LENGTH OF PERIOD SIMPLE CONTINUED FRACTION EXPANSION OF $\sqrt{d}$

DEAN ROBERT HICKERSON
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In this article, the length, $p(d)$, of the period of the simple continued fraction (s.c.f.) for $\sqrt{d}$ is discussed, where $d$ is a positive integer, not a perfect square. In particular, it is shown that

$$p(d) < d^{1/2 + \log 2/\log \log d + 0 \log \log \log d / (\log \log d)^2}.$$  

In addition, some properties of the complete quotients of the s.c.f. expansion of $\sqrt{d}$ are developed.

It is well known that the s.c.f. expansion for $\sqrt{d}$ is periodic if $d$ is a positive integer, not a perfect square. Throughout this paper, $p(d)$ will denote the length of this period. It is shown in [2] (page 294), that $p(d) < 2d$. Computer calculation of $p(d)$ originally suggested that $p(d) \leq 2[\sqrt{d}]$. This was shown to be false for $d = 1726$, for which $p(d) = 88$ and $2[\sqrt{d}] = 82$. Further calculation revealed 3 more counterexamples for $d \leq 3000$. They were $p(2011) = 94$ while $2[\sqrt{2011}] = 88$, $p(2566) = 102$ while $2[\sqrt{2566}] = 100$, and $p(2671) = 104$ while $2[\sqrt{2671}] = 102$.

This suggests as a conjecture that

$$p(d) = O(d^{1/2}) \quad \text{and} \quad p(d) \neq o(d^{1/2}).$$

It follows from the corollary to Theorem 2 that

$$p(d) = O(d^{1/2 + \epsilon})$$

or more precisely, that

$$p(d) < d^{1/2 + \log 2/\log \log d + 0 \log \log \log d / (\log \log d)^2}.$$  

We will need the following results which are given in or follow from §§ 7.1-7.4 and 7.7 of [1].

1. Any periodic s.c.f. is a quadratic irrational number, and conversely.

2. The s.c.f. expansion of the real quadratic irrational number $(a + \sqrt{b})/c$ is purely periodic if and only if $(a + \sqrt{b})/c > 1$ and $-1 < (a - \sqrt{b})/c < 0$.

3. Any quadratic irrational number $\xi_0$ may be put in the form $\xi_0 = (m_0 + \sqrt{d})/q_0$, where $d$, $m_0$, and $q_0$ are integers, $q_0 \neq 0$, $d \geq 1$, $d$ is not a perfect square, and $q_0 \mid (d - m_0^2)$. We may then define infinite
sequences \( m_i, q_i, a_i, \) and \( \xi_i \) by the equations \( \xi_i = (m_i + \sqrt{d})/q_i, a_i = [\xi_i], m_{i+1} = a_i q_i - m_i, \) and \( q_{i+1} = (d - m_{i+1})q_i. \) Then, for \( i \geq 0, m_i, q_i, \) and \( a_i \) are integers, \( q_i \neq 0, \) and \( q_i \mid (d - m_i). \) Also, for \( i \geq 1, a_i \) and \( \xi_i \) are positive.

(4) In the notation of (3) above, we have for \( i \geq 0, \) \( \xi_i = \langle a_i, a_{i+1}, a_{i+2}, \cdots \rangle. \) In particular, \( \xi_0 = \langle a_0, a_1, a_2, \cdots \rangle. \)

(5) There is a positive integer \( N \) such that, if \( i > N, \) then \( q_i > 0. \)

(6) There exist nonnegative integers \( j \) and \( k \) such that \( j < k, m_j = m_k, \) and \( q_j = q_k. \) We may choose \( j \) to be the smallest integer such that for some \( k > j, m_j = m_k \) and \( q_j = q_k. \) We may then choose \( k \) to be the smallest integer such that \( j < k, m_j = m_k, \) and \( q_j = q_k. \) Then, if \( t \) is a nonnegative integer, then \( m_{j+t} = m_{k+t}, q_{j+t} = q_{k+t}, a_{j+t} = a_{k+t}, \) and \( \xi_{j+t} = \xi_{k+t}. \) Therefore, if \( i < j, \) then

\[
\xi_i = \langle a_i, a_{i+1}, \cdots, a_{j-1}, a_j, \cdots, a_{k-1}, \rangle,
\]

while if \( i \geq j, \) then \( \xi_i = \langle a_{i'}, a_{i'+1}, \cdots, a_{k-1}, a_k, a_j, a_{j+1}, \cdots, a_{i'-1}, \rangle, \) where \( i' \) is the integer such that \( j \leq i' \leq k - 1 \) and \( i \equiv i' \pmod{(k-j)}. \) In particular, \( \xi_0 = \langle a_0, a, \cdots, a_{j-1}, a_j, \cdots, a_{k-1}, \rangle. \)

