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**INTEGRABILITY THEOREMS FOR POWER SERIES  
EXPANSIONS OF TWO VARIABLES**

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## INTEGRABILITY THEOREMS FOR POWER SERIES EXPANSIONS OF TWO VARIABLES

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Let  $f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n}x^m y^n$  in the triangle  $x + y \leq 1$ ,  $x, y \geq 0$ , or in the quarter-disk  $x^2 + y^2 < 1$ ,  $x, y \geq 0$ . This paper shows some relations between  $L$ -integrability of  $f(x, y)$ , with certain multipliers, and the coefficients  $a_{m,n}$ .

1. DEFINITION. A real-valued function  $f(x, y)$  is said to be harmonic in a domain  $D$  in  $R^2$  if it is 2-times continuously differentiable in  $D$  and satisfies Laplace's equation

$$\Delta f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{for any } (x, y) \in D.$$

Throughout the paper, the letter  $C$ , with or without a suffix, denotes a positive constant, not necessarily the same at each appearance.

Heywood [3] proved a result as follows:

Suppose that  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  for  $0 \leq x < 1$ , that  $\gamma < 1$ , and that there are positive numbers  $\varepsilon, C$  such that  $a_n \geq -Cn^{-(\gamma+\varepsilon)}$  for all sufficiently large  $n$ . Then  $(1-x)^{-\gamma} f(x) \in L(0, 1)$  if and only if  $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$  converges absolutely.

We shall show two analogues of his result for power series expansions of two variables.

Kiselman [4] proved the following theorem.

**THEOREM A.** *If  $f(x, y)$  is harmonic in the disk  $x^2 + y^2 < r_0^2$  ( $r_0 > 0$ ), but not in any open disk of larger radius centred on the origin, then the power series expansion*

$$(1) \quad f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n$$

*converges absolutely in the square  $K: |x| + |y| < r_0$ , uniformly on every compact subset of  $K$ . It diverges at all points exterior to  $K$  for which  $x \neq 0$ , and  $y \neq 0$ .*

Further, the following theorem is known (see [2, p. 189 and 200] and [4]).

**THEOREM B.** *Suppose that  $f(x, y)$  is harmonic in the disk*

$$x^2 + y^2 < r_0^2,$$

and that  $f(x, y)$  has the power series expansion (1) in the square  $K$ , where  $K$  is defined as in Theorem A. Let  $P_N(x, y)$  be defined by

$$P_N(x, y) = \sum_{m+n=N} a_{m,n} x^m y^n \quad (N = 0, 1, 2, \dots).$$

Then the polynomial expansion

$$f(x, y) = \sum_{N=0}^{\infty} P_N(x, y)$$

of  $f(x, y)$  converges uniformly and absolutely in  $x^2 + y^2 \leq r^2$  for any  $0 < r < r_0$ , where  $P_N(x, y)$  are harmonic.

We give the following four theorems.

**THEOREM 1.** Suppose that a double power series (1) converges absolutely in the triangle

$$(2) \quad T: x + y < 1, \quad x, y \geq 0,$$

that  $\gamma < 1$ , and that there are positive numbers  $\varepsilon, C$  such that

$$(3) \quad a_{m,n} \geq -C(m+n+1)^{m+n-\gamma-\varepsilon+1/2} (m+1)^{-(m+1/2)} (n+1)^{-(n+1/2)}$$

for all sufficiently large  $m+n$ . Then  $(1-x-y)^{-\gamma} f(x, y)$  is Lebesgue-integrable on  $T$  if and only if

$$(4) \quad \sum_{m,n=0}^{\infty} (m+n+1)^{-m-n+\gamma-5/2} (m+1)^{m+1/2} (n+1)^{n+1/2} a_{m,n}$$

converges absolutely.

