# NEW PROOFS OF BING'S APPROXIMATION THEOREMS FOR SURFACES

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# THEOREMS FOR SURFACES

## J. W. Cannon

We give alternative proofs of Bing's Side Approximation
and Homeomorphic Approximation Theorems for surfaces in
3-manifolds.

0. **Introduction.** We give in §1 an alternative proof of Bing's Side Approximation Theorem (S.A.T.) [4] [7] [8]:

S.A.T. Suppose that $S$ is a 2-sphere topologically embedded in $E^3$. Then, for each $\varepsilon > 0$, there are disjoint $\varepsilon$-disks $D_1, \cdots, D_n$ in $S$ and an $\varepsilon$-homeomorphism $h$ from $S$ into $E^3$ such that

(1) $h(S)$ is polyhedral, and

(2) $S - \bigcup_i D_i \subset \operatorname{Ext} h(S)$.

One may require alternatively that

(2') $S - \bigcup_i D_i \subset \operatorname{Int} h(S)$.

In §2 we deduce from the above the slightly stronger version of the S.A.T. (also due to Bing) which one generally encounters in applications.

In the strict sense, Bing's proof of the S.A.T. is completely elementary. Our proof on the other hand uses freely local homology and homotopy properties, Dehn's lemma and related theorems, and homology linking theory. We have attempted to substitute simplicity of *outline* for simplicity of technique. This has the advantage that the outline serves almost without change in a proof of the Moise [15]–Bing [6] theorem on the triangulability of 3-manifolds and the Armentrout [3] theorem on approximating cellular maps between 3-manifolds by homeomorphisms. We do not go into detail on these latter theorems. Some details of the proof which are unique to the S.A.T. (and which, in fact, make it harder than the triangulation and cellular map theorems) have been proved in their more natural setting, "ULC properties of embedded curves and surfaces in $E^3$" [10, §2 and the first half of §3]. This allows us to make the outline even more transparent and unencumbered by detail. The necessary information from [10] is summarized in four lemmas at the end of this introduction. This quick summary allows the present paper to be read independently of [10]. We give a short proof in §3 of another of Bing's important theorems on surfaces, namely, the Homeomorphic Approximation Theorem (H.A.T.) [5]. The shortness arises from the fact that we use the S.A.T. to do the work that Bing did from scratch.

H.A.T. Suppose that $S$ is a topologically embedded 2-sphere in

$E^3$ and that, for each $\varepsilon > 0$, there is an $\varepsilon$-homeomorphism from $S$ into Int $S$. Then $S \cup$ Int $S$ is a 3-cell.

Papers [9] and [10] are the basic references for this paper: Reference [9] includes descriptions of the Euclidean spaces ($E^1$, $E^2$, $E^3$, $\cdots$), $n$-cells ($B^0$, $B^1$, $B^2$, $\cdots$ and their homeomorphic images), $n$-spheres ($S^0 = $ Bd $B^1$, $S^1 = $ Bd $B^2$, $\cdots$ and their homeomorphic images), complexes, manifolds, disks (2-cells), arcs (1-cells), simple closed curves and loops, finite graphs (1-complexes), polyhedra, and general position and cut-and-paste techniques. It also gives statements of Dehn's Lemma [16] (cf. [9, 4.5.1 and Addendum to 4.5.1], the Loop Theorem [17], and the Sphere Theorem [16]). Such concepts are widely known and need only be checked by the beginner. Reference [10] contains technical material less widely known and summarized here:

NOTATION. We use $\rho$ for the Euclidean metric, Diam for diameter, Cl for closure, Bd for boundary (point set or combinatorial), Int and Ext for interior and exterior, $N(X, \varepsilon)$ for the (open) $\varepsilon$-neighborhood of $X$ in $E^3$. An $\varepsilon$-set has diameter less than $\varepsilon$; an $\varepsilon$-map or homeomorphism moves no point as far as $\varepsilon$. If $T$ is a triangulation (possibly curvilinear) of some topologically embedded polyhedron, then $T^i$ denotes the $i$-skeleton of $T$, and $|T^i|$ denotes the underlying point set which carries $T^i$ (i.e., the union of the simplexes in $T^i$).

*Property $ULC^n$.* Let $A$ be a subset of $E^3$ and $n$ a nonnegative integer. Then $A$ is $ULC^n$ if the following condition is satisfied: For each $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 \leq i \leq n$ and $f: S^i \to A$ is a map from the $i$-sphere $S^i$ into a $\delta$-subset of $A$, then there is a map $f^*: B^{i+1} \to A$ into an $\varepsilon$-subset of $A$ which extends $f$ (recall that $S^i = $ Bd $B^{i+1}$).

0.1. [19, p. 66]. If $S$ is a 2-sphere in $E^3$ and $U$ is a component of $E^3 - S$, then $U$ is $ULC^0$.

0.2. *Existence of nice arcs in 2-spheres.* Suppose $S$ is a 2-sphere in $E^3$, $U_1$ and $U_2$ are the components of $E^3 - S$, $G$ is a finite graph in $S$, and $\varepsilon > 0$. Then there is an $\varepsilon$-homeomorphism $h: S \to S$ such that Cl $(U_1) - h(G)$ and Cl $(U_2) - h(G)$ are $ULC^2$.

*Proof.* [10; 2C. 7(2). 2, 2C. 7(3). 2, 2C. 7(1), 2C. 3].

0.3. Suppose $S$ is a 2-sphere in $E^3$, $U_1$ and $U_2$ are the components of $E^3 - S$, and $J$ is a simple closed curve in $S$ such that Cl$(U_1) - J$ and Cl $(U_2) - J$ are $ULC^2$. Then $J$ pierces an almost-polyhedral disk at each point of $J$. (A 2-manifold is said to be almost-polyhedral if it is locally polyhedral except at finitely many points.)

*Proof.* [10; 3.2, 3.3].

0.4. *Adjustments of maps in ULC sets.* Suppose $C \subset E^3$ and $C$ is $ULC^n$ ($n$ a nonnegative integer). Suppose $P$ is an $(n + 1)$-complex, $B$ is a closed subset of $P$, and $F: P \to \operatorname{Cl} C$ is a map. Then, for each $\varepsilon > 0$, there is a map $f^*: P \to \operatorname{Cl} C$ such that

$$f^* \mid B = f \mid B,$$
$$f^*(P - B) \subset C, \text{ and}$$
$$\rho(f^*(x), f(x)) < \varepsilon, \text{ for each } x \in P.$$

*Proof.* [10, 2C. 2].

The following information on linking is classical and is included for the use of the reader in verifying certain intuitive claims made in the proof of the S.A.T. and H.A.T. We give one example of such a verification in an appendix following § 3.

*Linking of Simple Closed Curves in $E^3$* (cf. [1, Chapter 15] and [18, § 77] for details). We use the notation $L(J, K)$ for the (homological) linking number (integer coefficients) of disjoint oriented simple closed curves or loops in $E^3$. The integer $L(J, K)$ may be calculated as follows. Choose polygonal oriented simple closed curves $J'$ and $K'$ such that $J$ is homologous to $J'$ in $E^3 - K$ and $K$ is homologous to $K'$ in $E^3 - J'$. Let $D$ be a polyhedral singular oriented surface in $E^3$ bounded by the oriented curve $J'$ and in general position with respect to $K'$. Assume $E^3$ oriented. Then each oriented simplex in $D$ can be assigned a positive and a negative side in $E^3$ (intuitively, by the right-hand rule). Then $L(J, K)$ is the algebraic number of times $K'$ cuts through $D$: an intersection at which the oriented $K'$ passes from the negative to positive side of $D$ is counted as $+1$, from the positive to the negative as $-1$. Then the following properties hold.

0.5. $L(J, K) = L(K, J)$ and this integer is independent of the choice of $J'$, $K'$, and $D$ (in particular depends only on the homology class $[K]$ of $K$ in $E^3 - J$).

0.6. If $J$ is an oriented simple closed curve in $E^3$, then there is an isomorphism $H_1(E^3 - J; \mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}$ which sends $[K]$ to $L(J, K)$.

