

Pacific Journal of Mathematics

INTEGRABILITY THEOREMS FOR POWER SERIES EXPANSIONS OF TWO VARIABLES

YOSHIMITSU HASEGAWA

INTEGRABILITY THEOREMS FOR POWER SERIES EXPANSIONS OF TWO VARIABLES

YOSHIMITHU HASEGAWA

Let $f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n}x^m y^n$ in the triangle $x + y \leq 1$, $x, y \geq 0$, or in the quarter-disk $x^2 + y^2 < 1$, $x, y \geq 0$. This paper show some relations between L -integrability of $f(x, y)$, with certain multipliers, and the coefficients $a_{m,n}$.

1. DEFINITION. A real-valued function $f(x, y)$ is said to be harmonic in a domain D in R^2 if it is 2-times continuously differentiable in D and satisfies Laplace's equation

$$\Delta f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{for any } (x, y) \in D.$$

Throughout the paper, the letter C , with or without a suffix, denotes a positive constant, not necessarily the same at each appearance.

Heywood [3] proved a result as follows:

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $0 \leq x < 1$, that $\gamma < 1$, and that there are positive numbers ε, C such that $a_n \geq -Cn^{-(\gamma+\varepsilon)}$ for all sufficiently large n . Then $(1-x)^{-\gamma} f(x) \in L(0, 1)$ if and only if $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges absolutely.

We shall show two analogues of his result for power series expansions of two variables.

Kiselman [4] proved the following theorem.

THEOREM A. *If $f(x, y)$ is harmonic in the disk $x^2 + y^2 < r_0^2$ ($r_0 > 0$), but not in any open disk of larger radius centred on the origin, then the power series expansion*

$$(1) \quad f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n$$

converges absolutely in the square $K: |x| + |y| < r_0$, uniformly on every compact subset of K . It diverges at all points exterior to K for which $x \neq 0$, and $y \neq 0$.

Further, the following theorem is known (see [2, p. 189 and 200] and [4]).

THEOREM B. *Suppose that $f(x, y)$ is harmonic in the disk*

$$x^2 + y^2 < r_0^2,$$

and that $f(x, y)$ has the power series expansion (1) in the square K , where K is defined as in Theorem A. Let $P_N(x, y)$ be defined by

$$P_N(x, y) = \sum_{m+n=N} a_{m,n} x^m y^n \quad (N = 0, 1, 2, \dots).$$

Then the polynomial expansion

$$f(x, y) = \sum_{N=0}^{\infty} P_N(x, y)$$

of $f(x, y)$ converges uniformly and absolutely in $x^2 + y^2 \leq r^2$ for any $0 < r < r_0$, where $P_N(x, y)$ are harmonic.

We give the following four theorems.

THEOREM 1. *Suppose that a double power series (1) converges absolutely in the triangle*

$$(2) \quad T: x + y < 1, \quad x, y \geq 0,$$

that $\gamma < 1$, and that there are positive numbers ε, C such that

$$(3) \quad a_{m,n} \geq -C(m + n + 1)^{m+n-\gamma-\varepsilon+1/2} (m + 1)^{-(m+1/2)} (n + 1)^{-(n+1/2)}$$

for all sufficiently large $m + n$. Then $(1 - x - y)^{-\gamma} f(x, y)$ is Lebesgue-integrable on T if and only if

$$(4) \quad \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2} (m + 1)^{m+1/2} (n + 1)^{n+1/2} a_{m,n}$$

converges absolutely.

THEOREM 2. *Suppose that $f(x, y)$ is harmonic in the quarter-disk*

$$(5) \quad Q: x^2 + y^2 < 1, \quad x, y \geq 0,$$

and that $f(x, y)$ has the power series expansion (1) in the triangle T , where T is defined by (2). Then, under the assumption (3), the function $(1 - x - y)^{-\gamma} f(x, y)$, $\gamma < 1$, is Lebesgue-integrable on T if and only if the series (4) converges absolutely.

Theorem 2 is an obvious consequence of Theorem A ($r_0 = 1$) and Theorem 1, and so we omit the proof.

