline{\beta_\Omega P(A_2K)} - \varepsilon(A_2) .$$

(Let the sum of the underlined measures be called the *canonical part* of P_e.)

It is clear that the extension, P_e, of Proposition 1 is canonical if and only if the signed measure ε is identically zero.

LEMMA 2. *The signed measure ε is absolutely continuous with respect to P.*

Proof. Let $B \in \mathfrak{A}$ be a positive set for ε according to its Jordan decomposition and let $A \in \mathfrak{A}$ with $P(A) = 0$. Then,

$$P_e(ABHK) \leqq P(ABK) \leqq P(A) = 0 \tag{2.1}$$

and so $\varepsilon(AB) = 0$. If $C(=B^c)$ is a negative set for ε then it follows that $\varepsilon(AC) = 0$ where one merely inserts C for B in (2.1). Hence $\varepsilon \ll P$.

LEMMA 3. *If $\Omega_0 \in \mathfrak{A}$ with $P(\Omega_0) = 1$ then $\varepsilon(\Omega_0) = 0$.*

Proof. Immediate.

The following lemma is needed before the main result can be presented.

LEMMA 4. *Let $(\Omega, \mathfrak{A}, P)$ be a probability space with $\mathfrak{B} \subset \mathfrak{A}$. Let P_0 be another measure on $\mathfrak{A}$ with $P = P_0$ on $\mathfrak{B}$ and $P \ll P_0$. Suppose $P_0^{\mathfrak{B}}$ is regular. Then, $P^{\mathfrak{B}}$ is regular.*

Proof. Let $p_0(\cdot\,, \cdot \mid \mathfrak{B})$ be a version of $P_0^{\mathfrak{B}}$ such that $p_0(\omega, \cdot \mid \mathfrak{B})$ is a measure ($P_0|_{\mathfrak{B}}$ a.e.). Also, let $X = dP/dP_0$ where for all $A \in \mathfrak{A}$

$$P(A) = \int_A X\, dP_0 .$$

Hence, define

$$h(\omega, A) = \int_A X(\omega') p_0(\omega, d\omega' \mid \mathfrak{B}) . \tag{4.1}$$

From (4.1) it is immediate that $h(\cdot, A)$ is $\mathfrak{B}$-measurable for every $A \in \mathfrak{A}$ and for fixed $\omega \in \Omega$, $h(\omega, \cdot)$ is a measure on $\mathfrak{A}$. It remains to show that for any $B \in \mathfrak{B}$

$$\int_B h(\omega, A) P(d\omega) = P(AB) .$$

To show this, begin by establishing that

$$X \in L_1(\Omega, \mathfrak{A}, p_0(\omega, \cdot \mid \mathfrak{B}))\ P_0|_{\mathfrak{B}} \text{ a.e.}$$

This follows at once by observing that

$$\int_\Omega X(\omega') p(\omega, d\omega' \mid \mathfrak{B}) = (E^{\mathfrak{B}} X)(\omega)$$

and

$$\int_\Omega (E^{\mathfrak{B}} X)(\omega) P_0(d\omega) = \int_\Omega X(\omega) P_0(d\omega) = 1 .$$

Next, write

$$X = \lim_{n\to\infty} X_n \quad \text{where} \quad X_n = \sum_{k=1}^{m_n} \zeta_{k,n} (\Psi A_{k,n}) \quad \text{where}$$

$\zeta_{k,n}$ is a real constant, $(\Psi A_{k,n})$ is the characteristic function of $A_{k,n} \in \mathfrak{A}$ and $\{X_n\}_{n\geq 1}$ is an increasing sequence.

Finally, since $X \in L_1(\Omega, \mathfrak{A}, p_0(\omega, \cdot \mid \mathfrak{B}))\ P_0|_{\mathfrak{B}}$ a.e., the monotone convergence theorem can be used on the following chain of equalities to give the desired result:

$$\begin{aligned}
\int_B h(\omega', A) P(d\omega') &= \int_B \left\{ \int_A X(\omega) p_0(\omega', d\omega \mid \mathfrak{B}) \right\} P(d\omega') \\
&= \int_B \left\{ \lim_{n\to\infty} \sum_{k=1}^{m_n} \zeta_{k,n} p_0(\omega', A_{k,n} A \mid \mathfrak{B}) \right\} P(d\omega') \\
&= \lim_{n\to\infty} \left\{ \int_B \sum_{k=1}^{m_n} \zeta_{k,n}\, p_0(\omega', A A_{k,n} \mid \mathfrak{B}) \right\} P(d\omega') \\
&\qquad\qquad\qquad\qquad (\text{since } P = P_0 \text{ on } \mathfrak{B}) \\
&= \lim_{n\to\infty} \left\{ \int_B \sum_{k=1}^{m_n} \zeta_{k,n} p_0(\omega', A A_{k,n} \mid \mathfrak{B}) \right\} P_0(d\omega') \\
&= \lim_{n\to\infty} \left\{ \sum_{k=1}^{m_n} \zeta_{k,n} P_0(A A_{k,n} B) \right\} \\
&= \lim_{n\to\infty} \left\{ \int_{AB} \sum_{k=1}^{m_n} \zeta_{k,n} (\Psi A_{k,n})(\omega) P_0(d\omega) \right\} \\
&= \int_{AB} X(\omega) P_0(d\omega) = P(AB) .
\end{aligned}$$

Lemma 4 gives immediately

THEOREM 5. *Let* $(\Omega, \mathfrak{A}, P)$, $\mathfrak{B} \subset \mathfrak{A}$, *and* $\mathfrak{A}'$ *be given. Let* P_e *be any simple extension of* P *to* $\mathfrak{A}'$. *Let* $P^{\mathfrak{B}}$ *be regular. A sufficient condition that* $P_e^{\mathfrak{B}}$ *be regular is that* $P_c^{\mathfrak{B}}$ *be regular where* P_c *is the canonical part of* P_e. (*Let* K *be the set which extends* P *canonically to* $\mathfrak{A}'$ *as in* [1].)