0.7. If $D$ is a disk in $E^3$ (not necessarily polyhedral) and $p \in \operatorname{Int} D$, then there is a simple closed curve $J$ in $(E^3 - D) \cup \{p\}$ for which $L(J, \operatorname{Bd} D) \neq 0$. Furthermore, if $J$ is any such loop and $J_1$ is any loop in $E^3 - \operatorname{Bd} D$, then $J_1$ is homologous in $E^3 - \operatorname{Bd} D$ to some multiple of $J$. In particular $|L(J, \operatorname{Bd} D)| = 1$ by (0.6).

Finally, we mention that we shall use freely the standard cut-and-paste procedures of 3-space topology without being too explicit about the details. For the uninitiated (and even for the initiated who finds the details of the proof of Lemma (4.2), § 4, somewhat outside the range of his experience) we recommend the note by Griffith and Harrold in the Proceedings of the National Academy of Science [12] where the procedure is carried out in some detail. (Note the comments about [12] at the end of our paper.)

1. *Proof of the S.A.T.* Throughout this section, $S$ will denote a topologically embedded 2-sphere in Euclidean 3-dimensional space $E^3$ and $\varepsilon$ a positive number.

Choose a triangulation $T$ of $S$ which has mesh so small that the simplicial neighborhood in $T$ of each simplex has diameter less than $\varepsilon/8$. By (0.2), we may assume that the carrier $G = |T^1|$ of the 1-skeleton $T^1$ of $T$ is a finite graph such that $\mathrm{Cl}\,(\mathrm{Int}\,S) - G$ and $\mathrm{Cl}\,(\mathrm{Ext}\,S) - G$ are $ULC^2$.

*Step 1. Building an almost-polyhedral 2-sphere $S(v)$ about a vertex $v$ of $T$.* We choose from the interior of the carrier of each 1-simplex $\sigma$ of $T$ a point $p(\sigma)$. By (0.3), there is a very small almost-polyhedral disk $D(\sigma)$ pierced by $|T^1|$ at $p(\sigma)$. Let $v$ be a vertex of $T$ and $\sigma_1, \cdots, \sigma_n = \sigma_0$ the 1-simplexes of $T$ which emanate from $v$. We assume $\sigma_1, \cdots, \sigma_n = \sigma_0$ indexed so that $\sigma_i$ and $\sigma_j$ are faces of a common 2-simplex $\tau_{ij}$ of $T$ if $i - j = 1$. Let $s_i$ be the arc in $\sigma_i$ bounded by $v$ and $p(\sigma_i)$. Let $q(\sigma_i)$ be a point of $(\mathrm{Int}\,\sigma_i) - s_i$ ($q(\sigma_n) = q(\sigma_0)$). Let $s_i'$ be the arc in $\sigma_i$ bounded by $v$ and $q(\sigma_i)$. Let $A_{ij}(i - j = 1)$ denote a spanning arc of $\tau_{ij}$ with endpoints $q(\sigma_i)$ and $q(\sigma_j)$. We require that $A_{ij}$ be very near homeomorphically to the arc $s_i' \cup s_j'$. Then $J = \bigcup_{i-j=1} A_{ij}$ is a simple closed curve in $S$ which bounds a small disk
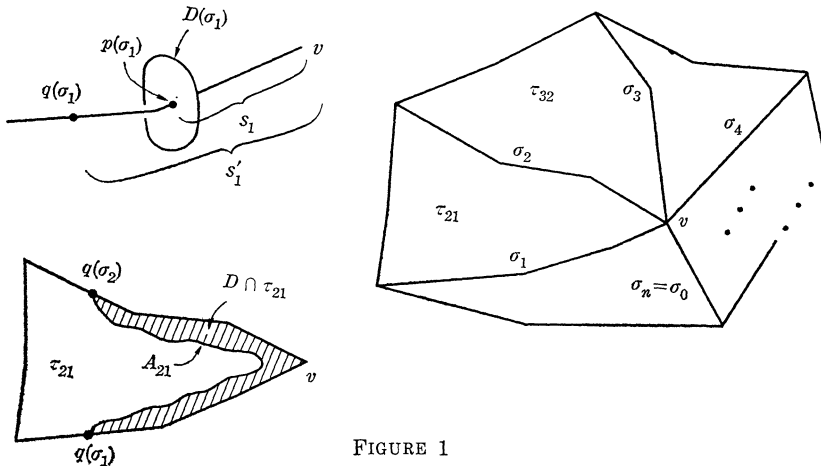


FIGURE 1

$D$ in $S$ with $\bigcup_i s_i \subset \text{Int } D$. We may require that $D \cap D(\sigma_i) = D \cap \text{Int } D(\sigma_i)$ and that this intersection lie very close to $p(\sigma_i)$. (See Figure 1.)

Since $\text{Cl}(\text{Int } S) - |T^1|$ and $\text{Cl}(\text{Ext } S) - |T^1|$ are $ULC^1$, it follows from (0.4) that there are maps $f: D \to S \cup \text{Int } S$ and $g: D \to S \cup \text{Ext } S$ which fix $\text{Bd } D$, move points of $\text{Int } D$ only very slightly, and take $\text{Int } D$ into the complement of $|T^1|$. We may think of $f(D) \cup g(D)$ as a singular 2-sphere in $E^3$. After slight adjustment we may assume that $f(D) \cup g(D)$ is in general position with respect to $\bigcup_i D(\sigma_i)$. We require that $f$ and $g$ be chosen so near to the identity that $(f(D) \cup g(D)) \cap \bigcup_i \text{Bd } D(\sigma_i) = \varnothing$. We leave it to the reader to establish the following claim (e.g., by a linking argument):

*Claim.* If, for each $i$ and $j$ with $i - j = 1$, the arc $A_{ij}$ is very close homeomorphically to the arc $s_i' \cup s_j'$ and if the maps $f$ and $g$ are very near the identity map, then there is in $f(D) \cup g(D)$ (as adjusted for general position) a singular disk-with-holes $D_0$ which has the following properties:

( 1 )    $D_0 \cap \bigcup_i D(\sigma_i) = \text{Bd } D_0$, and

( 2 )    $D_0$ has in each $D(\sigma_i)$ precisely one boundary curve which is not homotopically trivial in $D(\sigma_i) - \{p(\sigma_i)\}$, and that boundary curve is homotopic in $D(\sigma_i) - \{p(\sigma_i)\}$ to a simple closed curve.

If we start with the disk-with-holes $D_0$ supplied by the claim, then the boundary curves of $D_0$ which are trivial in some $D(\sigma_i) - \{p(\sigma_i)\}$ may be filled in by singular disks in $D(\sigma_i) - \{p(\sigma_i)\}$, and these singular disks may then be pushed slightly to one side of $D(\sigma_i)$ so that (1) and (2) are still satisfied but so that the nontrivial boundary curves promised by (2) are the only boundary curves of the adjusted $D_0$. These nontrivial boundary curves may in turn be replaced by simple closed curves $J_i$ in $D(\sigma_i) - \{p(\sigma_i)\}$, by (2), so that (1) and (2) are still satisfied. We continue to call the adjusted disk-with-holes $D_0$.

Let $D_i(i = 1, \cdots, n)$ be the disk in $D(\sigma_i)$ bounded by $J_i$. Then $D_0 \cup D_1 \cup \cdots \cup D_n$ is an almost-polyhedral singular 2-sphere in $E^3$ with no singularities near $p(\sigma_1), \cdots, p(\sigma_n)$. By Dehn's Lemma([16]; note also [9, Addendum to Theorem 4.5.1]), there is a nonsingular 2-sphere $S(v)$ in the union of $D_1 \cup \cdots \cup D_n$ and an arbitrarily small neighborhood of $\text{Int } D_0$ such that $v \in \text{Int } S(v)$ and such that $S(v)$ is polyhedral except possibly at the points $p(\sigma_1), \cdots, p(\sigma_n)$. It follows that, if $D_0$ is sufficiently close to $D$, we must have $S(v) \cap |T^1| = \{p(\sigma_1), \cdots, p(\sigma_n)\}$ with $|T^1|$ piercing $S(v)$ at each of the points $p(\sigma_i)$.