(7) If \( \xi_0 = \sqrt{d}, \) then we may take \( m_0 = 0 \) and \( q_0 = 1 \) in (3). In (6), we have \( j = 1 \) and \( k = r + 1 \) for some positive integer \( r. \) Then \( \xi_0 = \langle a_0, a, \cdots, a_r, \rangle \) and, for \( i \geq 1, \) \( \xi_i = \langle a_{i'}, \cdots, a_r, a_i, \cdots, a_{i'-1}, \rangle, \) where \( i' \) is such that \( 1 \leq i' \leq r \) and \( i \equiv i' \pmod{r}. \)

(8) In (7), if \( t \geq 0 \) then \( m_{i+t} = m_{r+i+t}, q_{i+t} = q_{r+i+t}, a_{i+t} = a_{r+i+t}, \) and \( \xi_{i+t} = \xi_{r+i+t}. \) It follows from this that if \( i \geq 1 \) and \( s \geq 0, \) then \( m_{i+s} = m_i, q_{i+s} = q_i, a_{i+s} = a_i, \) and \( \xi_{i+s} = \xi_i. \)

Throughout this paper it will be assumed that \( d \) is a positive integer, not a perfect square. The period \( r \) of the s.c.f. expansion of \( \sqrt{d} \) will be denoted by \( p(d). \)

2. Preliminary results. In this section, \( m_i, q_i, a_i, \) and \( \xi_i \) will refer to the sequences defined in (3)-(8) above, with \( \xi_0 = \sqrt{d}, m_0 = 0, \) and \( q_0 = 1. \)

**Lemma 1.** If \( i \geq 0, \) then \( q_i > 0. \)

**Proof.** From (5), there is an \( N \) such that, if \( i > N, \) then \( q_i > 0. \) Suppose \( i \geq 1. \) Then there is an integer \( s \) such that \( i + rs > N. \) By (8), \( q_i = q_{i+rs}. \) But since \( i + rs > N, q_{i+rs} > 0. \) Therefore, \( q_i > 0. \) That is, if \( i \geq 1, \) we are done. Since \( q_0 = 1, \) this result holds for \( i = 0 \) also, so the proof is complete.

**Theorem 1.** If \( i \geq 1, \) then \( 0 < m_i < \sqrt{d} \) and \( \sqrt{d} - m_i < q_i < \sqrt{d} + m_i. \)
Proof. From (7), if $i \geq 1$, then $\xi_i = \langle a_{i'}, \cdots, a_r, a_i, \cdots, a_{i'-1} \rangle$ so the s.c.f. for $\xi_i$ is purely periodic. But $\xi_i = (m_i + \sqrt{d})/q_i$, so from (2), $(m_i + \sqrt{d})/q_i > 1$ and $-1 < (m_i - \sqrt{d})/q_i < 0$. Since, from Lemma 1, $q_i > 0$, we obtain $m_i + \sqrt{d} > q_i$ and $-q_i < m_i - \sqrt{d} < 0$. This yields $m_i < \sqrt{d}$ and $\sqrt{d} - m_i < q_i < \sqrt{d} + m_i$.

Thus $-m_i < m_i$ and $m_i > 0$, so the proof is complete.

For given $d$, let $T = T(d)$ be the set of ordered pairs $(m, q)$ which satisfy $m < \sqrt{d}$, $\sqrt{d} - m < q < \sqrt{d} + m$, and $q | (d - m^2)$. That is, $T = \{(m, q) | m < \sqrt{d}, \sqrt{d} - m < q < \sqrt{d} + m, q | (d - m^2)\}$. Let $g(d) = c(T)$, the cardinality of $T$.

From (6) and (7) of Section 1, if $1 \leq i < l \leq r$ then $(m_i, q_i) \neq (m_l, q_l)$. Therefore, the set $U = \{(m_i, q_i) | 1 \leq i \leq r\}$ has exactly $r$ elements. By Theorem 1, $U \subset T$ so $r = c(U) \leq c(T) = g(d)$. Since $r = p(d)$, we obtain

**Lemma 2.** $p(d) \leq g(d)$.

3. An upper bound on $g(d)$.

**Theorem 2.** $g(d) < d^{1/2+\log 2/\log \log d+O(\log \log \log d/\log \log d)^2}$.

**Proof.**

\[
g(d) = c(T) = c((m, q) | 0 < m < \sqrt{d}, \sqrt{d} - m < q < \sqrt{d} + m, q | (d - m^2))
\]
\[
= \sum_{m=1}^{[\sqrt{d}]} c((q \mid \sqrt{d} - m < q < \sqrt{d} + m, q \mid (d - m^2))) \leq \sum_{m=1}^{[\sqrt{d}]} \tau(d - m^2),
\]

where $\tau(n)$ denotes the number of divisors of $n$.

