

# Pacific Journal of Mathematics

**LENGTH OF PERIOD SIMPLE CONTINUED FRACTION  
EXPANSION OF  $\sqrt{d}$**

DEAN ROBERT HICKERSON

## LENGTH OF PERIOD OF SIMPLE CONTINUED FRACTION EXPANSION OF $\sqrt{d}$

DEAN R. HICKERSON

**In this article, the length,  $p(d)$ , of the period of the simple continued fraction (s.c.f.) for  $\sqrt{d}$  is discussed, where  $d$  is a positive integer, not a perfect square. In particular, it is shown that**

$$p(d) < d^{1/2 + \log 2 / \log \log d + O(\log \log \log d / (\log \log d)^2)}.$$

**In addition, some properties of the complete quotients of the s.c.f. expansion of  $\sqrt{d}$  are developed.**

It is well known that the s.c.f. expansion for  $\sqrt{d}$  is periodic if  $d$  is a positive integer, not a perfect square. Throughout this paper,  $p(d)$  will denote the length of this period. It is shown in [2] (page 294), that  $p(d) < 2d$ . Computer calculation of  $p(d)$  originally suggested that  $p(d) \leq 2[\sqrt{d}]$ . This was shown to be false for  $d = 1726$ , for which  $p(d) = 88$  and  $2[\sqrt{d}] = 82$ . Further calculation revealed 3 more counterexamples for  $d \leq 3000$ . They were  $p(2011) = 94$  while  $2[\sqrt{2011}] = 88$ ,  $p(2566) = 102$  while  $2[\sqrt{2566}] = 100$ , and  $p(2671) = 104$  while  $2[\sqrt{2671}] = 102$ .

This suggests as a conjecture that

$$p(d) = O(d^{1/2}) \quad \text{and} \quad p(d) \neq o(d^{1/2}).$$

It follows from the corollary to Theorem 2 that

$$p(d) = O(d^{1/2+\epsilon})$$

or more precisely, that

$$p(d) < d^{1/2 + \log 2 / \log \log d + O(\log \log \log d / (\log \log d)^2)}.$$

We will need the following results which are given in or follow from §§ 7.1-7.4 and 7.7 of [1].

(1) Any periodic s.c.f. is a quadratic irrational number, and conversely.

(2) The s.c.f. expansion of the real quadratic irrational number  $(a + \sqrt{b})/c$  is purely periodic if and only if  $(a + \sqrt{b})/c > 1$  and  $-1 < (a - \sqrt{b})/c < 0$ .

(3) Any quadratic irrational number  $\xi_0$  may be put in the form  $\xi_0 = (m_0 + \sqrt{d})/q_0$ , where  $d$ ,  $m_0$ , and  $q_0$  are integers,  $q_0 \neq 0$ ,  $d \geq 1$ ,  $d$  is not a perfect square, and  $q_0 \mid (d - m_0^2)$ . We may then define infinite

sequences  $m_i, q_i, a_i$ , and  $\xi_i$  by the equations  $\xi_i = (m_i + \sqrt{d})/q_i, a_i = [\xi_i]$ ,  $m_{i+1} = a_i q_i - m_i$ , and  $q_{i+1} = (d - m_{i+1}^2)/q_i$ . Then, for  $i \geq 0$ ,  $m_i, q_i$ , and  $a_i$  are integers,  $q_i \neq 0$ , and  $q_i \mid (d - m_i^2)$ . Also, for  $i \geq 1$ ,  $a_i$  and  $\xi_i$  are positive.

(4) In the notation of (3) above, we have for  $i \geq 0$ ,  $\xi_i = \langle a_i, a_{i+1}, a_{i+2}, \dots \rangle$ . In particular,  $\xi_0 = \langle a_0, a_1, a_2, \dots \rangle$ .

(5) There is a positive integer  $N$  such that, if  $i > N$ , then  $q_i > 0$ .

(6) There exist nonnegative integers  $j$  and  $k$  such that  $j < k$ ,  $m_j = m_k$ , and  $q_j = q_k$ . We may choose  $j$  to be the smallest integer such that for some  $k > j$ ,  $m_j = m_k$  and  $q_j = q_k$ . We may then choose  $k$  to be the smallest integer such that  $j < k$ ,  $m_j = m_k$ , and  $q_j = q_k$ . Then, if  $t$  is a nonnegative integer, then  $m_{j+t} = m_{k+t}$ ,  $q_{j+t} = q_{k+t}$ ,  $a_{j+t} = a_{k+t}$ , and  $\xi_{j+t} = \xi_{k+t}$ . Therefore, if  $i < j$ , then

$$\xi_i = \langle a_i, a_{i+1}, \dots, a_{j-1}, \overline{a_j, \dots, a_{k-1}} \rangle,$$

while if  $i \geq j$ , then  $\xi_i = \langle \overline{a_{i'}, a_{i'+1}, \dots, a_{k-2}, a_{k-1}, a_j, a_{j+1}, \dots, a_{i'-1}} \rangle$ , where  $i'$  is the integer such that  $j \leq i' \leq k-1$  and  $i \equiv i' \pmod{(k-j)}$ . In particular,  $\xi_0 = \langle a_0, a_1, \dots, a_{j-1}, \overline{a_j, \dots, a_{k-1}} \rangle$ .