*Step 2. Building a polyhedral handlebody $H$ about $|T^1|$.* It is clear from the construction described in Step 1 that, if $v_1, \cdots, v_m$ are the vertices of $T$, then we may choose the 2-spheres $S(v_1), \cdots, S(v_m)$ so as to satisfy the following requirements:

(3)  $[S(v_i) \cup \text{Int } S(v_i)] \cap [S(v_j) \cup \text{Int } S(v_j)] = S(v_i) \cap S(v_j)(i \neq j)$; and this intersection is either $\varnothing$ or it is a subdisk of some $D(\sigma)$ and contains the corresponding $p(\sigma)$.

(4)  $[S(v_i) \cup \text{Int } S(v_i)] \cap \tau \neq \varnothing$ if and only if $v_i$ is a vertex of $\tau$, for each $\tau \in T$.

(5)  Diam $S(v_i) < \varepsilon/8$.

Let $H = \bigcup_{i=1}^{m} S(v_i) \cup \text{Int } S(v_i)$. We see that $H$ is a polyhedral handlebody as follows. Suppose $S(v_i) \cap S(v_j)(i > j)$ is nonempty, hence a disk $D'_{ij}$. Then $D'_{ij}$ is polyhedral modulo its one-point intersection with $|T^1|$. Use Dehn's lemma to replace $D'_{ij}$ by a polyhedral disk $D_{ij}$ which differs from $D'_{ij}$ only very near $D'_{ij} \cap |T^1|$. The iteration of this process changes each $S(v_i)$ into a polyhedral 2-sphere $T(v_i)$. Then $H = \bigcup_{i=1}^{m} T(v_i) \cup \text{Int } T(v_i)$, and $H$ is clearly a handlebody since each of the sets $C(v_i) = T(v_i) \cup \text{Int } T(v_i)$ is a polyhedral 3-cell. If $v_r$, $v_s$, and $v_t$ ($r > s > t$) are the vertices of a single 2-simplex $\tau$ in $T$, then

$$X(\tau) = C(v_r) \cup C(v_s) \cup C(v_t)$$

is a polyhedral solid torus. The disks $D_{rs}$, $D_{st}$, and $D_{rt}$ are called the *distinguished meridional disks* of $X(\tau)$ and the other disks $D_{ij}$ which lie on Bd $X(\tau)$ are called the *distinguished boundary disks* of $X(\tau)$. The totality of disks $D_{ij}$ are called the *distinguished disks* of $H$.

*Step. 3.  Spanning polyhedral disks across the "holes" in H.* Let $\tau$ and $X(\tau)$ be as in the preceding paragraph. We may certainly require that Bd $\tau \subset \text{Int } X(\tau)$ and that Bd $\tau$ be homotopic in Int $X(\tau)$ to a centerline of $X(\tau)$. As one consequence we find that there are simple closed curves $J$ and $K$ in Int $\tau$ such that Bd $\tau$ and $K$ bound an annulus $A$ in $\tau \cap \text{Int } X(\tau)$ with $J$ separating Bd $\tau$ from $K$ in $A$. By (0.2), we may assume that Cl (Int $S$) $- K$ and Cl (Int $S$) $- K$ are $ULC^1$. Let $D$ be the subdisk of $\tau$ bounded by $J$. By (0.4), there are maps $f: D \rightarrow S \cup \text{Int } S$ and $g: D \rightarrow S \cup \text{Ext } S$ which fix Bd $D$, move points of Int $D$ only slightly, and take Int $D$ into the complement of $K$. We think of $f(D) \cup g(D)$ as a singular 2-sphere and use the Sphere Theorem [16] to replace $f(D) \cup g(D)$ by a nonsingular polyhedral 2-sphere $S(\tau)$ which lies arbitrarily close to $f(D) \cup g(D)$ and has $K$ in its interior. We may assume that $S(\tau)$ is in general position with respect to Bd $H$. We leave it to the reader to establish the following claim. (Again we suggest a linking argument. Cf. also the argument involving continua $K_i$ later in this section: if a curve in Bd $D_0(\tau)$ bounds a disk in Bd $X(\tau)$, then it does so in (Bd $H$) $\cap X(\tau)$.)

*Claim.* If $f$ and $g$ are sufficiently near the identity and $S(\tau)$ is

sufficiently near $f(D) \cup g(D)$, then there is a disk-with-holes $D_0(\tau)$ in $S(\tau)$ which has the following properties:

(6)   $D_0(\tau) \cap H = \text{Bd } D_0(\tau) = D_0(\tau) \cap X(\tau)$;

(7)   $D_0(\tau)$ has precisely one boundary component which does not bound a disk in $(\text{Bd } H) \cap X(\tau)$ and that component is homotopic in $X(\tau)$ to $\text{Bd } \tau$.

Let $\tau_1, \cdots, \tau_p$ be the 2-simplexes of $T$ and $D_0(\tau_1), \cdots, D_0(\tau_p)$ disks-with-holes whose existence is assured by the construction just described and the claim. We may clearly assume the following conditions satisfied:

(8)   $D_0(\tau_i) \cap D_0(\tau_j) = \varnothing$ if $i \neq j$;

(9)   Diam $[D_0(\tau_i) \cup X(\tau_i)] < \varepsilon/2$.

We now proceed to adjust the $D_0(\tau_i)$'s so that they are disks. If the $D_0(\tau_i)$'s are not already disks, then there is an $i$ such that one of the boundary components of $D_0(\tau_i)$ bounds a disk $E$ in $(\text{Bd } H) \cap X(\tau_i)$ (cf.(7) above) and $(\text{Int } E) \cap \bigcup_j D_0(\tau_j) = \varnothing$. Add $E$ to $D_0(\tau_i)$ and push that part of the new $D_0(\tau_i)$ which is near $E$ slightly into $E^3 - H$. An iteration of this procedure changes each $D_0(\tau_i)$ into a disk. Note that we may maintain conditions (8) and (9).

We now adjust the disks $D_0(\tau_1), \cdots, D_0(\tau_p)$ so that, for each $i$, $\text{Bd } D_0(\tau_i)$ runs "straight around" $\text{Bd } X(\tau_i)$ in $\text{Bd } H$. The problem here is descriptive rather than inherent. We therefore urge the reader to examine carefully Figures 2 and 3 before attempting to read the description which follows. We fix for consideration a 2-simplex $\tau$ of
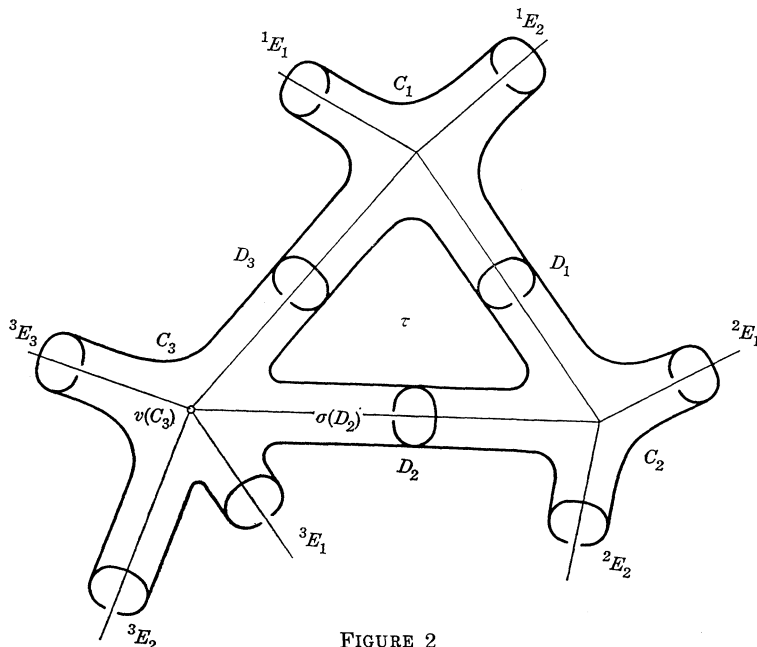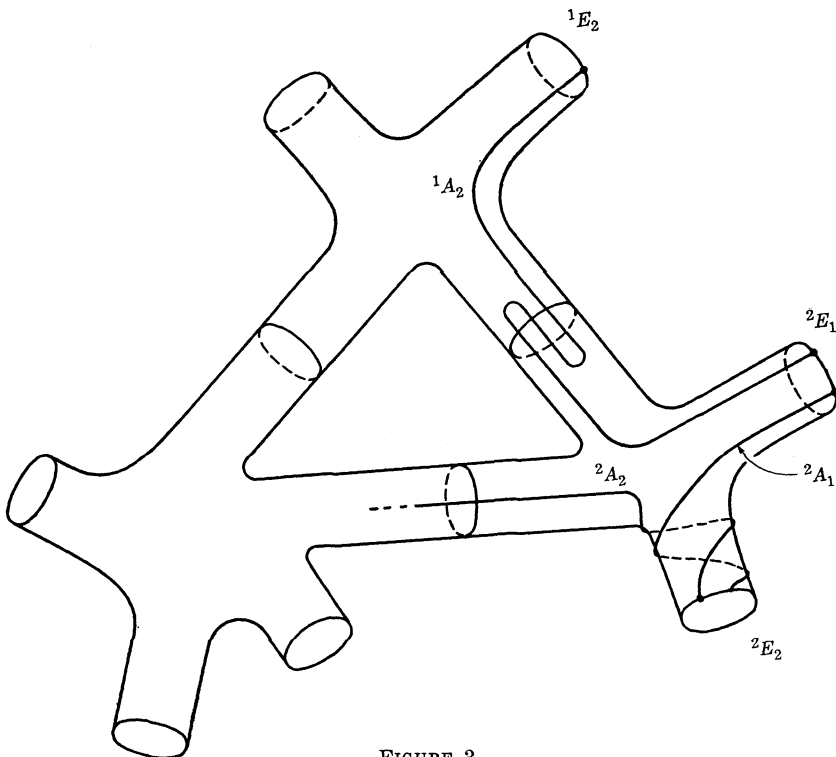