THEOREM 3. *Suppose that a double power series (1) converges absolutely in the quarter-disk Q , where Q is defined by (5), that $\gamma < 1$, and that there are positive numbers ε, C such that*

$$(6) \quad a_{m,n} \cong \begin{cases} -C(m+n+1)^{(m+n+1)/2-\gamma-\varepsilon}(m+1)^{-(m+1)/2} \\ \quad \times (n+1)^{-(n+1)/2} & \text{(even } m, n) \\ -C(m+n+1)^{(m+n)/2-\gamma-\varepsilon}(m+1)^{-m/2} \\ \quad \times (n+1)^{-(n+1)/2} & \text{(odd } m \text{ and even } n) \\ -C(m+n+1)^{(m+n)/2-\gamma-\varepsilon}(m+1)^{-(m+1)/2} \\ \quad \times (n+1)^{-n/2} & \text{(even } m \text{ and odd } n) \\ -C(m+n+1)^{(m+n-1)/2-\gamma-\varepsilon}(m+1)^{-m/2} \\ \quad \times (n+1)^{-n/2} & \text{(odd } m, n) \end{cases}$$

for all sufficiently large $m+n$. Then the function

$$\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$$

is Lebesgue-integrable on Q if and only if the series

$$(7) \quad \sum_{m,n=0}^{\infty} (m+n+1)^{-(m+n+3)/2+\gamma} (m+1)^{m/2} (n+1)^{n/2} a_{m,n}$$

converges absolutely.

REMARK 1. In Theorem 3, it is easily seen that (6) may be replaced by a stronger condition

$$a_{m,n} \geq -C(m+n+1)^{(m+n-1)/2-\gamma-\varepsilon}(m+1)^{-m/2}(n+1)^{-n/2} \quad (m, n = 0, 1, 2, \dots)$$

for all sufficiently large $m+n$.

THEOREM 4. Suppose that $f(x, y)$ is harmonic in the quarter-disk Q , where Q is defined by (5), and that $f(x, y)$ has the power series expansion (1) in the triangle T , where T is defined by (2). Then, under the assumption (6), the function $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$, $\gamma < 1$, is Lebesgue-integrable on Q if and only if the series (7) converges absolutely.

Theorem 4 is a consequence of Theorem B ($r_0 = 1$) and Theorem 3. In § 2, we shall prove Theorem 1 and give an example for Theorem 2. Further, in § 3, we shall prove Theorems 3 and 4.

2. Proof of Theorem 1. First, suppose that $(1 - x - y)^{-\gamma} f(x, y)$ is Lebesgue-integrable on T . Without loss of generality, we suppose that $\gamma + \varepsilon$ is a noninteger value < 1 . For, we get

$$a_{m,n} \geq -C(m+n+1)^{m+n-\gamma-\varepsilon'+1/2}(m+1)^{-(m+1/2)}(n+1)^{-(n+1/2)}$$

for $0 < \varepsilon' < \varepsilon$. We have, for any $(x, y) \in T$,

$$\begin{aligned}
 (1 - x - y)^{\gamma + \varepsilon - 1} &= \sum_{N=0}^{\infty} \frac{\Gamma(N + 1 - \gamma - \varepsilon)}{\Gamma(N + 1) \Gamma(1 - \gamma - \varepsilon)} (x + y)^N \\
 &= \frac{1}{\Gamma(1 - \gamma - \varepsilon)} \sum_{N=0}^{\infty} \frac{\Gamma(N + 1 - \gamma - \varepsilon)}{\Gamma(N + 1)} \\
 (8) \quad &\times \sum_{\substack{m+n=N \\ m, n \geq 0}} \binom{m+n}{n} x^m y^n \\
 &= \frac{1}{\Gamma(1 - \gamma - \varepsilon)} \sum_{m, n=0}^{\infty} \frac{\Gamma(m+n-\gamma-\varepsilon+1)}{\Gamma(m+1)\Gamma(n+1)} x^m y^n \\
 &= \frac{1}{\Gamma(1 - \gamma - \varepsilon)} \sum_{m, n=0}^{\infty} b_{m, n} x^m y^n,
 \end{aligned}$$

say, where $\Gamma(u)$ is the Gamma function. By Stirling's formula (see e.g. [1, p. 24])

$$\Gamma(u) = \sqrt{2\pi} u^{u-1/2} e^{-u+\gamma/12u} \quad \text{for any } u > 0,$$

where γ is a number independent of u between 0 and 1, we obtain

$$(9) \quad C_1 u^{u-1/2} e^{-u} \leq \Gamma(u) \leq C_2 u^{u-1/2} e^{-u} \quad \text{for any } u \geq u_0$$

if u_0 is a fixed positive number. Hence we get easily

$$(10) \quad C_3 \lambda_{m, n} \leq b_{m, n} \leq C_4 \lambda_{m, n} \quad \text{for all } m, n \geq 0,$$

where

$$\lambda_{m, n} = (m + n + 1)^{m+n-\gamma-\varepsilon+1/2} (m + 1)^{-(m+1/2)} (n + 1)^{-(n+1/2)}$$

