ON REALIZING HNN GROUPS IN 3-MANIFOLDS

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In this paper we suppose that the fundamental group of a 3-manifold $M$ has a presentation as an HNN group. We then show that under suitable conditions we can realize this presentation by embedding a closed, connected incompressible surface in $M$.

In [2], [3], and [4] we show that if $\pi_1(M)$ is constructed in certain ways, one can realize this construction by a surface embedded in $M$. In this paper we show that one can realize the HNN construction when certain relationships between $\pi_1(M)$ and $M$ are present. The results in this paper are related to Theorem 2.4 in [10].

In this paper all spaces will be simplicial complexes, all maps will be piecewise linear, and all 3-manifolds will be 3-manifolds with boundary. However the boundary may be vacuous. Let $X$ be a connected subspace of a space $Y$. As usual we shall denote the boundary, closure, and interior of $X$ in $Y$ by $\text{bd}(X)$, $\text{cl}(X)$, and $\text{int}(X)$ respectively. The natural inclusion map from $X$ into $Y$ will be denoted by $\rho$ and the induced homomorphism from $\pi_1(X)$ into $\pi_1(Y)$ by $\rho_*$. Let $S$ be a closed connected surface other than the 2-sphere of projective plane embedded in a space $Y$. Then $S$ is incompressible in $Y$ if $\rho_*: \pi_1(S) \to \pi_1(Y)$ is one-to-one. If $S$ is a closed surface embedded in $Y$, then $S$ is incompressible in $Y$ if each component of $S$ is incompressible in $Y$. Irreducible and $P^2$-irreducible are defined as in [7]. We denote the unit interval $[0, 1]$ by $I$ throughout.

DEFINITION 1. Let $K$ be a group and $A$ a subgroup of $K$. Let $S$ be a closed connected surface other than the projective plane or 2-sphere. Let $A_j \equiv \pi_1(S)$ and $A_j \subset A$ for $j = 1, 2$. Let $k$ be an element of $K$ not in $A$ such that $A_i = k^{-1}A_2k$. Then if $A$ and $k$ generate $K$ and all relations of $K$ are consequences of the relations of $A$ together with the relations $k$ induces between the elements of $A$, and $A_2$, we shall say that $K$ is an extension of $A$ by $k$ across $A_1$ and $A_2$. The reader will note that the class of groups defined above is a subclass of the Higmann, Neumann, Neumann (H.N.N.) groups [8].

Let $M$ be a 3-manifold, $x$ a point in $M$, and $S$ an incompressible surface in $M$ such that $M - S$ is connected. Then it is a consequence of Van Kampen’s Theorem that $\pi_1(M, x)$ is an extension of $\pi_1(M - S, x)$ by some element of $\pi_1(M, x)$ across appropriate subgroups of $\pi_1(M, x)$. One might then wonder “If $\pi_1(M, x)$ were such an extension, could we embed in $M$ an incompressible surface which realizes this exten-
We will show blow that this can, in fact, be done. Let $M$ be a compact 3-manifold and $x$ a point of $M$. We suppose that $\pi_1(M, x)$ is an extension of $A$ by $k$ across $A_1$ and $A_2$ as given in Definition 1 above. We can represent this extension by an ordered sequence \( \langle \pi_1(M, x), A, A_1, A_2, k \rangle \). If for each component $F$ of the boundary of $M$ some conjugate $\rho_\ast \pi_1(F)$ is contained in $A$, we shall say that the extension preserves the peripheral structure of $M$. Suppose a second representation of $\pi_1(M, x)$ is given by \( \langle \pi_1(M, x), B, B_1, B_2, \hat{k} \rangle \) and this extension of $B$ is induced by an incompressible, closed, two-sided surface $S$ embedded in $M$ and a loop $l$ meeting $S$ in the single point $x$, i.e., $B$ is generated by the elements of $\pi_1(M, x)$ having representative loops which do not cross $S$, $\hat{k} = [l]$, $B_1 = \rho_\ast \pi_1(S, x)$ and $B_2 = [l]B_1[l]^{-1}$. We shall say that $S$ realizes the extension of $B$ if there is an isomorphism 

\[
\Phi: \pi_1(M, x) \longrightarrow \pi_1(M, x)
\]

such that

1. $\Phi(A) = B$
2. $\Phi(A_j) = B_j$, $j = 1, 2$
3. $\Phi(k) = \hat{k}$.