Proof. It is immediate that $P_c|_{\mathfrak{B}} = P = P_e|_{\mathfrak{B}}$. Thus the proof will be complete by Lemma 4 if it can be shown that $P_e \ll P_c$. To do so, suppose $A' \in \mathfrak{A}'$ with $A' = A_1H + A_2H^c$ and $A_i \in \mathfrak{A}$, $i = 1, 2$. If $P_c(A') = 0$, it follows that $P(A_1K) = P(A_2K) = 0$. Thus

$$\varepsilon(A_1K) = \varepsilon(A_2K) = 0$$

by Lemma 2. But, by Proposition 1 it follows that $\varepsilon(A) = \varepsilon(AK)$ for all $A \in \mathfrak{A}$; hence $\varepsilon(A_1) = \varepsilon(A_2) = 0$ and thus $P_e(A') = 0$.

COROLLARY 6. *With the notation of Theorem* 5, *assume* $P_c^{\mathfrak{B}}$ *is regular with* $0 < \alpha_\Omega < 1$. *Let* $P_{c'}$ *be any other canonical extension of* P *to* $\mathfrak{A}'$, *then* $P_{c'}^{\mathfrak{B}}$ *is regular.*

Proof. $P_{c'} \ll P_c$ and the proof is complete by Lemma 4.

The representation of an arbitrary simple extension as constructed in Proposition 1 helps establish the following interesting

PROPOSITION 7. *Let* $(\Omega, \mathfrak{A}, P)$ *be given with* $\mathfrak{A}$ *countably generated and* $\{\omega\} \in \mathfrak{A}$ *for all* $\omega \in \Omega$. *Suppose* $H \notin \mathfrak{A}$ *with* $P_*(H) = 0$ *and* $P^*(H) = 1$. *Then there exists no simple extension* P_e *of* P *to* $\mathfrak{A}' \equiv \sigma(\mathfrak{A}, H)$ *such that* $P_e^{\mathfrak{A}}$ *is regular.*

Proof. With H so chosen, it follows that the set K associated with the canonical part of P_e has P-measure one.

By Proposition 1 write

$$P_e(A') = \alpha_\Omega P(A_1K) + \varepsilon(A_1) + \beta_\Omega P(A_2K) - \varepsilon(A_2)$$

for any $A' \in \mathfrak{A}'$ with $A' \in A_1H + A_2H^c$ and $A_i \in \mathfrak{A}$, $i = 1, 2$. It may be assumed that $0 < \alpha_\Omega < 1$; otherwise, P_e would be canonical (see [1]) and the result would follow directly as in [3], p. 210.

Suppose there exists a version of $P_e^{\mathfrak{A}}$, $p_e(\cdot\,, \cdot \mid \mathfrak{A})$, such that $p_e(\omega, \cdot \mid \mathfrak{A})$ is a measure on $\mathfrak{A}'$. Define

$$B \equiv \{\omega \mid p_e(\omega, H \mid \mathfrak{A}) = 0\}\,.$$

It follows that $P(B) < 1$, otherwise write

$$0 = \int_B p_e(\omega, H \mid \mathfrak{A}) P_e(d\omega) = P_e(BH) = \alpha_\Omega P(BK) + \varepsilon(B)$$
$$= \alpha_\Omega P(B) + \varepsilon(B) = \alpha_\Omega \, ,$$

where $P(B) = 1$ and $\varepsilon(B) = 0$ by Lemma 3, and get $\alpha_\Omega = 0$, a contradiction.

Define a set E where E is the set of points ω for which it is not true that $p_e(\omega, D \mid \mathfrak{A}) = (\psi D)(\omega)$ identically for all $D \in \mathfrak{A}$ (where ψD is the characteristic function of D). Since $\mathfrak{A}$ is countably generated, $P(E) = 0$ (see [3, p. 210]).

It then follows that $(E \cup B)^c \subset H$. Suppose otherwise; that is, $\omega \in (E \cup B)^c$ and $\omega \in H^c$ and get

$$p_e(\omega, \{\omega\} \cup H \mid \mathfrak{A}) = p_e(\omega, \{\omega\} \mid \mathfrak{A}) + p_e(\omega, H \mid \mathfrak{A})$$
$$= (\psi\{\omega\})(\omega) + p_e(\omega, H \mid \mathfrak{A}) > 1 \, ,$$

a contradiction.

But $P((E \cup B)^c) > 0$ and $(E \cup B)^c \subset H$. This contradicts construction of H and so $P_e^{\mathfrak{A}}$ cannot be regular for any simple extension of P to $\mathfrak{A}'$.

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Received February 22, 1972.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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