(notice $u_0 \geq \min(1 - \gamma - \varepsilon, 1)$). Let

$$g(x, y) = C_5 \Gamma(1 - \gamma - \varepsilon) (1 - x - y)^{\gamma + \varepsilon - 1}, \quad C_5 \geq C/C_3.$$

Then, it is clear that $(1 - x - y)^{-\gamma} g(x, y)$ is Lebesgue-integrable on T . Thus, by assumption,

$$\begin{aligned}
 (1 - x - y)^{-\gamma} \{f(x, y) + g(x, y)\} &= (1 - x - y)^{-\gamma} \\
 &\times \sum_{m, n=0}^{\infty} (a_{m, n} + C_5 b_{m, n}) x^m y^n
 \end{aligned}$$

is Lebesgue-integrable on T . By (3) and (10), we have

$$a_{m, n} + C_5 b_{m, n} \geq a_{m, n} + C \lambda_{m, n} \geq 0$$

for all sufficiently large $m + n$. Hence we get

$$\begin{aligned}
 (11) \quad \iint_T (1 - x - y)^{-\gamma} \left\{ \sum_{m, n=0}^{\infty} (a_{m, n} + C_5 b_{m, n}) x^m y^n \right\} dx dy \\
 = \sum_{m, n=0}^{\infty} (a_{m, n} + C_5 b_{m, n}) \iint_T (1 - x - y)^{-\gamma} x^m y^n dx dy,
 \end{aligned}$$

where the right-side series converges absolutely. Using the change of variable $x = (1 - y)u$, we have, for all $m, n \geq 0$,

$$\begin{aligned} & \iint_T (1 - x - y)^{-\gamma} x^m y^n dx dy \\ &= \int_0^1 dy \int_0^{1-y} (1 - x - y)^{-\gamma} x^m y^n dx \\ &= \int_0^1 (1 - y)^{m+1-\gamma} y^n dy \int_0^1 (1 - u)^{-\gamma} u^m du \\ &= \frac{\Gamma(n + 1)\Gamma(m + 2 - \gamma)}{\Gamma(m + n + 3 - \gamma)} \cdot \frac{\Gamma(m + 1)\Gamma(1 - \gamma)}{\Gamma(m + 2 - \gamma)} \\ &= \Gamma(1 - \gamma) \cdot \frac{\Gamma(m + 1)\Gamma(n + 1)}{\Gamma(m + n + 3 - \gamma)}. \end{aligned}$$

Hence, from (9), we get

$$\begin{aligned} & C_6(m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} \\ (12) \quad & \leq \iint_T (1 - x - y)^{-\gamma} x^m y^n dx dy \\ & \leq C_7(m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} \end{aligned}$$

for all $m, n \geq 0$. Thus, by (11) and (12),

$$(13) \quad \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2}(a_{m,n} + C_5 b_{m,n})$$

converges absolutely. Further, from (10)

$$\begin{aligned} (14) \quad & \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2}(m + 1)^{m+1/2}(n + 1)^{n+1/2} b_{m,n} \\ & \leq C_4 \sum_{m,n=0}^{\infty} (m + n + 1)^{-2-\varepsilon} < \infty. \end{aligned}$$

By (3) and (10), we get

$$|a_{m,n}| \leq a_{m,n} + 2C\lambda_{m,n} \leq a_{m,n} + 2C_5 b_{m,n} \quad (C_5 \geq C/C_3)$$

for all sufficiently large $m + n$. Hence, from (13) and (14), the series (4) converges absolutely.