**Theorem 1.** Let $M$ be a compact 3-manifold such that $\pi_2(M) = 0$. Let $S$ be a closed connected surface other than the 2-sphere or projective plane. Suppose $\pi_1(M, x)$ has a representation given by 

\[ \langle \pi_1(M, x), A, A_1, A_2, k \rangle \]

where $A_i \equiv \pi_1(S)$ and the extension above preserves the peripheral structure of $M$. Then there is an embedding of $S$ in $M$ which realizes the given extension.

The proof of Theorem 1 above is similar in many respects to the proof of Theorem 1 in [3]. One first constructs a complex $X$ having the same fundamental group as $M$. One then finds a map $f: M \longrightarrow X$ inducing an isomorphism from $\pi_1(M)$ to $\pi_1(X)$. The complex $X$ is constructed to contain an embedded surface $S$ realizing the given extension. One shows that there is a map $g$ homotopic to $f$ such that $g^{-1}(S)$ is an incompressible, connected, closed surface in $M$ and that $g^{-1}(S)$ realizes the given extension.

The following three lemmas appear in [4]. We omit the proofs which are not difficult.

**Lemma 1.** Let $M$ be a compact, connected 3-manifold such that $\pi_2(M) = 0$. Let $X$ be a connected complex and $S$ a closed incompressi-
ble surface embedded in $X$ and having a neighborhood homeomorphic to $S \times I$. We suppose that no component of $S$ is a 2-sphere or projective plane. Let $X_k, k = 1, \ldots, n$ be the components of $X - S$. We suppose that $\pi_i(X) = \pi_i(X_k) = 0$ for $i \geq 2$ and $k = 1, \ldots, n$. Let $f: M \to X$ be a map such that $f^*: \pi_i(M) \to \pi_i(X)$ is one-to-one. If $\text{bd}(M)$ does not meet $S$. Then there is a homotopy, constant on $\text{bd}(M)$, of $f$ to a map $g$ such that $g^{-1}(S)$ is an incompressible surface in $M$.

**Lemma 2.** Let $S_1$ and $S_2$ be disjoint, incompressible, connected, two-sided surfaces which are embedded in a $P^2$-irreducible 3-manifold $M$. Then if $S_1$ is homotopic to $S_2$ in $M$, $S_1 \cup S_2$ bounds an $S_1 \times I$ embedded in $M$.

**Lemma 3.** Let $M$ be a compact, connected, $3$-manifold, $X$ a connected complex, and $F$ and $S$ incompressible connected surfaces in $M$ and $X$ respectively. We suppose that $S$ is neither a 2-sphere or projective plane and $\pi_i(X) = 0$ for $i \geq 2$.

Let $f: (M, F) \to (X, S)$ be a map of pairs such that for some $x \in F$

$$f_*\pi_1(M, x) \subset \pi_1(S, f(x)).$$

Then $f$ is homotopic under a deformation, constant on $F$, to a map into $S$.