FIGURE 2

FIGURE 3

$T$. Recall that $X(\tau)$ is a solid torus divided by its distinguished meridional disks, which we denote by $D_1$, $D_2$, and $D_3$, into 3-cells $C_1$, $C_2$, and $C_3$, where $D_1 = C_1 \cap C_2$, $D_2 = C_2 \cap C_3$, and $D_3 = C_3 \cap C_1$. We denote the distinguished boundary disks of $X(\tau)$ by

$$^1E_1, \cdots, {}^1E_{r(1)}, {}^2E_1, \cdots, {}^2E_{r(2)}, {}^3E_1, \cdots, {}^3E_{r(3)} ,$$

where those with superscript $i$ lie in Bd $C_i$.

Each of the cells $C_i$ encloses a unique vertex $v(C_i)$ of $T$; each distinguished disk $\underline{D}$ of $H$ intersects a unique 1-simplex $\sigma(\underline{D})$ of $T$. We assume the indices on the distinguished disks chosen so that each successive pair from the sequence

(10)    $\sigma(^1E_1), \cdots, \sigma(^1E_{r(1)}), \sigma(^2E_1), \cdots, \sigma(^2E_{r(2)}), \sigma(^3E_1), \cdots, \sigma(^3E_{r(3)})$,

$\sigma(^1E_1)$ is a pair of edges from a single 2-simplex of $T$. It follows from (7) that there are disjoint arcs $^1A_1, \cdots, {}^1A_{r(1)}, {}^2A_1, \cdots, {}^2A_{r(2)}$, $^3A_1, \cdots, {}^3A_{r(3)}$ in Bd $X(\tau)$ joining successive pairs of distinguished disks from the list (10) (with $\sigma$'s removed) and lying, for each pair, in Bd $D_0(\tau')$, where $\tau'$ is the 2-simplex of $T$ which has the corresponding pair of edges in its boundary. Note that a single component $K_i$ of $C_i \cap (\bigcup^j A_k \bigcup^j E_k)$ contains the disks $\bigcup_k {}^iE_k$ and intersects both of the distinguished meridional disks of $X(\tau)$ which lie in Bd $C_i$. With our

descriptive apparatus finally set up, we can define our task precisely: reduce $D_0(\tau) \cap (D_1 \cup D_2 \cup D_3)$ to exactly three points. The procedure is straightforward and is contained in the next paragraph.

We may suppose $\operatorname{Bd} D_0(\tau)$ in general position with respect to the union $\operatorname{Bd} D_1 \cup \operatorname{Bd} D_2 \cup \operatorname{Bd} D_3$. If the intersection of $\operatorname{Bd} D_0(\tau)$ with this union consists of more than three points, then since $\operatorname{Bd} D_0(\tau)$ is homotopic in $X(\tau)$ to a centerline of $X(\tau)$, there is an arc $A$ in $\operatorname{Bd} D_0(\tau)$ which lies in a single $C_i$ and has its endpoints on a single $\operatorname{Bd} D_j$. The existence of the continuum $K_i$ (see the previous paragraph) forces $A$ to bound, together with an arc in $\operatorname{Bd} D_j$, a disk $E$ in $\operatorname{Bd} C_i$ whose interior misses each distinguished disk in $H$. Then each of the sets $D_0(\tau_k)$ which intersects $E$ can be adjusted by cut and paste near $E$ so as to have no intersection with $E$. In particular, this reduces the number of intersections of $D_0(\tau)$ with $\operatorname{Bd} D_j$ by at least two. Note that these changes may be made so as to preserve the essential properties of the disks $D_0(\tau_1), \cdots, D_0(\tau_p)$ as described in (6), (7), (8), (9), and the paragraph following (9). The reduction of $D_0(\tau) \cap (D_1 \cup D_2 \cup D_3)$ to three points therefore follows by induction (for each $\tau$).

*Step 4. Building a polyhedral $\varepsilon$-approximation to $S$.* On the boundary of each distinguished disk $D_j$ of $H$ there are precisely two points which lie in $\bigcup_k D_0(\tau_k)$. Let $A_j$ be a polygonal spanning arc of $D_j$ which has those two points as endpoints. Let $C_i$ be any of the polyhedral cells of which $H$ is composed. Then $[\bigcup_j A_j \cup \bigcup_k D_0(\tau_k)] \cap C_i$ is a single polygonal simple closed curve $J_i$ in $\operatorname{Bd} C_i$. Let $P_i$ be a polyhedral spanning disk of $C_i$ with boundary $J_i$. Then $P = (\bigcup_i P_i) \cup (\bigcup_k D_0(\tau_k))$ is a polyhedral 2-sphere that is homeomorphically within $\varepsilon$ of $S$. We leave the easy construction of the homeomorphism to the reader.

*Step 5. Identifying polyhedral $\varepsilon$-side approximations to $S$.* One sees immediately that the polyhedral 2-sphere $P$ of Step 4 locally separates $\operatorname{Bd} H$ into two components. Since $P$ is a 2-sphere in $E^3$ (as opposed to a one-sided surface in a 3-manifold), the separation is also global. Let $H_1$ and $H_2$ be the components of $\operatorname{Bd} H - P$ in $\operatorname{Int} P$ and $\operatorname{Ext} P$, respectively. Then

$$S_1 = H_1 \cup (\bigcup_k D_0(\tau_k)) \quad \text{and} \quad S_2 = H_2 \cup (\bigcup_k D_0(\tau_k))$$

are polyhedral 2-spheres which also homeomorphically within $\varepsilon$ of $S$. Let $D_k$ be a subdisk of $\tau_k$ in $\operatorname{Int} \tau_k$ which contains $(S_1 \cup S_2) \cap \tau_k$. Then the disks $D_1, \cdots, D_p$ are the $\varepsilon$-disks whose existence is asserted by the S.A.T. and $S_1$ and $S_2$ satisfy

$$S - \bigcup_k D_k \subset \operatorname{Ext} S_1 ,$$

and

$$S - \bigcup_k D_k \subset \operatorname{Int} S_2 .$$

2. **Improving the S.A.T.** The following lemma is an alternative to [7, Theorem 14, p. 181]. Let $H(X)$ denote the convex hull of any set $X$ in $E^3$.