**THEOREM 2.** Suppose that  $f(x, y)$  is harmonic in the quarter-disk

$$(5) \quad Q: x^2 + y^2 < 1, \quad x, y \geq 0,$$

and that  $f(x, y)$  has the power series expansion (1) in the triangle  $T$ , where  $T$  is defined by (2). Then, under the assumption (3), the function  $(1-x-y)^{-\gamma} f(x, y)$ ,  $\gamma < 1$ , is Lebesgue-integrable on  $T$  if and only if the series (4) converges absolutely.

Theorem 2 is an obvious consequence of Theorem A ( $r_0 = 1$ ) and Theorem 1, and so we omit the proof.

**THEOREM 3.** Suppose that a double power series (1) converges absolutely in the quarter-disk  $Q$ , where  $Q$  is defined by (5), that  $\gamma < 1$ , and that there are positive numbers  $\varepsilon, C$  such that

$$(6) \quad a_{m,n} \cong \begin{cases} -C(m+n+1)^{(m+n+1)/2-\gamma-\varepsilon}(m+1)^{-(m+1)/2} \\ \qquad \qquad \qquad \times (n+1)^{-(n+1)/2} & \text{(even } m, n) \\ -C(m+n+1)^{(m+n)/2-\gamma-\varepsilon}(m+1)^{-m/2} \\ \qquad \qquad \qquad \times (n+1)^{-(n+1)/2} & \text{(odd } m \text{ and even } n) \\ -C(m+n+1)^{(m+n)/2-\gamma-\varepsilon}(m+1)^{-(m+1)/2} \\ \qquad \qquad \qquad \times (n+1)^{-n/2} & \text{(even } m \text{ and odd } n) \\ -C(m+n+1)^{(m+n-1)/2-\gamma-\varepsilon}(m+1)^{-m/2} \\ \qquad \qquad \qquad \times (n+1)^{-n/2} & \text{(odd } m, n) \end{cases}$$

for all sufficiently large  $m + n$ . Then the function

$$\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$$

is Lebesgue-integrable on  $Q$  if and only if the series

$$(7) \quad \sum_{m,n=0}^{\infty} (m+n+1)^{-(m+n+3)/2+\gamma}(m+1)^{m/2}(n+1)^{n/2}a_{m,n}$$

converges absolutely.

REMARK 1. In Theorem 3, it is easily seen that (6) may be replaced by a stronger condition

$$a_{m,n} \cong -C(m+n+1)^{(m+n-1)/2-\gamma-\varepsilon}(m+1)^{-m/2}(n+1)^{-n/2} \quad (m, n = 0, 1, 2, \dots)$$

for all sufficiently large  $m + n$ .

THEOREM 4. Suppose that  $f(x, y)$  is harmonic in the quarter-disk  $Q$ , where  $Q$  is defined by (5), and that  $f(x, y)$  has the power series expansion (1) in the triangle  $T$ , where  $T$  is defined by (2). Then, under the assumption (6), the function  $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$ ,  $\gamma < 1$ , is Lebesgue-integrable on  $Q$  if and only if the series (7) converges absolutely.

Theorem 4 is a consequence of Theorem B ( $r_0 = 1$ ) and Theorem 3. In § 2, we shall prove Theorem 1 and give an example for Theorem 2. Further, in § 3, we shall prove Theorems 3 and 4.

2. Proof of Theorem 1. First, suppose that  $(1 - x - y)^{-\gamma} f(x, y)$  is Lebesgue-integrable on  $T$ . Without loss of generality, we suppose that  $\gamma + \varepsilon$  is a noninteger value  $< 1$ . For, we get

$$a_{m,n} \cong -C(m+n+1)^{m+n-\gamma-\varepsilon'+1/2}(m+1)^{-(m+1/2)}(n+1)^{-(n+1/2)}$$