Conversely we suppose that the series (4) converges absolutely, and will deduce that $(1 - x - y)^{-\gamma} f(x, y)$ is Lebesgue-integrable on T . For this part of the argument we do not assume (3). We have in fact

$$\begin{aligned}
 & \iint_T (1 - x - y)^{-\gamma} |f(x, y)| \, dx dy \\
 & \leq \iint_T (1 - x - y)^{-\gamma} \left\{ \sum_{m,n=0}^{\infty} |a_{m,n}| x^m y^n \right\} \, dx dy \\
 & = \sum_{m,n=0}^{\infty} |a_{m,n}| \iint_T (1 - x - y)^{-\gamma} x^m y^n \, dx dy \\
 & \leq C_7 \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2} (m + 1)^{m+1/2} (n + 1)^{n+1/2} |a_{m,n}| < \infty
 \end{aligned}$$

by (12). Thus Theorem 1 is proved.

EXAMPLE FOR THEOREM 2. Let

$$f(x, y) = \Re(1 - z)^{-2} = \frac{(1 - x)^2 - y^2}{\{(1 - x)^2 + y^2\}^2} \quad (z = x + iy, i = \sqrt{-1}).$$

Then $f(x, y)$ is harmonic in the disk $x^2 + y^2 < 1$. Since

$$f(x, y) = \Re \sum_{N=0}^{\infty} (N + 1)z^N = \sum_{N=0}^{\infty} (N + 1) \sum_{m+2n=N} (-1)^n \binom{m + 2n}{2n} x^m y^{2n}$$

in the disk $x^2 + y^2 < 1$, we get

$$f(x, y) = \sum_{m,n=0}^{\infty} (-1)^n \frac{\Gamma(m + 2n + 2)}{\Gamma(m + 1)\Gamma(2n + 1)} x^m y^{2n}$$

in the square $|x| + |y| < 1$, by Theorem A. When $a_{m,n}$ denote the (m, n) th coefficients of this power series expansion, we have, from (9),

$$\begin{aligned}
 & C_1(m + 2n + 1)^{m+2n+3/2} (m + 1)^{-(m+1/2)} (2n + 1)^{-(2n+1/2)} \\
 & \leq |a_{m,2n}| \leq C_2(m + 2n + 1)^{m+2n+3/2} (m + 1)^{-(m+1/2)} (2n + 1)^{-(2n+1/2)}
 \end{aligned}$$

and $a_{m,2n+1} = 0$. First we put $\gamma < -1$. Then the sequence $\{a_{m,n}\}$ satisfies (3) for $\varepsilon = -(\gamma + 1)/2$. Now we have

$$\begin{aligned}
 & \iint_T (1 - x - y)^{-\gamma} |f(x, y)| \, dx dy \\
 & = \int_0^1 (1 - x)^{-\gamma-1} dx \int_0^1 \frac{(1 - u)^{-\gamma+1} (1 + u)}{(1 + u^2)^2} du < \infty
 \end{aligned}$$

by the change of variable $y = (1 - x)u$. Further we get

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} (m + n + 1)^{-m-n+\gamma-5/2} (m + 1)^{m+1/2} (n + 1)^{n+1/2} |a_{m,n}| \\
 & \leq C_2 \sum_{m,n=0}^{\infty} (m + n + 1)^{\gamma-1} < \infty.
 \end{aligned}$$

Next we set $\gamma = -1$. Then $\{a_{m,n}\}$ does not satisfy (3), but we notice $\varepsilon = 0$. It is clear that

$$\iint_T (1 - x - y) |f(x, y)| dx dy = \int_0^1 \frac{(1-u)^2(1+u)}{(1+u^2)^2} du < \infty .$$

But we get

$$\begin{aligned} & \sum_{m,n=0}^{\infty} (m+n+1)^{-m-n-7/2} (m+1)^{m+1/2} (n+1)^{n+1/2} |a_{m,n}| \\ & \geq C_1 \sum_{m,n=0}^{\infty} (m+2n+1)^{-2} > \frac{C_1}{4} \sum_{m,n=0}^{\infty} (m+n+1)^{-2} = \infty . \end{aligned}$$

Thus this example ($\gamma = -1$) show that we cannot set $\varepsilon = 0$ in (3) without destroying the validity of Theorem 2.