2.1. Suppose that $p \in E^3$ and that $X_1, \cdots, X_n$ are compact subsets of $E^3$ satisfying

(1)  If $i, j$, and $k$ are distinct indices, then $X_i \cap X_j \cap X_k = \varnothing$.

(2)  If $X_i \cap X_j \neq \varnothing$, then $H(X_i \cup X_j) \subset E^3 - \{p\}$.

Then $X = \bigcup_i X_i$ does not separate $p$ from infinity.

*Proof.* By [13, Theorem VI 10, p. 97], it suffices to show that $X$ is nullhomotopic in $E^3 - \{p\}$. The sets $X_1, \cdots, X_n$ cover $X$; let $N$ be the nerve of that covering: one vertex $v_i$ for each nonempty $X_i$, one 1-simplex $v_i v_j$ for each nonempty intersection $X_i \cap X_j$. The nerve $N$ contains no higher dimensional simplexes by (1). The vertex $v_i$ can be realized geometrically as a point of $X_i$, the simplex $v_i v_j$ as the straight line segment in $H(X_i \cup X_j)$ joining $v_i$ and $v_j$. Map the sets $X_i \cap X_j$ to the midpoints of the 1-simplexes $v_i v_j$ and extend these partial maps on the sets $X_i \cap X_j$ to maps from the sets $X_k$ into the stars of the vertices $v_k$ in $N$ (the star is an absolute retract). This map can be realized as the final stage of a straight line homotopy in $E^3$ from $X$ into the (possibly singular) geometric realization of $N$ in $\bigcup \{ H(X_i \cup X_j) \mid X_i \cap X_j \neq \varnothing \}$. The homotopy moves $X$ only in $\bigcup \{ H(X_i \cup X_j) \mid X_i \cap X_j \neq \varnothing \} \subset E^3 - \{p\}$. Since the geometric realization of $N$ is nullhomotopic in $E^3 - \{p\}$, the proof is complete.

2.2. Suppose $S$ is a 2-sphere in $E^3$ and $\varepsilon > 0$. Then there is a $\delta > 0$ such that if $K$ is a compact set in $S$ and $K$ has no component of diameter $> \delta$, there exists a finite number of disjoint $\varepsilon$-disks $E_1, \cdots, E_n$ in $S$ such that $K \subset \bigcup_i \operatorname{Int} E_i$.

*Proof.* This can be deduced easily either from [9, Theorem 4.8.7] or its proof.

2.3. *Addendum to the* S.A.T. [7, § 7]. The following conclusion may be added to the *S.A.T.*: There are disjoint $\varepsilon$-disks $E_1, \cdots, E_m$ on $h(S)$ such that

(3)  $h(S) - \bigcup_j E_j \subset \operatorname{Int} S$.

*Proof.* Choose $p \in \text{Int } S$. We may assume that $\rho(p, S) > 2\varepsilon$. By (2.2), there is a $\delta$, $0 < \delta < \varepsilon$, such that if $K$ is a compact subset in $S$ having no component of diameter greater than $3\delta$, there exist disjoint $\varepsilon/3$-disks $E_1, \cdots, E_m$ in $S$ such that $K \subset \bigcup_i \text{Int } E_i$.

By the S.A.T., there are disjoint $\delta$-disks $D_1, \cdots, D_n$ in $S$ and a $\delta$-homeomorphism from $S$ into $E^3$ such that

( 1 )  $h(S)$ is polyhedral, and

( 2 )  $S - \bigcup_i D_i \subset \text{Ext } h(S)$.

Let $K_0 = h(S) \cap \bigcup_i D_i$ and $K = h^{-1}(K_0)$. Then $K$ has no component of diameter greater than $3\delta$ and is covered by the interiors of disjoint $\varepsilon/3$-disks $E_1', \cdots, E_m'$ in $S$. Let $E_j = h(E_j')$. Then $E_1, \cdots, E_m$ are disjoint $\varepsilon$-disks on $h(S)$ such that $S \cap h(S) \subset \bigcup_j \text{Int } E_j$. It remains only to show that $h(S) - \bigcup_j E_j \subset \text{Int } S$. Since $\rho(p, S \cup h(S)) > \varepsilon$ and $S$ is homeomorphically within $\varepsilon$ of $S$, $p \in \text{Int } h(S)$ [13, Theorem VI 10, p. 97]. Since $\rho(p, S \cup h(S)) > \varepsilon$ and each $D_i$ and $E_j$ has diameter less than $\varepsilon$, it follows from (2.1) that $(\bigcup_i D_i) \cup (\bigcup_j E_j)$ does not separate $p$ from infinity. Let $R$ be a ray from $p$ to infinity in $E^3 - [(\bigcup_i D_i) \cup (\bigcup_j E_j)]$. Let $q$ be the first point of $R \cap (S \cup h(S))$. If one had $q \in S$, then one would have a contradiction to (2) $S - \bigcup_i D_i \subset \text{Ext } h(S)$. Hence $q \in h(S)$ and (3) $h(S) - \bigcup_j E_j \subset \text{Int } S$. This completes the proof.

*S.A.T. for open subsets of spheres* [7, § 8], [8]. If $U$ is an open subset of a 2-sphere in $E^3$ and $f$ is a positive continuous real function on $U$, then there exists a homeomorphism $h$ of $S$ into $E^3$, a locally finite collection of disjoint disks $\{D_i\}$ in $U$ and a locally finite collection of disjoint disks $\{E_j\}$ in $h(U)$ such that:

( 1 )  $h$ is the identity on $S - U$.

( 2 )  $\rho(x, h(x)) < f(x)$ for each $x \in U$.

( 3 )  $h(U)$ is locally polyhedral.

( 4 )  $U - \bigcup \text{Int } D_i \subset \text{Ext } h(S)$.

( 5 )  $h(U) - \bigcup \text{Int } E_i \subset \text{Int } S$.

( 6 )  $\text{Diam } D_i < \text{min value of } f \text{ on } D_i$.

( 7 )  $\text{Diam } E_i < \text{min value of } f \text{ on } h^{-1}(E_i)$.

*Proof.* The proof proceeds almost without change from that of the S.A.T. One chooses an infinite triangulation of $U$ rather than a finite triangulation of $S$. Otherwise the argument is entirely local and the desired result follows.

3.  **Proof of the H.A.T.** (statement appears in §1.)  Throughout this section we assume that $S$ is a 2-sphere in $E^3$ which satisfies the hypothesis of the H.A.T.; i.e., for each $\varepsilon > 0$, there is an $\varepsilon$-homeomorphism from $S$ into $\text{Int } S$. By the S.A.T., we may assume that the

image of $S$ is a *polyhedral* 2-sphere in $\operatorname{Int} S$ which we call a *polyhedral $\varepsilon$-approximation to $S$ in* $\operatorname{Int} S$.

DEFINITION. Suppose $K$ is a (curvilinear) simplicial complex in $S$. We say that $K$ can be polyhedrally collared from $\operatorname{Int} S$ if there is an embedding $h\colon K \times [0, 1] \to S \cup \operatorname{Int} S$ such that

(1)   $h(k, 0) = k$   if   $k \in K$.
(2)   $h(K \times (0, 1]) \subset \operatorname{Int} S$.
(3)   $h \mid K \times (0, 1]$ is locally piecewise linear. Then $h$ is called a *polyhedral collar* on $K$ from $\operatorname{Int} S$. (We identify $K \times [0, 1]$ with some abstract simplicial complex for the purpose of defining piecewise linear.)

3.1.   Each simple closed curve $J$ in $S$ can be polyhedrally collared from $\operatorname{Int} S$.

*Proof.* In outline, the S.A.T. for open subsets of spheres (§ 2) implies that we can raise a blister near one of the disks on $S$ which is bounded by $J$; we can then use the polyhedral approximations to $S$ in $\operatorname{Int} S$ supplied by the hypothesis of the H.A.T. to cut off the feelers of the blister which reach back through $S$ (see Figure 4).