for  $0 < \varepsilon' < \varepsilon$ . We have, for any  $(x, y) \in T$ ,

$$\begin{aligned}
(1-x-y)^{\gamma+\varepsilon-1} &= \sum_{N=0}^{\infty} \frac{\Gamma(N+1-\gamma-\varepsilon)}{\Gamma(N+1)\Gamma(1-\gamma-\varepsilon)} (x+y)^N \\
&= \frac{1}{\Gamma(1-\gamma-\varepsilon)} \sum_{N=0}^{\infty} \frac{\Gamma(N+1-\gamma-\varepsilon)}{\Gamma(N+1)} \\
(8) \quad &\times \sum_{\substack{m+n=N \\ m, n \geq 0}} \binom{m+n}{n} x^m y^n \\
&= \frac{1}{\Gamma(1-\gamma-\varepsilon)} \sum_{m, n=0}^{\infty} \frac{\Gamma(m+n-\gamma-\varepsilon+1)}{\Gamma(m+1)\Gamma(n+1)} x^m y^n \\
&= \frac{1}{\Gamma(1-\gamma-\varepsilon)} \sum_{m, n=0}^{\infty} b_{m, n} x^m y^n,
\end{aligned}$$

say, where  $\Gamma(u)$  is the Gamma function. By Stirling's formula (see e.g. [1, p. 24])

$$\Gamma(u) = \sqrt{2\pi} u^{u-1/2} e^{-u+\eta/12u} \quad \text{for any } u > 0,$$

where  $\eta$  is a number independent of  $u$  between 0 and 1, we obtain

$$(9) \quad C_1 u^{u-1/2} e^{-u} \leq \Gamma(u) \leq C_2 u^{u-1/2} e^{-u} \quad \text{for any } u \geq u_0$$

if  $u_0$  is a fixed positive number. Hence we get easily

$$(10) \quad C_3 \lambda_{m, n} \leq b_{m, n} \leq C_4 \lambda_{m, n} \quad \text{for all } m, n \geq 0,$$

where

$$\lambda_{m, n} = (m+n+1)^{m+n-\gamma-\varepsilon+1/2} (m+1)^{-(m+1/2)} (n+1)^{-(n+1/2)}$$

(notice  $u_0 \geq \min(1-\gamma-\varepsilon, 1)$ ). Let

$$g(x, y) = C_5 \Gamma(1-\gamma-\varepsilon) (1-x-y)^{\gamma+\varepsilon-1}, \quad C_5 \geq C/C_3.$$

Then, it is clear that  $(1-x-y)^{-\gamma} g(x, y)$  is Lebesgue-integrable on  $T$ . Thus, by assumption,

$$\begin{aligned}
(1-x-y)^{-\gamma} \{f(x, y) + g(x, y)\} &= (1-x-y)^{-\gamma} \\
&\times \sum_{m, n=0}^{\infty} (a_{m, n} + C_5 b_{m, n}) x^m y^n
\end{aligned}$$

is Lebesgue-integrable on  $T$ . By (3) and (10), we have

$$a_{m, n} + C_5 b_{m, n} \geq a_{m, n} + C \lambda_{m, n} \geq 0$$

for all sufficiently large  $m+n$ . Hence we get

$$\begin{aligned}
(11) \quad \iint_T (1-x-y)^{-\gamma} \left\{ \sum_{m, n=0}^{\infty} (a_{m, n} + C_5 b_{m, n}) x^m y^n \right\} dx dy \\
= \sum_{m, n=0}^{\infty} (a_{m, n} + C_5 b_{m, n}) \iint_T (1-x-y)^{-\gamma} x^m y^n dx dy,
\end{aligned}$$

where the right-side series converges absolutely. Using the change of variable  $x = (1 - y)u$ , we have, for all  $m, n \geq 0$ ,

$$\begin{aligned} & \iint_T (1 - x - y)^{-\gamma} x^m y^n dx dy \\ &= \int_0^1 dy \int_0^{1-y} (1 - x - y)^{-\gamma} x^m y^n dx \\ &= \int_0^1 (1 - y)^{m+1-\gamma} y^n dy \int_0^1 (1 - u)^{-\gamma} u^m du \\ &= \frac{\Gamma(n + 1)\Gamma(m + 2 - \gamma)}{\Gamma(m + n + 3 - \gamma)} \cdot \frac{\Gamma(m + 1)\Gamma(1 - \gamma)}{\Gamma(m + 2 - \gamma)} \\ &= \Gamma(1 - \gamma) \cdot \frac{\Gamma(m + 1)\Gamma(n + 1)}{\Gamma(m + n + 3 - \gamma)}. \end{aligned}$$