**Proof of Theorem 1.** It is a consequence of Remark 1 in [9] that we may assume that $M$ is irreducible.

Let $(M_A, \hat{x}, p)$ be the covering space of $(M, x)$ associated with $A \subset \pi_1(M, x)$. Let $f_1, f_2: (S, y) \to (M, x)$ be maps such that $f_j^*\pi_1(S, y) = A_j$ for $j = 1, 2$. Since $f_j^*\pi_1(S, y) \subset p_*\pi_1(M_A, \hat{x})$, there is a map $\hat{f}_j: (S, y) \to (M_A, \hat{x})$ such that $p\hat{f}_j = f_j$ for $j = 1, 2$. Let $X$ be the union of $M_A$ and $S \times I$ with identifications $\hat{f}_j(s) = (s, 0)$ and $\hat{f}_j(s) = (s, 1)$. We note that the arc $\{y\} \times [0, 1] \subset S \times I$ becomes a simple loop $\hat{l}$ after the identification above since $\hat{f}_j(y) = \hat{f}_j(y) = \hat{x}$. Let $\Phi: A \cup \{k\} \to \pi_1(X, \hat{x})$ be a function defined by $\Phi(k) = [l]$ and $\Phi(a) = P^{*a}(a)$ for $a \in A$. Then $\Phi$ can be extended to an isomorphism of $\pi_1(M, x)$ onto $\pi_1(X, \hat{x})$ since $X$ has been constructed so that $\pi_1(X, \hat{x})$ will have a presentation identical to the given presentation of $\pi_1(M, x)$.

It can be shown as in the proof of the theorem in [2] that $\pi_i(X) = \pi_i(X - S) = 0$ for $i \geq 2$.

We denotes $S \times \{1/2\} \subset X$ by $S$.

Let the boundary of $M$ be expressed as $\bigcup_{m=1}^n F_m$ where $F_m$ is a closed connected 2-manifold. Then some conjugate of $\rho_\ast \pi_1(F_m)$ is contained in $A$ for $m = 1, \ldots, n$. Thus we can find a collection $\{\alpha_m | m = 1, \ldots, n\}$ of simple arcs embedded in $M$ such that intersec-
tion of each pair of these arcs is \( x \), \( \alpha_m \) meets \( F_m \) in a single point, and there is a map \( \hat{\rho} : \bigcup_{m=1}^n (F_m \cup \alpha_m) \to M \) such that \( p \hat{\rho} = \rho \). Note that for each loop \( l_0 \) in \( \bigcup_{m=1}^n (F_m \cup \alpha_m) \) based at \( x \), \([\hat{\rho} l_0] = \Phi[l_0]\). Since

\[
\hat{\rho} \rho_* = \Phi \rho_* : \pi_1(\bigcup_{m=1}^n (F_m \cup \alpha_m), x) \to \pi_1(x, \hat{x})
\]

we can extend \( \hat{\rho} \) to a map \( f : M \to X \) such that \( \Phi = f_* : \pi_1(M, x) \to \pi_1(X, \hat{x}) \) by using standard techniques from obstruction theory. (See [2] or [3] for the details of this construction.) It is a consequence of Lemma 1 that there is a map \( g \), homotopic to \( f \) such that \( g_\gamma^-(S) \) is an incompressible surface in \( M \) and \( g_1 = f \) on the boundary of \( M \).

Since \( g_\gamma^-(S) \) and \( S \) are incompressible in \( M \) and \( X \) respectively, if \( S_0 \) is any component of \( g_\gamma^-(S) \), the homomorphism \( (g_1 | S_0)_* : \pi_1(S_0) \to \pi_1(S) \) is one-to-one. Thus by Theorem 1 in [6] \( g_1 | S_0 \) is homotopic to a covering map. Thus after a deformation, constant outside of a small neighborhood of \( S_0 \), we may assume that \( g_1 | S_0 \) is a local homeomorphism. Thus we may assume that \( g_1 \) is a local homeomorphism on \( g_\gamma^-(S) \).

Let \( z \) be a point on \( S_0 \). Suppose that the isomorphism \( \Phi_z = g_1 | z : \pi_1(M, z) \to \pi_1(X, g_1(z)) \) does not carry \( \pi_1(S_0, z) \) onto \( \pi_1(S, g_1(z)) \). It is a consequence of the result in [1] that \( M \) is \( P^2 \)-irreducible. Since \( \Phi_z^{-1} \pi_1(S, g_1(z)) \) would properly contain \( \pi_1(S_0, z) \), we would have by Theorem 6 in [7] that \( S_0 \) bounds a twisted line bundle \( N \subset M \). One can easily show using the techniques of [7], as has been done in [5], that \( \rho_* \pi_1(N, z) \) may be taken to be \( \Phi_z^{-1}(\rho_* \pi_1(S, g_1(z))) \). It follows from Lemma 3 that there is a deformation of \( g \), to a map \( g_z \) which pushes \( g_1(N) \) first onto \( S \) and then to one side of \( S \) so that \( g_\gamma^-(S) = g_\gamma^-(S) - S_0 \). Thus we can assume that \( (g_1 | S_0)_* : \pi_1(S_0) \to \pi_1(S) \) is an epimorphism for each component \( S_0 \) of \( g_\gamma^-(S) \).