In more detail, let $D$ be a disk in $S$ bounded by $J$. Let $f(x) = \rho(x, S - D)$ for each $x \in \operatorname{Int} D$. By the S.A.T. for open subsets of spheres (§ 2), there exists a homeomorphism of $S$ into $E^3$, a locally finite collection of disjoint disks $\{D_i\}$ in $\operatorname{Int} D$, and a locally finite collection of disjoint disks $\{E_j\}$ in $h(\operatorname{Int} D)$ such that:

(1)   $h \mid S - \operatorname{Int} D = $ identity.
(2)   $\rho(x, h(x)) < f(x)$ for each $x \in \operatorname{Int} D$.
(3)   $h(\operatorname{Int} D)$ is locally polyhedral.
(4)   $\operatorname{Int} D - \bigcup \operatorname{Int} D_i \subset \operatorname{Ext} h(S)$.
(5)   $h(\operatorname{Int} D) - \bigcup \operatorname{Int} E_i \subset \operatorname{Int} S$.
(6)   $\operatorname{Diam} D_i < $ min value of $f$ on $D_i$.
(7)   $\operatorname{Diam} E_i < $ min value of $f$ on $h^{-1}(E_i)$.
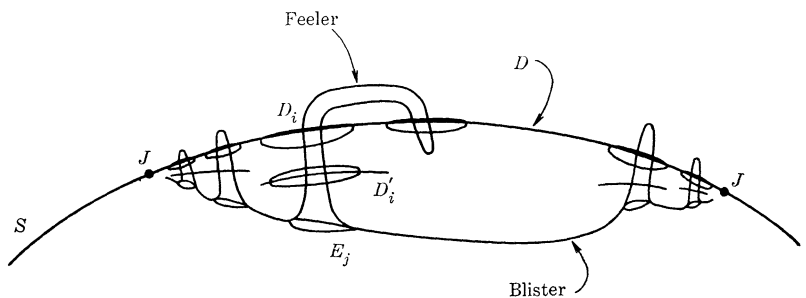


FIGURE 4

Let $V$ denote the component of $h(\text{Int } D) - S$ which has limit points in $J$. Note that $h(\text{Int } D) - \bigcup \text{Int } E_j \subset V$. By the hypothesis of the H.A.T., $S$, and hence also each disk $D_i$, can be approximated very closely by a polyhedral set in Int $S$. It follows easily that there is a null sequence $D_1', D_2', \cdots$ of disjoint polyhedral disks in Int $S$, each $D_i'$ approximating the corresponding disk $D_i$ and each in general position with respect to $h(S)$ such that

(8) $D_i' \cap h(S) \subset \bigcup \text{Int } E_j$.

(9) No component of $V - \bigcup D_i'$ has limit points both in $J$ and in Int $D$.

Let $W$ be the component of $V - \bigcup D_i'$ which has limit points in $J$. Note that $h(\text{Int } D) - \bigcup \text{Int } E_j \subset W$. Consider the set Cl $(W) - W$. The components of this set form a null sequence $J, J_1, J_2, \cdots$ of simple closed curves, each $J_i$ lying in some Int $D_j'$ and only finitely many of the $J_i$ in any one $D_j'$. For those curves $J_i$ which lie in Int $D_1'$, we fill in the holes in Cl $(W)$ bounded by the $J_i$ with disks $D(J_i)$ in $D_1'$ and push these disks slightly to one side of $D_1'$, pushing those furthest which are innermost on $D_1'$. If this is continued sequentially (say, in the order $D_1', D_2', \cdots$) and if care is exercised in the adjustments which push the disks $D(J_i)$ away from the disks $D_j'$, one will obtain thereby a disk $E = \text{Cl } (W) \cup D(J_i)$ (adjusted) which has $J$ as boundary and has interior which is a locally polyhedral subset of Int $S$. A polyhedral subset of Int $S$. A polyhedral collar for $J$ from Int $S$ can easily be identified in $E$.

3.2. If $K$ is a $\theta$-curve in $S$ (i.e., the union of three arcs with common endpoints and disjoint interiors) and $J$ is a simple closed curve in $K$, then any polyhedral collar on $J$ from Int $S$ can be extended to a polyhedral collar on $K$ from Int $S$.

*Proof.* Let $D$ be the disk in $S$ which is bounded by $J$ and contains the arc $\alpha = \text{Cl } (K - J)$. Let $\beta$ be an arc which has the same endpoints $p$ and $q$ that $\alpha$ has and which lies except for $p$ and $q$ in $S - D$. Then $J' = \alpha \cup \beta$ is a simple closed curve in $S$. Let $D'$ be a disk bounded by $J'$ in $S$.

We are given by hypothesis an annulus $A$ which polyhedrally collars $J$ from Int $S$. The annulus $A$ comes equipped with an assigned product structure $A = J \times [0, 1]$ $(J = J \times \{0\})$ which we shall consider in more detail at the end of the proof. The proof of (3.1) showed how to construct a disk $E$ very near the disk $D'$ such that $J' = \text{Bd } E$, Int $E \subset \text{Int } S$, and Int $E$ is locally polyhedral. It is clear from the proof of (3.1) (choose $f$ appropriately) that we may further require that $E \cap (J \times \{1\}) = \varnothing$. We require that Int $E$ and $A$ be in general position.

The inexperienced reader should compare the following discussion with [12]: Since $\operatorname{Int} E$ and $A$ are in general position, there are only countably many components $K_1$, $K_2$, $\cdots$ of $A \cap \operatorname{Int} E$ and they form a null sequence, locally finite except possibly at $p$ and $q$ (the endpoints of $\alpha$), and satisfying for each individual $i$ either

(1) $\operatorname{Cl}(K_i) = K_i$ is a simple closed curve in $\operatorname{Int} E$,

(2) $\operatorname{Cl}(K_i) = K_i \cup \{p\}$ or $K_i \cup \{q\}$ is a simple closed curve containing $p$ or $q$, or

(3) $\operatorname{Cl}(K_i) = K_i \cup \{p, q\}$ is an arc from $p$ to $q$.

If $E$ is chosen near enough to $D'$, then is follows that each component $K_i$ of type (1) or (2) bounds a (unique) disk $A(K_i)$ in $A$. Such a $K_i$ also bounds a (unique) disk $E(K_i)$ in $E$. Let

$E_0 = E - \bigcup \{\operatorname{Int} E(K_i) \mid K_i$ is of type (1) or (2)$\}$, and

$E' = E_0 \cup [\bigcup \{\operatorname{Int} A(K_i) \mid K_i$ is of type (1) or (2) and $K_i \subset E_0\}]$. If $K_i \cup K_j \subset E_0 (i \neq j)$, $K_i$ and $K_j$ are of type (1) or (2), and $A(K_i) \subset A(K_j)$, then one is to understand that $E'$ contains two copies of $A(K_i) - \{p, q\}$, one corresponding to $K_i$, the other lying in $A(K_j)$ and corresponding to $K_j$. With this understanding, $E'$ is a singular disk which may be made nonsingular by pushing the sets $A(K_i) - \{p, q\}$ slightly to one side of $A$ in $\operatorname{Int} S$. If $A(K_i) \subset A(K_j)$, then the copy of $A(K_i)$ corresponding to $K_i$ must be pushed further away from $A$ than is the copy of $A(K_i)$ in $A(K_j)$.

If each of the sets $A(K_i) - \{p, q\}$ is pushed to the appropriate side of $A$ and other appropriate care is taken, the nonsingular $E'$ obtained will satisfy all of the conditions that the original $E$ satisfied except that there will be only finitely many components of $A \cap \operatorname{Int} E'$ and each will be of type (3).

Since $J$ separates $\operatorname{Int} \alpha = \alpha - \{p, q\}$ from $\operatorname{Int} \beta$ in $S$, it follows that $A \cap \operatorname{Int} E' \neq \varnothing$, hence that there is a disk $E''$ in $E'$ such that $E'' \cap A$ is a single arc in $A \cap \operatorname{Bd} E''$ from $p$ to $q$ and such that the complementary arc in $\operatorname{Bd} E''$ is the original arc $\alpha$. Standard cut and paste techniques allow one to adjust $E''$ so that the arc $A \cap \operatorname{Bd} E''$ runs up the fiber $\{p\} \times [0, 1]$ in $A = J \times [0, 1]$, arond the upper rim $J \times \{1\}$ of $A$, and back down the fiber $\{q\} \times [0, 1]$. The desired polyhedral collar for $K$ from $\operatorname{Int} S$ can be identified in $A \cup E''$ (adjusted). This completes the proof of (3.2).