Hence, from (9), we get

$$\begin{aligned} (12) \quad & C_6(m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} \\ & \leq \iint_T (1 - x - y)^{-\gamma} x^m y^n dx dy \\ & \leq C_7(m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} \end{aligned}$$

for all  $m, n \geq 0$ . Thus, by (11) and (12),

$$(13) \quad \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2}(a_{m,n} + C_5 b_{m,n})$$

converges absolutely. Further, from (10)

$$\begin{aligned} (14) \quad & \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} b_{m,n} \\ & \leq C_4 \sum_{m,n=0}^{\infty} (m + n + 1)^{-2-\varepsilon} < \infty. \end{aligned}$$

By (3) and (10), we get

$$|a_{m,n}| \leq a_{m,n} + 2C\lambda_{m,n} \leq a_{m,n} + 2C_5 b_{m,n} \quad (C_5 \geq C/C_3)$$

for all sufficiently large  $m + n$ . Hence, from (13) and (14), the series (4) converges absolutely.

Conversely we suppose that the series (4) converges absolutely, and will deduce that  $(1 - x - y)^{-\gamma} f(x, y)$  is Lebesgue-integrable on  $T$ . For this part of the argument we do not assume (3). We have in fact

$$\begin{aligned}
 & \iint_T (1-x-y)^{-\gamma} |f(x,y)| dx dy \\
 & \leq \iint_T (1-x-y)^{-\gamma} \left\{ \sum_{m,n=0}^{\infty} |a_{m,n}| x^m y^n \right\} dx dy \\
 & = \sum_{m,n=0}^{\infty} |a_{m,n}| \iint_T (1-x-y)^{-\gamma} x^m y^n dx dy \\
 & \leq C_\gamma \sum_{m,n=0}^{\infty} (m+n+1)^{-m-n+\gamma-5/2} (m+1)^{m+1/2} (n+1)^{n+1/2} |a_{m,n}| < \infty
 \end{aligned}$$

by (12). Thus Theorem 1 is proved.

EXAMPLE FOR THEOREM 2. Let

$$f(x,y) = \Re(1-z)^{-2} = \frac{(1-x)^2 - y^2}{\{(1-x)^2 + y^2\}^2} \quad (z = x + iy, i = \sqrt{-1}).$$

Then  $f(x,y)$  is harmonic in the disk  $x^2 + y^2 < 1$ . Since

$$f(x,y) = \Re \sum_{N=0}^{\infty} (N+1)z^N = \sum_{N=0}^{\infty} (N+1) \sum_{m+2n=N} (-1)^n \binom{m+2n}{2n} x^m y^{2n}$$

in the disk  $x^2 + y^2 < 1$ , we get

$$f(x,y) = \sum_{m,n=0}^{\infty} (-1)^n \frac{\Gamma(m+2n+2)}{\Gamma(m+1)\Gamma(2n+1)} x^m y^{2n}$$

in the square  $|x| + |y| < 1$ , by Theorem A. When  $a_{m,n}$  denote the  $(m,n)$ th coefficients of this power series expansion, we have, from (9),

$$\begin{aligned}
 & C_1(m+2n+1)^{m+2n+3/2} (m+1)^{-(m+1/2)} (2n+1)^{-(2n+1/2)} \\
 & \leq |a_{m,2n}| \leq C_2(m+2n+1)^{m+2n+3/2} (m+1)^{-(m+1/2)} (2n+1)^{-(2n+1/2)}
 \end{aligned}$$

and  $a_{m,2n+1} = 0$ . First we put  $\gamma < -1$ . Then the sequence  $\{a_{m,n}\}$  satisfies (3) for  $\varepsilon = -(\gamma+1)/2$ . Now we have