Since \( \pi_1(M) \not\subset A \), \( g_\gamma^-(S) \) is not empty.

Let \( S_0 \) and \( S_1 \) be components of \( g_\gamma^-(S) \). We claim that \( S_0 \cup S_1 \) bounds a copy of \( S_0 \times [0, 1] \) embedded in \( M \). Since \( M \) is \( P^2 \)-irreducible, this will follow from Lemma 2 after we show that \( S_0 \) and \( S_1 \) are homotopic. Let \( z_0 \) be a point on \( S_0 \). Since \( g_1 | S_0 \) and \( g_1 | S_1 \) are assumed to be homeomorphisms, there is a unique point \( z_1 \) on \( S_1 \) such that \( g_1(z_0) = g_1(z) \). Let \( \alpha \) be an arc running from \( z_0 \) to \( z_1 \). Since \( g_1 \) is an isomorphism, we can find a loop \( l_1 \) based at \( z_0 \) such that the loops \( g_1(l_1) \) and \( g_1(\alpha) \) represent the same element in \( \pi_1(X, g_1(z_0)) \). Thus we may assume that \( [g_1(\alpha)] = 1 \in \pi_1(X) \). Let \( \lambda_0 \) be a loop on \( S_0 \) based at \( z_0 \) and \( \lambda_1 \) a loop on \( S_1 \) such that \( g_1(\lambda_0) = g_1(\lambda_1) \). Since the loop \( g_1(\lambda_0) g_1(\alpha) (g_1(\lambda_0))^{-1} (g_1(\alpha))^{-1} \) is nullhomotopic and \( \pi_1(X) = 0 \), we can show as in the proof of Theorem 1 in [3] that \( S_0 \) and \( S_1 \) are homotopic. Our claim follows.

We wish to show that we may assume \( g_\gamma^-(S) \) contains exactly one component.

Suppose there is more than one component in \( g_\gamma^-(S) \) and that the
number of components of \( g_T^S \) cannot be decreased by a small deformation of \( g \). Let \( l : S^1 \to M \) be a loop in \( M \) such that \( g_\ast [l] = [\hat{l}] \). We may assume that

(i) \( g_\ast (l) \) meets \( S \) since the intersection number of \([\hat{l}]\) and \( S \) is one. Thus we can take our basepoint to lie on one of the surfaces in \( g_T^S \).

(ii) \( l \) crosses \( g_T^{-1}(S) \) at each point in \( l \cap g_T^{-1}(S) \) and thus \( (g_\ast l)^{-1}(S) \) is a finite set whose cardinality cannot be reduced.

(iii) \( g_\ast (l \cap g_T^{-1}(S)) \) is a single point.

Let \( D \) be a disk and \( \beta_1 \) and \( \beta_2 \) arcs in the boundary of \( D \) such that \( \beta_1 \cap \beta_2 = \text{bd} (\beta_1) \). Then we can define a map \( \gamma : D \to X \) such that \( \gamma (\beta_1) \) is the loop \( g_\ast \mu (S^1) \) and \( \gamma (\beta_2) \) is the loop \( \hat{l} \).

We wish to show that \( g_T^{-1}(S) \) may be taken to be homeomorphic to \( S \) (connected). Assume that \( g_T^{-1}(S) \) is not connected; then it has been shown that each pair of distinct surfaces in \( g_T^{-1}(S) \) bounds a copy of \( S \times I \) embedded in \( M \). If this is the case, it is clear that \( l^{-1}g_T^{-1}(S) \) contains more than one point. Let \( \delta_1, \ldots, \delta_s \) be the closures of the components of \( S^1 - l^{-1}g_T^{-1}(S) \). After a general position argument we may assume \( \gamma^{-1}(S) \) contains an arc \( \beta_3 \) which cuts off an arc \( \beta_4 \subset \beta_1 \) and that \( g_\ast l(\delta_i) = \gamma (\beta_4) \). Now \( l \) carries \( \text{bd} (\delta_i) \) to one or two components of \( g_T^{-1}(S) \).