**3.3. COROLLARY TO (3.2).** *If $T$ is a curvilinear triangulation of $S$, then the 1-skeleton of $T$ can be polyhedrally collared from $\operatorname{Int} S$. If $T'$ is a curvilinear derived subdivision of $T$, then any polyhedral collar from $\operatorname{Int} S$ on the 1-skeleton of $T$ can be extended to a polyhedral collar from $\operatorname{Int} S$ on the 1-skeleton of $T'$.*

*Proof.* The 1-skeleton of a triangulation can be built up from an

initial simple closed curve by adding arcs spanning the 2-cells of in-
creasingly fine cellular subdivisions of $S$. Thus the corollary follows
directly from (3.1) and (3.2).

3.4.    Suppose $T$ is a curvilinear triangulation of $S$, $h$: $|T^1| \times [0, 1] \rightarrow$
$S \cup \text{Int } S$ is a polyhedral collar on the 1-skeleton $T^1$ of $T$ from $\text{Int } S$,
and $\alpha$ and $\beta$ are positive numbers less than 1. Then there is a poly-
hedral embedding $f: S \rightarrow \text{Int } S$ such that:
   ( 1 )   $f(S)$ separates $h(|T^1| \times \{1\})$ from $S$ in $E^3$,
   ( 2 )   $f(t) = h(t, \alpha)$ for each $t \in |T^1|$, and
   ( 3 )   $f(\text{Int } \sigma) \subset N(\sigma \cup h(\text{Bd } \sigma \times [0, \alpha]); \beta) - h(|T^1| \times [0, 1])$ for each
2-simplex $\sigma \in T^2$.

*Proof.* Suppose that, for each 2-simplex $\sigma \in T^2$, we have shown
how to construct a polyhedral disk $\sigma'$ such that $\text{Bd } \sigma = h(\text{Bd } \sigma \times \{\alpha\})$
and $\text{Int } \sigma' \subset N(\sigma \cup h(\text{Bd } \sigma \times [0, \alpha]); \beta) - h(|T^1| \times [0, 1])$. Then, for $\beta$
sufficiently small (which we may certainly assume), $S' = \bigcup \{\sigma' \mid \sigma \in T^2\}$
will automatically be a polyhedral 2-sphere in $E^3$ which separates
$h(|T^1| \times \{1\})$ from $S$ in $E^3$; and a homeomorphism $f: S \rightarrow S'$ satisfying
the requirements of (3.4) can be constructed by defining $f \mid |T^1|$ by
requirement (2) and then, for each $\sigma \in T^2$, extending $f \mid \text{Bd } \sigma$ across
$\text{Int } \sigma$ to take $\sigma$ to $\sigma'$. Thus it suffices to construct $\sigma'$.

Suppose that $\sigma$ is a 2-simplex in the 2-skeleton $T^2$ of $T$. Let $S_0$
be a polyhedral $\beta$-approximation to $S$ from $\text{Int } S$ with $h(\text{Bd } \sigma \times [0, 1])$
and $S_0$ in general position. For $\beta$ sufficiently small it will be true
that
   ( 4 )   $S_0 \cap h(\text{Bd } \sigma \times [0, 1]) \subset h(\text{Bd } \sigma \times (0, \alpha))$,
   ( 5 )   $S_0$ separates $h(\text{Bd } \sigma \times \{1\})$ from $S$, and
   ( 6 )   some component $C$ of $S - h(|T^1| \times [0, 1])$ will lie in $N(\sigma; \beta)$,
have its boundary components in $N(\text{Bd } \sigma \times [0, 1])$, and have precisely
one boundary component which is not contractible in the annulus
$h(\text{Bd } \sigma \times [0, 1])$.

Fill in those boundary components of $C$ which are contractible in
$h(\text{Bd } \sigma \times [0, 1])$ with the disks they bound in $h(\text{Bd } \sigma \times [0, \alpha])$ and push
them to the "$\sigma$"-side of $h(\text{Bd } \sigma \times [0, \alpha])$. This yields a disk $\sigma'$ with
$\text{Bd } \sigma' = K$ and $\text{Int } \sigma' \subset \text{Int } S - h(|T^1| \times [0, 1])$. Since the simple closed
curves $K$ and $h(\text{Bd } \sigma \times \{\alpha\})$ are parallel in $h(\text{Bd } \sigma \times (0, \alpha])$, cut and
paste can be used to slide $K$ upward in the annulus $h(\text{Bd } \sigma \times [0, \alpha])$
so as to coincide with $h(\text{Bd } \sigma \times \{\alpha\})$ as required. This completes the
construction of $\sigma'$ and the proof of (3.4).

*Proof of the H.A.T.* We now construct a homeomorphism from
$S \cup \text{Int } S$ to the unit ball $B^3$ in $E^3$ centered at the origin.

Let $T_1$ denote a curvilinear triangulation of $S$ and $T_2, T_3, \cdots$

successive curvilinear derived subdivisions of $T_1$ so chosen that mesh $T_i \to 0$ as $i \to \infty$.

By (3.3), there exist for $i = 1, 2, \cdots$ polyhedral collars $h_i \colon |(T_i)^1| \times [0, 1] \to S \cup \operatorname{Int} S$ on the 1-skeletons $(T_i)^1$ of these triangulations, $h_{i+1}$ extending $h_i$. We may clearly require that, for each $t \in |(T_i)^1|$, Diam $h_i(\{t_i\} \times [0, 1/i]) < 1/i$.

By (3.4), there exist for $i = 1, 2, \cdots$ polyhedral embeddings $f_i \colon S \to \operatorname{Int} S$ such that:

(1)  $f_i(S)$ separates $h_i(|(T_i)^1| \times \{1\})$ from $S$ in $E^3$,

(2)  $f_i(t) = h_i(t, 1/2i)$ for each $t \in |(T_i)^1|$, and

(3)  $f_i(\operatorname{Int} \sigma) \subset N(\sigma, 1/2i) - h_i(|(T_i)^1| \times [0, 1])$ for each 2-simplex $\sigma \in (T_i)^2$.

Let $n_1 = 1, n_2, \cdots$ be a subsequence of $1, 2, \cdots$ so chosen that $f_{n_i}(S) \cap f_{n_j}(S) = \varnothing$ for $i \neq j$. Let $H_{n_i}$ be the restriction of the polyhedral collar $h_{n_i}$ to the set $|(T_{n_i})^1| \times [0, 1/2n_i]$. Then

$$X = S \cup \{ \textstyle\bigcup_i f_{n_i}(S) \} \cup \{ \textstyle\bigcup_i \operatorname{Image} H_{n_i} \}$$

is the template which we shall use to describe a homeomorphism

$$h \colon S \cup \operatorname{Int} S \longrightarrow B^3 \,.$$

*Description of* $h \mid S$.  Let $h \mid S \colon S \to (S^2 = \operatorname{Bd} B^3)$ be any homeomorphism.

*Description of* $h \mid \operatorname{Image} H_{n_i}$.  If $t \in |(T_{n_i})^1|$ and $\alpha \in [0, 1/2n_i]$, let $h(H_{n_i}(t, \alpha))$ be on the radius joining $h(t)$ and the origin in $B^3$ and at a distance $\alpha$ from $h(t)$.

*Description of* $h \mid f_{n_i}(S)$.  If $s \in S$, let $h(f_{n_i}(s))$ be on the radius joining $h(s)$ and the origin in $B^3$ and at a distance $1/2n_i$ from $h(s)$.