(7) If  $\xi_0 = \sqrt{d}$  then we may take  $m_0 = 0$  and  $q_0 = 1$  in (3). In (6), we have  $j = 1$  and  $k = r + 1$  for some positive integer  $r$ . Then  $\xi_0 = \langle a_0, \overline{a_1, \dots, a_r} \rangle$  and, for  $i \geq 1$ ,  $\xi_i = \langle \overline{a_{i'}, \dots, a_r, a_1, \dots, a_{i'-1}} \rangle$ , where  $i'$  is such that  $1 \leq i' \leq r$  and  $i \equiv i' \pmod{r}$ .

(8) In (7), if  $t \geq 0$  then  $m_{1+t} = m_{r+1+t}$ ,  $q_{1+t} = q_{r+1+t}$ ,  $a_{1+t} = a_{r+1+t}$ , and  $\xi_{1+t} = \xi_{r+1+t}$ . It follows from this that if  $i \geq 1$  and  $s \geq 0$ , then  $m_{i+rs} = m_i$ ,  $q_{i+rs} = q_i$ ,  $a_{i+rs} = a_i$ , and  $\xi_{i+rs} = \xi_i$ .

Throughout this paper it will be assumed that  $d$  is a positive integer, not a perfect square. The period  $r$  of the s.c.f. expansion of  $\sqrt{d}$  will be denoted by  $p(d)$ .

2. Preliminary results. In this section,  $m_i, q_i, a_i$ , and  $\xi_i$  will refer to the sequences defined in (3)-(8) above, with  $\xi_0 = \sqrt{d}$ ,  $m_0 = 0$ , and  $q_0 = 1$ .

LEMMA 1. *If  $i \geq 0$ , then  $q_i > 0$ .*

*Proof.* From (5), there is an  $N$  such that, if  $i > N$ , then  $q_i > 0$ . Suppose  $i \geq 1$ . Then there is an integer  $s$  such that  $i + rs > N$ . By (8),  $q_i = q_{i+rs}$ . But since  $i + rs > N$ ,  $q_{i+rs} > 0$ . Therefore,  $q_i > 0$ . That is, if  $i \geq 1$ , we are done. Since  $q_0 = 1$ , this result holds for  $i = 0$  also, so the proof is complete.

THEOREM 1. *If  $i \geq 1$ , then  $0 < m_i < \sqrt{d}$  and  $\sqrt{d} - m_i < q_i < \sqrt{d} + m_i$ .*

*Proof.* From (7), if  $i \geq 1$ , then  $\xi_i = \langle \overline{a_{i'}, \dots, a_r, a_1, \dots, a_{i'-1}} \rangle$  so the s.c.f. for  $\xi_i$  is purely periodic. But  $\xi_i = (m_i + \sqrt{d})/q_i$ , so from (2),  $(m_i + \sqrt{d})/q_i > 1$  and  $-1 < (m_i - \sqrt{d})/q_i < 0$ . Since, from Lemma 1,  $q_i > 0$ , we obtain  $m_i + \sqrt{d} > q_i$  and  $-q_i < m_i - \sqrt{d} < 0$ . This yields  $m_i < \sqrt{d}$  and  $\sqrt{d} - m_i < q_i < \sqrt{d} + m_i$ .

Thus  $-m_i < m_i$  and  $m_i > 0$ , so the proof is complete.

For given  $d$ , let  $T = T(d)$  be the set of ordered pairs  $(m, q)$  which satisfy  $m < \sqrt{d}$ ,  $\sqrt{d} - m < q < \sqrt{d} + m$ , and  $q \mid (d - m^2)$ . That is,  $T = \{(m, q) \mid m < \sqrt{d}, \sqrt{d} - m < q < \sqrt{d} + m, q \mid (d - m^2)\}$ . Let  $g(d) = c(T)$ , the cardinality of  $T$ .

From (6) and (7) of Section 1, if  $1 \leq i < l \leq r$  then  $(m_i, q_i) \neq (m_l, q_l)$ . Therefore, the set  $U = \{(m_i, q_i) \mid 1 \leq i \leq r\}$  has exactly  $r$  elements. By Theorem 1,  $U \subset T$  so  $r = c(U) \leq c(T) = g(d)$ . Since  $r = p(d)$ , we obtain

LEMMA 2.  $p(d) \leq g(d)$ .