3. In order to prove Theorem 3, we need the following lemma.

LEMMA. Suppose that $\mu < 1$, and that $A(x, y)$ is defined by

$$A(x, y) = (1 + x + y + xy)(1 - x^2 - y^2)^{\mu-1}$$

in the quarter-disk Q , where Q is defined by (5). Then $A(x, y)$ has the power series expansion

$$(15) \quad A(x, y) = \sum_{m,n=0}^{\infty} d_{m,n} x^m y^n, \quad C_1 \delta_{m,n} \leq d_{m,n} \leq C_2 \delta_{m,n} \quad (C_1, C_2 > 0)$$

in Q , where

$$\delta_{m,n} = \begin{cases} (m+n+1)^{(m+n+1)/2-\mu} (m+1)^{-(m+1)/2} & \\ \quad \times (n+1)^{-(n+1)/2} & \text{(even } m, n) \\ (m+n+1)^{(m+n)/2-\mu} (m+1)^{-m/2} & \\ \quad \times (n+1)^{-(n+1)/2} & \text{(odd } m \text{ and even } n) \\ (m+n+1)^{(m+n)/2-\mu} (m+1)^{-(m+1)/2} & \\ \quad \times (n+1)^{-n/2} & \text{(even } m \text{ and odd } n) \\ (m+n+1)^{(m+n-1)/2-\mu} (m+1)^{-m/2} & \\ \quad \times (n+1)^{-n/2} & \text{(odd } m, n) . \end{cases}$$

Proof. We have, for any $(x, y) \in Q$,

$$\begin{aligned} (1 - x^2 - y^2)^{\mu-1} &= \sum_{N=0}^{\infty} \frac{\Gamma(N+1-\mu)}{\Gamma(N+1)\Gamma(1-\mu)} (x^2 + y^2)^N \\ &= \sum_{N=0}^{\infty} \frac{\Gamma(N+1-\mu)}{\Gamma(N+1)\Gamma(1-\mu)} \sum_{\substack{m+n=N \\ m,n \geq 0}} \binom{m+n}{m} x^{2m} y^{2n} \\ &= \sum_{m,n=0}^{\infty} \frac{1}{\Gamma(1-\mu)} \cdot \frac{\Gamma(m+n+1-\mu)}{\Gamma(m+1)\Gamma(n+1)} x^{2m} y^{2n} \\ &= \sum_{m,n=0}^{\infty} p_{m,n} x^{2m} y^{2n}, \end{aligned}$$

say. Then we get

$$(16) \quad A(x, y) = \sum_{m, n=0}^{\infty} p_{m, n}(x^{2m}y^{2n} + x^{2m+1}y^{2n} + x^{2m}y^{2n+1} + x^{2m+1}y^{2n+1}).$$

We put

$$d_{m, n} = \begin{cases} p_{m/2, n/2} & (\text{even } m, n) \\ p_{(m-1)/2, n/2} & (\text{odd } m \text{ and even } n) \\ p_{m/2, (n-1)/2} & (\text{even } m \text{ and odd } n) \\ p_{(m-1)/2, (n-1)/2} & (\text{odd } m, n). \end{cases}$$

Now, from (16) and (9), we get easily (15). Thus the Lemma is proved.

Proof of Theorem 3. First, suppose that $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$ is Lebesgue-integrable on Q . Without loss of generality, we may suppose that $\gamma + \varepsilon$ is a noninteger < 1 . Let

$$(17) \quad h(x, y) = (1 + x + y + xy)(1 - x^2 - y^2)^{\gamma + \varepsilon - 1}$$

in Q . Then, by the Lemma ($\mu = \gamma + \varepsilon$), we have

$$(18) \quad h(x, y) = \sum_{m, n=0}^{\infty} k_{m, n} x^m y^n, \quad C_1 \theta_{m, n} \leq k_{m, n} \leq C_2 \theta_{m, n}$$

in Q , where $k_{m, n}$ and $\theta_{m, n}$ are defined respectively like $d_{m, n}$ and $\delta_{m, n}$ in the Lemma with $\mu = \gamma + \varepsilon$. Clearly, the function

$$\begin{aligned} & \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} h(x, y) \\ &= (1 + x + y + xy) \{1 + (x^2 + y^2)^{1/2}\}^{\gamma + \varepsilon - 1} \{1 - (x^2 + y^2)^{1/2}\}^{\varepsilon - 1} \end{aligned}$$

is Lebesgue-integrable on Q . Hence, by assumption, the function

$$\begin{aligned} & \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \{f(x, y) + C_3 h(x, y)\} \\ &= \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \sum_{m, n=0}^{\infty} (a_{m, n} + C_3 k_{m, n}) x^m y^n \end{aligned}$$