*Description of* $h \mid (S \cup \operatorname{Int} S) - X$.  Let $U$ be a component of $(S \cup \operatorname{Int} S) - X$. Then $\operatorname{Bd} U$ is a polyhedral 2-sphere in $X$ of which $U$ is the interior. By Alexander's Theorem (see [2], [11] or [14]), $U \cup \operatorname{Bd} U$ is a 3-cell. The map $h \mid \operatorname{Bd} U$ is already defined and takes $\operatorname{Bd} U$ homeomorphically onto a curvilinear 2-sphere $h(\operatorname{Bd} U)$ in $\operatorname{Int} B^3$. Obviously, $h(\operatorname{Bd} U) \cup \operatorname{Int} h(\operatorname{Bd} U)$ is also a 3-cell. Since both $U \cup \operatorname{Bd} U$ and $h(\operatorname{Bd} U) \cup \operatorname{Int} h(\operatorname{Bd} U)$ are 3-cells, there is an extension of $h \mid \operatorname{Bd} U$ which takes $U \cup \operatorname{Bd} U$ homeomorphically onto $h(\operatorname{Bd} U) \cup \operatorname{Int} h(\operatorname{Bd} U)$. We take $h \mid U \cup \operatorname{Bd} U$ to be this homeomorphism.

One checks easily that $h$ defined piece by piece above defines a homeomorphism from $S \cup \operatorname{Int} S$ onto $B^3$. This completes the proof of the H.A.T.

APPENDIX. We give an example of the type of argument one might use in establishing such things as the claims in Steps 1 and 3 of the proof of the S.A.T., Property (9) in the proof of (3.1), and the statements involving "for $\beta$ sufficiently small" in the proof of (3.4). Such details are often omitted as essentially obvious in the mathematical literature (or hidden under the rug, depending on one's viewpoint).

EXAMPLE (cf. the second paragraph of the proof of (3.4)). Let $S$ be a 2-sphere in $E^3$, $D$ a disk in $S$; and suppose $J = \text{Bd}\, D$ has a polyhedral collar $h\colon J \times [0,1] \to S \cup \text{Int}\, S$ from $\text{Int}\, S$. For each $\varepsilon > 0$, there is a $\delta > 0$ such that if $S$ has a polyhedral $\delta$-approximation $f\colon S \to \text{Int}\, S$ from $\text{Int}\, S$ with $f(S)$ and $h(J \times [0,1])$ in general position, then:

(1)  $f(S)$ separates $h(J \times [0,1])$ from $S$ in $E^3$, and

(2)  there is a component $C$ of $f(S) - h(J \times [0,1])$ such that $C \subset N(D, \varepsilon)$ and $\text{Cl}\,(C) - C$ has precisely one component in $h(J \times [0,1])$ which does not bound a disk in $h(J \times [0,1])$.

(Note that if $f(S) \cap h(J \times \{1\}) \neq \varnothing$ and if $C'$ is any component of $f(S) - h(J \times [0,1])$, then $\text{Cl}\,(C')$ is a disk-with-holes with interior equal to $C'$ and boundary curves in $h(J \times [0,1])$.)

*Proof.* We first require an auxiliary construction: let $\alpha$ and $\beta$ be arcs with common endpoints $p \in \text{Int}\, D$ and $g \in S - D$ such that $\alpha - \{p, q\} \subset \text{Int}\, S - h(J \times [0,1])$ and $\beta - \{p, q\} \subset \text{Ext}\, S$. Let $D_p \subset \text{Int}\, D$ and $D_q \subset S - D$ be disks such that $p \in \text{Int}\, D_p$ and $q \in \text{Int}\, D_q$. We choose $\delta > 0$ and smaller than the following numbers

$$\delta_1 = \rho[S, h(J \times \{1\})]$$
$$\delta_2 = \rho[D_p \cup D_q, h(J \times [0,1])]$$
$$\delta_3 = \min\{\rho(p, S - D), \rho(q, D)\}$$
$$\delta_4 = \min\{\varepsilon/2, \rho[S - N(D, \varepsilon/2), h(J \times [0,1])]\}$$
$$\delta_5 = \rho(S - (\text{Int}\, D_p \cup \text{Int}\, D_q), \alpha \cup \beta]\,.$$

(a)  $f(S)$ *separates* $h(J \times \{1\})$ *from* $S$ *in* $E^3$: Indeed, $\delta < \delta_1$ implies that $S$ and $f(S)$ are homotopic in $E^3 - h(J \times \{1\})$, hence that $h(J \times \{1\}) \subset \text{Int}\, f(S)$ by [13, Theorem VI 10, p. 94]. Since $S \subset \text{Ext}\, f(S)$, the result follows.

(b)  $f(D_p) \cap h(J \times [0,1]) = \varnothing = f(D_q) \cap h(J \times [0,1])$ since $\delta < \delta_2$.

(c)  *The component* $C$ *of* $f(S) - h(J \times [0,1])$ *which contains* $f(D_p)$ (such exists by (b)) *does not contain* $f(D_q)$: Indeed, suppose $\{f(p), f(q)\} \subset C$. Let $\gamma$ be an arc in $C$ from $f(p)$ to $f(q)$. Let $\gamma_p$ and $\gamma_q$ be the straight line segments in $E^3$ joining $p$ to $f(p)$ and $f(q)$ to $q$, respectively. Let $\gamma'$ be an arc from $q$ to $p$ in $\text{Ext}(S) \cup \{p, q\}$. Because $\delta < \delta_3, \gamma_p \cap S \subset \text{Int}\, D$ and $\gamma_q \cap S \subset S - D$. It follows that the loop $\Gamma$ which

traverses $\gamma$, $\gamma_q$, $\gamma'$, and $\gamma_p$ (in that order) has linking number $\pm 1$ with $J$. Since $\delta < \delta_2$, $\Gamma \cap h(J \times [0, 1]) = \varnothing$. Hence $L(\Gamma, h(J \times \{1\})) = \pm 1$ by (0.5) or (0.6), and $\Gamma$ is noncontractible in $E^3 - h(J \times \{1\})$ by (0.5). But $\Gamma$ is homotopic in $E^3 - h(J \times \{1\})$ to the loop $\Gamma''$ which traverses first $f^{-1}(\gamma)$ and then $\gamma'$ since $\delta < \delta_1$, and $\Gamma''$ is contractible in the simply connected set $S \cup \text{Ext}\, S$, a contradiction.

(d) $C \subset N(D, \varepsilon)$. The proof is like that of (c) with the following adjustments. One uses $q'$ such that $f(q') \in C - N(D, \varepsilon)$ instead of the original $q$ in the construction of $\Gamma$. One uses the fact that $\delta < \delta_4$ to prove that $\gamma_q' \cap S \subset S - D$ and that $\Gamma \cap h(J \times [0, 1]) = \varnothing$.

(e) $\text{Cl}\,(C) - C$ *has a component which bounds no disk in* $h(J \times [0, 1])$. Indeed, $L(\text{Bd}\, D_p, \alpha \cup \beta) = \pm 1$ by construction (cf. (0.7)). Also, $\text{Bd}\, D_p$ and $f(\text{Bd}\, D_p)$ are homotopic in $E^3 - (\alpha \cup \beta)$ since $\delta < \delta_5$. Thus $L(f(\text{Bd}\, D_p), \alpha \cup \beta) = \pm 1$. If each component of $\text{Cl}\,(C) - C$ bounded a disk in $h(J \times [0, 1])$, then $f(\text{Bd}\, D_p)$ would bound a singular disk in $[C - \text{Int}\, D_p] \cup h(J \times [0, 1]) \subset E^3 - (\alpha \cup \beta)$, a contradiction to (0.5).

(f) $\text{Cl}\,(C) - C$ *has at most one component which bounds* no disk in $h(J \times [0, 1])$. For otherwise, at least one such component would bound a disk in $f(S) - (D_p \cup D_q) \subset E^3 - (\alpha \cup \beta)$ even though it would link $\alpha \cup \beta$, a contradiction.

*Added in proof.* O. G. Harrold has pointed out an unwarranted assumption at the beginning of [12] (cf. paragraph 3 of the proof of our Lemma 3.2) which invalidates the main result of [12] but does not invalidate the very nice treatment of infinite cut and paste techniques in [12]. Indeed, the false assumption of [12] is valid in many situations, including ours (i.e., the paragraph cited above is correct). L. Keldyš has also claimed the main result of [12], her proof is also incorrect [Math. USSR Sbornik, 10 (1970), 267–287]. The status of the main result of [12] is apparently undecided.

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# PACIFIC JOURNAL OF MATHEMATICS

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## Vol. 46, No. 2     December, 1973