$$\begin{aligned}
 & \iint_T (1-x-y)^{-\gamma} |f(x,y)| dx dy \\
 & = \int_0^1 (1-x)^{-\gamma-1} dx \int_0^1 \frac{(1-u)^{-\gamma+1}(1+u)}{(1+u^2)^2} du < \infty
 \end{aligned}$$

by the change of variable  $y = (1-x)u$ . Further we get

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} (m+n+1)^{-m-n+\gamma-5/2} (m+1)^{m+1/2} (n+1)^{n+1/2} |a_{m,n}| \\
 & \leq C_2 \sum_{m,n=0}^{\infty} (m+n+1)^{\gamma-1} < \infty.
 \end{aligned}$$

Next we set  $\gamma = -1$ . Then  $\{a_{m,n}\}$  does not satisfy (3), but we notice  $\varepsilon = 0$ . It is clear that

$$\iint_T (1 - x - y) |f(x, y)| dx dy = \int_0^1 \frac{(1 - u)^2(1 + u)}{(1 + u^2)^2} du < \infty .$$

But we get

$$\begin{aligned} & \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n-7/2} (m + 1)^{m+1/2} (n + 1)^{n+1/2} |a_{m,n}| \\ & \geq C_1 \sum_{m,n=0}^{\infty} (m + 2n + 1)^{-2} > \frac{C_1}{4} \sum_{m,n=0}^{\infty} (m + n + 1)^{-2} = \infty . \end{aligned}$$

Thus this example ( $\gamma = -1$ ) show that we cannot set  $\varepsilon = 0$  in (3) without destroying the validity of Theorem 2.

3. In order to prove Theorem 3, we need the following lemma.

LEMMA. Suppose that  $\mu < 1$ , and that  $A(x, y)$  is defined by

$$A(x, y) = (1 + x + y + xy)(1 - x^2 - y^2)^{\mu-1}$$

in the quarter-disk  $Q$ , where  $Q$  is defined by (5). Then  $A(x, y)$  has the power series expansion

$$(15) \quad A(x, y) = \sum_{m,n=0}^{\infty} d_{m,n} x^m y^n, \quad C_1 \delta_{m,n} \leq d_{m,n} \leq C_2 \delta_{m,n} \quad (C_1, C_2 > 0)$$

in  $Q$ , where

$$\delta_{m,n} = \begin{cases} (m + n + 1)^{(m+n+1)/2-\mu} (m + 1)^{-(m+1)/2} & \\ \quad \times (n + 1)^{-(n+1)/2} & \text{(even } m, n) \\ (m + n + 1)^{(m+n)/2-\mu} (m + 1)^{-m/2} & \\ \quad \times (n + 1)^{-(n+1)/2} & \text{(odd } m \text{ and even } n) \\ (m + n + 1)^{(m+n)/2-\mu} (m + 1)^{-(m+1)/2} & \\ \quad \times (n + 1)^{-n/2} & \text{(even } m \text{ and odd } n) \\ (m + n + 1)^{(m+n-1)/2-\mu} (m + 1)^{-m/2} & \\ \quad \times (n + 1)^{-n/2} & \text{(odd } m, n) . \end{cases}$$

Proof. We have, for any  $(x, y) \in Q$ ,

$$\begin{aligned} (1 - x^2 - y^2)^{\mu-1} &= \sum_{N=0}^{\infty} \frac{\Gamma(N + 1 - \mu)}{\Gamma(N + 1)\Gamma(1 - \mu)} (x^2 + y^2)^N \\ &= \sum_{N=0}^{\infty} \frac{\Gamma(N + 1 - \mu)}{\Gamma(N + 1)\Gamma(1 - \mu)} \sum_{\substack{m+n=N \\ m,n \geq 0}} \binom{m + n}{m} x^{2m} y^{2n} \\ &= \sum_{m,n=0}^{\infty} \frac{1}{\Gamma(1 - \mu)} \cdot \frac{\Gamma(m + n + 1 - \mu)}{\Gamma(m + 1)\Gamma(n + 1)} x^{2m} y^{2n} \\ &= \sum_{m,n=0}^{\infty} p_{m,n} x^{2m} y^{2n} , \end{aligned}$$