It is shown in [3] that

\[
\log \tau(N) < \frac{\log 2 \log N}{\log \log N} + O\left(\frac{\log N \log \log \log N}{(\log \log N)^2}\right).
\]

It follows that

\[
\tau(N) < N^{\log 2/\log \log N+O(\log \log \log N/\log \log N)^2}.
\]

Therefore, for $m = 1, 2, \cdots, [\sqrt{d}]$,

\[
\tau(d - m^2) < d^{1/2+\log 2/\log \log d+O(\log \log \log d/\log \log d)^2},
\]

and the theorem follows by summing this expression over the $[\sqrt{d}] < d^{1/2}$ values of $m$.

**Corollary.** $p(d) < d^{1/2+\log 2/\log \log d+O(\log \log \log d/\log \log d)^2}$.

**Proof.** This follows immediately from Lemma 2 and Theorem 2.
4. A lower bound on the order of \( g(d) \). Theorem 2 shows that \( g(d) = O((d^{1/2} + \varepsilon)^2) \) for any \( \varepsilon > 0 \). It will follow from Theorem 3 that \( g(d) \neq o(d^{1/2}) \). Thus, Theorem 2 is almost best possible. This, however, is not necessarily true of its corollary.

**Theorem 3.** There exist infinitely many positive integers \( d \) for which \( g(d) > \sqrt{d} \).

**Proof.** Let \( n \) be an arbitrary positive integer. Let

\[
S = \{(m, q) \mid q - n \leq m, n + 1 - q \leq m, m \leq n\}.
\]

Then, for \( n^2 + 1 \leq d \leq n^2 + 2n \), \( T(d) = \{(m, q) \mid (m, q) \in S \text{ and } d \equiv m^2 \pmod{q}\} \). Given \((m, q) \in S\), let \( f(m, q) \) denote the number of integers \( d \) for which \( n^2 + 1 \leq d \leq n^2 + 2n \) and \( d \equiv m^2 \pmod{q}\). Then \( \sum_{d=n^2+1}^{n^2+2n} g(d) = \sum_{(m, q) \in S} f(m, q) \). However, it is easily seen that if \((m, q) \in S\), then \( f(m, q) \geq \lfloor 2n/q \rfloor \). Also, note that \( S = \{(m, q) \mid 1 \leq q \leq n, n + 1 - q \leq m \leq n\} \cup \{(m, q) \mid n + 1 \leq q \leq 2n, q - n \leq m \leq n\} \). If \( 1 \leq q \leq n \), then \( \lfloor 2n/q \rfloor > 2n/q - 1 \). If \( n + 1 \leq q \leq 2n \), then \( \lfloor 2n/q \rfloor = 1 \). Therefore,

\[
\sum_{d=n^2+1}^{n^2+2n} g(d) = \sum_{(m, q) \in S} f(m, q) \geq \sum_{(m, q) \in S} \left\lfloor \frac{2n}{q} \right\rfloor = \sum_{1 \leq q \leq n} \left\lfloor \frac{2n}{q} \right\rfloor + \sum_{n+1 \leq q \leq 2n} \left\lfloor \frac{2n}{q} \right\rfloor
\]

\[
= \sum_{q=1}^{n} q \left\lfloor \frac{2n}{q} \right\rfloor + \sum_{q=n+1}^{2n} (2n + 1 - q) \left\lfloor \frac{2n}{q} \right\rfloor \geq \sum_{q=1}^{n} q \left( \frac{2n}{q} - 1 \right)
\]

\[
+ \sum_{q=n+1}^{2n} (2n + 1 - q) = 2n^2.
\]

It follows from this inequality that at least one of the \( 2n \) numbers \( g(d) \) with \( n^2 + 1 \leq d \leq n^2 + 2n \) must be greater than \( (2n^2/2n) = n \). Since \( n = \lfloor \sqrt{d} \rfloor \) for any such \( d \), there is a \( d \) such that \( n = \lfloor \sqrt{d} \rfloor \) and \( g(d) > \sqrt{d} \). Since this is true for any positive \( n \), the theorem follows.

**References**


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