is Lebesgue-integrable on Q , where $C_3 \geq C/C_1$. Further, by (6) and (18), we have

$$(19) \quad a_{m, n} + C_3 k_{m, n} \geq a_{m, n} + C \theta_{m, n} \geq 0$$

for all sufficiently large $m + n$. Thus we get

$$(20) \quad \begin{aligned} & \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} \left\{ \sum_{m, n=0}^{\infty} (a_{m, n} + C_3 k_{m, n}) x^m y^n \right\} dx dy \\ &= \sum_{m, n=0}^{\infty} (a_{m, n} + C_3 k_{m, n}) \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dx dy, \end{aligned}$$

where the right-side series converges absolutely. By the change of variables

$$x = r \cos v, \quad y = r \sin v \quad (0 \leq r < 1, 0 \leq v \leq \pi/2),$$

we get

$$\begin{aligned} & \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dx dy \\ &= \int_0^1 (1 - r)^{-\gamma} r^{m+n+1} dr \int_0^{\pi/2} \sin^m v \cos^n v dv \\ &= \frac{\Gamma(m + n + 2)\Gamma(1 - \gamma)}{\Gamma(m + n + 3 - \gamma)} \cdot \frac{1}{2} \cdot \frac{\Gamma((m + 1)/2)\Gamma((n + 1)/2)}{\Gamma((m + n)/2 + 1)}. \end{aligned}$$

Thus, from (9), we get

$$\begin{aligned} (21) \quad & C_4(m + n + 1)^{-(m+n+3)/2+\gamma} (m + 1)^{m/2} (n + 1)^{n/2} \\ & \leq \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} x^m y^n dx dy \\ & \leq C_5(m + n + 1)^{-(m+n+3)/2+\gamma} (m + 1)^{m/2} (n + 1)^{n/2} \end{aligned}$$

for all $m, n \geq 0$. Hence, by (20),

$$(22) \quad \sum_{m, n=0}^{\infty} (m + n + 1)^{-(m+n+3)/2+\gamma} (m + 1)^{m/2} (n + 1)^{n/2} (a_{m, n} + C_3 k_{m, n})$$

converges absolutely. Further, by (18), we have

$$\begin{aligned} (23) \quad & \sum_{m, n=0}^{\infty} (m + n + 1)^{-(m+n+3)/2+\gamma} (m + 1)^{m/2} (n + 1)^{n/2} k_{m, n} \\ & \leq C_2 \sum_{m, n=0}^{\infty} \{ (m + n + 1)^{-i-\epsilon} (m + 1)^{-1/2} (n + 1)^{-1/2} \\ & \quad + (m + n + 1)^{-3/2-\epsilon} (n + 1)^{-1/2} + (m + n + 1)^{-3/2-\epsilon} (m + 1)^{-1/2} \\ & \quad + (m + n + 1)^{-2-\epsilon} \} < \infty. \end{aligned}$$

By (6) and (18), we get

$$|a_{m, n}| \leq a_{m, n} + 2C\theta_{m, n} \leq a_{m, n} + 2C_3 k_{m, n} \quad (C_3 \geq C/C_1)$$

for all sufficiently large $m + n$. Hence, from (22) and (23), the series (7) converges absolutely.

Conversely we suppose that series (7) converges absolutely, and will deduce that $\{1 - (x^2 + y^2)^{1/2}\}^{-\gamma} f(x, y)$ is Lebesgue-integrable on Q . For this part of the argument we do not assume (6). We have in fact

$$\begin{aligned}
 & \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\tau} |f(x, y)| \, dx dy \\
 & \leq \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\tau} \left\{ \sum_{m,n=0}^{\infty} |a_{m,n}| x^m y^n \right\} \, dx dy \\
 & = \sum_{m,n=0}^{\infty} |a_{m,n}| \iint_Q \{1 - (x^2 + y^2)^{1/2}\}^{-\tau} x^m y^n \, dx dy \\
 & \leq C_5 \sum_{m,n=0}^{\infty} (m + n + 1)^{-(m+n+3)/2+\tau} (m + 1)^{m/2} (n + 1)^{n/2} |a_{m,n}| < \infty
 \end{aligned}$$

by (21). Thus Theorem 3 is proved.

REMARK 2. From (17), it is easily seen that

$$C_1 h(x, y) \leq \{1 - (x^2 + y^2)^{1/2}\}^{\tau+\varepsilon-1} \leq C_2 h(x, y)$$

in Q .