say. Then we get

$$(16) \quad A(x, y) = \sum_{m, n=0}^{\infty} p_{m, n} (x^{2m}y^{2n} + x^{2m+1}y^{2n} + x^{2m}y^{2n+1} + x^{2m+1}y^{2n+1}) .$$

We put

$$d_{m, n} = \begin{cases} p_{m/2, n/2} & \text{(even } m, n) \\ p_{(m-1)/2, n/2} & \text{(odd } m \text{ and even } n) \\ p_{m/2, (n-1)/2} & \text{(even } m \text{ and odd } n) \\ p_{(m-1)/2, (n-1)/2} & \text{(odd } m, n) . \end{cases}$$

Now, from (16) and (9), we get easily (15). Thus the Lemma is proved.

*Proof of Theorem 3.* First, suppose that  $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$  is Lebesgue-integrable on  $Q$ . Without loss of generality, we may suppose that  $\gamma + \varepsilon$  is a noninteger  $< 1$ . Let

$$(17) \quad h(x, y) = (1 + x + y + xy)(1 - x^2 - y^2)^{\gamma + \varepsilon - 1}$$

in  $Q$ . Then, by the Lemma ( $\mu = \gamma + \varepsilon$ ), we have

$$(18) \quad h(x, y) = \sum_{m, n=0}^{\infty} k_{m, n} x^m y^n, \quad C_1 \theta_{m, n} \leq k_{m, n} \leq C_2 \theta_{m, n}$$

in  $Q$ , where  $k_{m, n}$  and  $\theta_{m, n}$  are defined respectively like  $d_{m, n}$  and  $\delta_{m, n}$  in the Lemma with  $\mu = \gamma + \varepsilon$ . Clearly, the function

$$\begin{aligned} & \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} h(x, y) \\ &= (1 + x + y + xy) \{1 + (x^2 + y^2)^{1/2}\}^{\gamma + \varepsilon - 1} \{1 - (x^2 + y^2)^{1/2}\}^{\varepsilon - 1} \end{aligned}$$

is Lebesgue-integrable on  $Q$ . Hence, by assumption, the function

$$\begin{aligned} & \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \{f(x, y) + C_3 h(x, y)\} \\ &= \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \sum_{m, n=0}^{\infty} (a_{m, n} + C_3 k_{m, n}) x^m y^n \end{aligned}$$

is Lebesgue-integrable on  $Q$ , where  $C_3 \geq C/C_1$ . Further, by (6) and (18), we have

$$(19) \quad a_{m, n} + C_3 k_{m, n} \geq a_{m, n} + C \theta_{m, n} \geq 0$$

for all sufficiently large  $m + n$ . Thus we get

$$(20) \quad \begin{aligned} & \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \left\{ \sum_{m, n=0}^{\infty} (a_{m, n} + C_3 k_{m, n}) x^m y^n \right\} dx dy \\ &= \sum_{m, n=0}^{\infty} (a_{m, n} + C_3 k_{m, n}) \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dx dy, \end{aligned}$$

where the right-side series converges absolutely. By the change of variables

$$x = r \cos v, \quad y = r \sin v \quad (0 \leq r < 1, 0 \leq v \leq \pi/2),$$

we get

$$\begin{aligned} & \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dx dy \\ &= \int_0^1 (1 - r)^{-\gamma} r^{m+n+1} dr \int_0^{\pi/2} \sin^m v \cos^n v dv \\ &= \frac{\Gamma(m + n + 2)\Gamma(1 - \gamma)}{\Gamma(m + n + 3 - \gamma)} \cdot \frac{1}{2} \cdot \frac{\Gamma((m + 1)/2)\Gamma((n + 1)/2)}{\Gamma((m + n)/2 + 1)}. \end{aligned}$$