### 3. An upper bound on $g(d)$ .

THEOREM 2.  $g(d) < d^{1/2 + \log 2 / \log \log d + O(\log \log \log d / (\log \log d)^2)}$ .

*Proof.*

$$\begin{aligned} g(d) &= c(T) = c(\{(m, q) \mid 0 < m < \sqrt{d}, \sqrt{d} - m < q < \sqrt{d} + m, q \mid (d - m^2)\}) \\ &= \sum_{m=1}^{[\sqrt{d}]} c(\{q \mid \sqrt{d} - m < q < \sqrt{d} + m, q \mid d - m^2\}) \leq \sum_{m=1}^{[\sqrt{d}]} \tau(d - m^2), \end{aligned}$$

where  $\tau(n)$  denotes the number of divisors of  $n$ .

It is shown in [3] that

$$\log \tau(N) < \frac{\log 2 \log N}{\log \log N} + O\left(\frac{\log N \log \log \log N}{(\log \log N)^2}\right).$$

It follows that

$$\tau(N) < N^{\log 2 / \log \log N + O(\log \log \log N / (\log \log N)^2)}.$$

Therefore, for  $m = 1, 2, \dots, [\sqrt{d}]$ ,

$$\tau(d - m^2) < d^{\log 2 / \log \log d + O(\log \log \log d / (\log \log d)^2)},$$

and the theorem follows by summing this expression over the  $[\sqrt{d}] < d^{1/2}$  values of  $m$ .

COROLLARY.  $p(d) < d^{1/2 + \log 2 / \log \log d + O(\log \log \log d / (\log \log d)^2)}$ .

*Proof.* This follows immediately from Lemma 2 and Theorem 2.

4. A lower bound on the order of  $g(d)$ . Theorem 2 shows that  $g(d) = O(d^{1/2+\varepsilon})$  for any  $\varepsilon > 0$ . It will follow from Theorem 3 that  $g(d) \neq o(d^{1/2})$ . Thus, Theorem 2 is almost best possible. This, however, is not necessarily true of its corollary.

**THEOREM 3.** *There exist infinitely many positive integers  $d$  for which  $g(d) > \sqrt{d}$ .*

*Proof.* Let  $n$  be an arbitrary positive integer. Let

$$S = \{(m, q) \mid q - n \leq m, n + 1 - q \leq m, m \leq n\}.$$

Then, for  $n^2 + 1 \leq d \leq n^2 + 2n$ ,  $T(d) = \{(m, q) \mid (m, q) \in S \text{ and } d \equiv m^2 \pmod{q}\}$ . Given  $(m, q) \in S$ , let  $f(m, q)$  denote the number of integers  $d$  for which  $n^2 + 1 \leq d \leq n^2 + 2n$  and  $d \equiv m^2 \pmod{q}$ . Then  $\sum_{d=n^2+1}^{n^2+2n} g(d) = \sum_{(m,q) \in S} f(m, q)$ . However, it is easily seen that if  $(m, q) \in S$ , then  $f(m, q) \geq [2n/q]$ . Also, note that  $S = \{(m, q) \mid 1 \leq q \leq n, n + 1 - q \leq m \leq n\} \cup \{(m, q) \mid n + 1 \leq q \leq 2n, q - n \leq m \leq n\}$ . If  $1 \leq q \leq n$ , then  $[2n/q] > 2n/q - 1$ . If  $n + 1 \leq q \leq 2n$ , then  $[2n/q] = 1$ . Therefore,

$$\begin{aligned} \sum_{d=n^2+1}^{n^2+2n} g(d) &= \sum_{(m,q) \in S} f(m, q) \geq \sum_{(m,q) \in S} \left[ \frac{2n}{q} \right] = \sum_{\substack{1 \leq q \leq n \\ n+1-q \leq m \leq n}} \left[ \frac{2n}{q} \right] + \sum_{\substack{n+1 \leq q \leq 2n \\ q-n \leq m \leq n}} \left[ \frac{2n}{q} \right] \\ &= \sum_{q=1}^n q \left[ \frac{2n}{q} \right] + \sum_{q=n+1}^{2n} (2n + 1 - q) \left[ \frac{2n}{q} \right] > \sum_{q=1}^n q \left( \frac{2n}{q} - 1 \right) \\ &\quad + \sum_{q=n+1}^{2n} (2n + 1 - q) = 2n^2. \end{aligned}$$

It follows from this inequality that at least one of the  $2n$  numbers  $g(d)$  with  $n^2 + 1 \leq d \leq n^2 + 2n$  must be greater than  $(2n^2/2n) = n$ . Since  $n = [\sqrt{d}]$  for any such  $d$ , there is a  $d$  such that  $n = [\sqrt{d}]$  and  $g(d) > \sqrt{d}$ . Since this is true for any positive  $n$ , the theorem follows.

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