Proof of Theorem 4. By Theorem B ($r_0 = 1$), we get

$$f(x, y) = \sum_{N=0}^{\infty} \sum_{m+n=N} a_{m,n} x^m y^n$$

in Q . We define $h(x, y)$ by (17). Then it is sufficient for us to notice that

$$\begin{aligned}
 f(x, y) + C_3 h(x, y) &= \sum_{N=0}^{\infty} \sum_{m+n=N} a_{m,n} x^m y^n + C_3 \sum_{m,n=0}^{\infty} k_{m,n} x^m y^n \\
 &= \sum_{N=0}^{\infty} \sum_{m+n=N} (a_{m,n} + C_3 k_{m,n}) x^m y^n \\
 &= \sum_{m,n=0}^{\infty} (a_{m,n} + C_3 k_{m,n}) x^m y^n
 \end{aligned}$$

in Q , in view of (18) and (19), where the last right-side series converges absolutely. Thus Theorem 4 is a consequence of Theorem 3.

The author wishes to thank the referee for several helpful suggestions.

REFERENCES

1. E. Artin, *The Gamma Function*, Holt, Rinehart and Winston, New York (1964).
2. M. Brelot, *Éléments de la théorie classique du potentiel*, Les cours de Sorbonne, 3rd edition, Paris (1965).
3. P. Heywood, *Integrability theorems for power series and Laplace transforms (II)*, J. London Math. Soc., **32** (1957), 22-27.
4. C. O. Kiselman, *Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants*, Bull. Soc. Math. France, **97** (1969), 329-356.

Received January 12, 1972 and in revised form July 24, 1972.

HIROSAKI UNIVERSITY, JAPAN

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Christopher Allday, <i>Rational Whitehead products and a spectral sequence of Quillen</i>	313
James Edward Arnold, Jr., <i>Attaching Hurewicz fibrations with fiber preserving maps</i>	325
Catherine Bandle and Moshe Marcus, <i>Radial averaging transformations with various metrics</i>	337
David Wilmot Barnette, <i>A proof of the lower bound conjecture for convex polytopes</i>	349
Louis Harvey Blake, <i>Simple extensions of measures and the preservation of regularity of conditional probabilities</i>	355
James W. Cannon, <i>New proofs of Bing's approximation theorems for surfaces</i>	361
C. D. Feustel and Robert John Gregorac, <i>On realizing HNN groups in 3-manifolds</i>	381
Theodore William Gamelin, <i>Iversen's theorem and fiber algebras</i>	389
Daniel H. Gottlieb, <i>The total space of universal fibrations</i>	415
Yoshimitsu Hasegawa, <i>Integrability theorems for power series expansions of two variables</i>	419
Dean Robert Hickerson, <i>Length of period simple continued fraction expansion of \sqrt{d}</i>	429
Herbert Meyer Kamowitz, <i>The spectra of endomorphisms of the disc algebra</i>	433
Dong S. Kim, <i>Boundedly holomorphic convex domains</i>	441
Daniel Ralph Lewis, <i>Integral operators on \mathcal{L}_p-spaces</i>	451
John Eldon Mack, <i>Fields of topological spaces</i>	457
V. B. Moscatelli, <i>On a problem of completion in bornology</i>	467
Ellen Elizabeth Reed, <i>Proximity convergence structures</i>	471
Ronald C. Rosier, <i>Dual spaces of certain vector sequence spaces</i>	487
Robert A. Rubin, <i>Absolutely torsion-free rings</i>	503
Leo Sario and Cecilia Wang, <i>Radial quasiharmonic functions</i>	515
James Henry Schmerl, <i>Peano models with many generic classes</i>	523
H. J. Schmidt, <i>The \mathcal{F}-depth of an \mathcal{F}-projector</i>	537
Edward Silverman, <i>Strong quasi-convexity</i>	549
Barry Simon, <i>Uniform crossnorms</i>	555
Surjeet Singh, <i>(KE)-domains</i>	561
Ted Joe Suffridge, <i>Starlike and convex maps in Banach spaces</i>	575
Milton Don Ulmer, <i>C-embedded Σ-spaces</i>	591
Wolmer Vasconcelos, <i>Conductor, projectivity and injectivity</i>	603
Hidenobu Yoshida, <i>On some generalizations of Meier's theorems</i>	609