Thus, from (9), we get

$$\begin{aligned} (21) \quad & C_4(m + n + 1)^{-(m+n+3)/2+\gamma} (m + 1)^{m/2} (n + 1)^{n/2} \\ & \leq \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dx dy \\ & \leq C_5(m + n + 1)^{-(m+n+3)/2+\gamma} (m + 1)^{m/2} (n + 1)^{n/2} \end{aligned}$$

for all  $m, n \geq 0$ . Hence, by (20),

$$(22) \quad \sum_{m,n=0}^{\infty} (m + n + 1)^{-(m+n+3)/2+\gamma} (m + 1)^{m/2} (n + 1)^{n/2} (a_{m,n} + C_3 k_{m,n})$$

converges absolutely. Further, by (18), we have

$$\begin{aligned} (23) \quad & \sum_{m,n=0}^{\infty} (m + n + 1)^{-(m+n+3)/2+\gamma} (m + 1)^{m/2} (n + 1)^{n/2} k_{m,n} \\ & \leq C_2 \sum_{m,n=0}^{\infty} \{(m + n + 1)^{-1-\epsilon} (m + 1)^{-1/2} (n + 1)^{-1/2} \\ & \quad + (m + n + 1)^{-3/2-\epsilon} (n + 1)^{-1/2} + (m + n + 1)^{-3/2-\epsilon} (m + 1)^{-1/2} \\ & \quad + (m + n + 1)^{-2-\epsilon}\} < \infty. \end{aligned}$$

By (6) and (18), we get

$$|a_{m,n}| \leq a_{m,n} + 2C\theta_{m,n} \leq a_{m,n} + 2C_3 k_{m,n} \quad (C_3 \geq C/C_1)$$

for all sufficiently large  $m + n$ . Hence, from (22) and (23), the series (7) converges absolutely.

Conversely we suppose that series (7) converges absolutely, and will deduce that  $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$  is Lebesgue-integrable on  $Q$ . For this part of the argument we do not assume (6). We have in fact

$$\begin{aligned}
& \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} |f(x, y)| dx dy \\
& \cong \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \left\{ \sum_{m,n=0}^{\infty} |a_{m,n}| x^m y^n \right\} dx dy \\
& = \sum_{m,n=0}^{\infty} |a_{m,n}| \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dx dy \\
& \leq C_5 \sum_{m,n=0}^{\infty} (m+n+1)^{-(m+n+3)/2+\gamma} (m+1)^{m/2} (n+1)^{n/2} |a_{m,n}| < \infty
\end{aligned}$$

by (21). Thus Theorem 3 is proved.

REMARK 2. From (17), it is easily seen that

$$C_1 h(x, y) \leq \{1 - (x^2 + y^2)^{1/2}\}^{\gamma+\varepsilon-1} \leq C_2 h(x, y)$$

in  $Q$ .

*Proof of Theorem 4.* By Theorem B ( $r_0 = 1$ ), we get

$$f(x, y) = \sum_{N=0}^{\infty} \sum_{m+n=N} a_{m,n} x^m y^n$$

in  $Q$ . We define  $h(x, y)$  by (17). Then it is sufficient for us to notice that

$$\begin{aligned}
f(x, y) + C_3 h(x, y) &= \sum_{N=0}^{\infty} \sum_{m+n=N} a_{m,n} x^m y^n + C_3 \sum_{m,n=0}^{\infty} k_{m,n} x^m y^n \\
&= \sum_{N=0}^{\infty} \sum_{m+n=N} (a_{m,n} + C_3 k_{m,n}) x^m y^n \\
&= \sum_{m,n=0}^{\infty} (a_{m,n} + C_3 k_{m,n}) x^m y^n
\end{aligned}$$

in  $Q$ , in view of (18) and (19), where the last right-side series converges absolutely. Thus Theorem 4 is a consequence of Theorem 3.

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