If \( l (\text{bd} (\delta_i)) \) is a single point, the loop \( l (\delta_i) \) is homotopic to a loop \( l_i \subset g_T^{-1}(S) \) such that \( g_\ast (l_i) = \gamma (\beta_3) \) since the restriction of \( g_\ast \) to each component of \( g_T^{-1}(S) \) is a homeomorphism and \( g_\ast \) is an isomorphism. It would follow that the number of points in \( l^{-1}g_T^{-1}(S) \) could have been reduced by a different choice of \( l \). Thus we conclude that \( l \) carries the points of \( \text{bd} (\delta_i) \) to distinct components of \( g_T^{-1}(S) \).

Let \( N \) be closure of the component of \( M - g_T^{-1}(S) \) which meets \( l (\delta_i) \). Let \( S_0 \) be a component of \( \text{bd} (N) \). Since \( g_\ast \mid S_0 \) is a homeomorphism and the loop \( g_\ast l (\delta_i) \) is homotopic to a loop in \( S_0 \) we may assume that the loop \( g_\ast l (\delta_i) \) is homotopic to a point. (One alters the image of \( l \) in a neighborhood of \( S_0 \).)

Since the loop \( g_\ast l (\delta_i) \) is nullhomotopic in \( X \), it can be shown that the map \( g_\ast \mid N \) is homotopic mod \( \text{bd} (N) \) to a map into \( S \); full details of a similar argument appear in [3]. It follows after an argument by induction that there exists a map \( g : M \to X \) homotopic to \( g_\ast \) mod \( \text{bd} (M) \) such that \( g^{-1}(S) \) contains exactly one component \( S_0 \) and \( g \mid S_0 \) is a homeomorphism. After an argument similar to the one given above, we can find a loop \( l \) meeting \( S_0 \) in a single point and based at \( x \in M \) such that \( g_\ast [l] = [\hat{l}] \).

We observe that \( S_0 \) and \( l \) induce an expression of \( \pi_1 (M, x) \) as an extension of a subgroup \( B \) of \( \pi_1 (M, x) \). Let \( B_1 \) and \( B_2 \) be the associated subgroups of \( \pi_1 (M, x) \). Then we see that our map \( g \) induces
an isomorphism \( g_\ast : \pi_1(M, x) \rightarrow \pi_1(M, x) \) such that

\[
\begin{align*}
(1) & \quad g_\ast(B) \subset A \\
(2) & \quad g_\ast(B_i) = A_i \\
(3) & \quad g_\ast B_i = A_i
\end{align*}
\]

Thus Theorem 1 is an immediate consequence of the remark preceding Lemma 2 on page 238 in [8] which shows that \( g_\ast \) sends \( B \) onto \( A \).

**Remark 1.** The remark mentioned above allows us to strengthen the statement of the theorem in [2] so that the splitting and the cutting are both actually realized.

**Remark 2.** We can also realize geometrically more general presentations of \( \pi_1(M) \) as an HNN group. In particular one might have that \( \pi_1(M) \) has a presentation as in the first definition in §4 in [8] where each of the subgroups \( L_i \) of \( K \) is isomorphic to the fundamental group of a closed connected surface other than \( S^2 \) or the projective plane and there are only finitely many of the \( t_i \). The proof of this result varies only slightly from the one given above.

**Remark 3.** Theorem 1 in this paper together with Theorem 1 in [3] or [4] give us a sort of converse to Van Kampen’s theorem as applied to a closed, connected, incompressible surface, other than \( S^2 \) or the projective plane, embedded in the interior of a compact 3-manifold.

**Remark 4.** This paper is in some sense a generalization of Stallings’ work in [11].

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