Pacific Journal of Mathematics

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Vol. 46, No. 2

December 1973

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In this note the spectra of certain endomorphisms of the disc algebra A are determined. Those endomorphisms T of A given by $Tf = f \circ \varphi$ for some $\varphi \in A$ with φ having a fixed point z_0 in the open unit disc are considered and it is shown that either the spectrum $\sigma(T)$ of T is the closed unit disc, or else $\sigma(T)$ is the closure of $\{(\varphi'(z_0))^n \mid n \text{ is a positive integer}\} \cup \{1\}.$

By an endomorphism of an algebra B we mean a linear map T of B into itself satisfying T(fg) = (Tf)(Tg) for all $f, g \in B$. We denote by $\sigma(T)$ those complex numbers λ for which $(\lambda - T)^{-1}$ does not exist.

Throughout this note A will denote the sup-norm algebra of functions continuous on the closed unit disc and analytic on the open unit disc. If $T(\neq 0)$ is a endomorphism of A, then there is a function φ in the unit ball of A for which $Tf = f \circ \varphi$ for all $f \in A$. Indeed, if z is the identity function in A, then $\varphi = Tz$. We call T the endomorphism of A induced by φ . Clearly $\sigma(T)$ depends on φ .

We remark that it follows from Schwarz's Lemma that if a function $\varphi \in A$, $||\varphi|| \leq 1$ has more than one fixed point in the open unit disc, then $\varphi(z) = z$ for all $z, |z| \leq 1$. It is well-known, however, that such $\varphi \in A$ can have infinitely many points on the unit circle with $\varphi(z) = z$ and yet φ need not be equal to the identity function z.

We begin by showing that if φ has a fixed point z_0 in the open unit disc, it is no restriction to assume that $z_0 = 0$.

LEMMA 1. Let $\varphi \in A$, $||\varphi|| \leq 1$ and T be the endomorphism of A induced by φ . Suppose $|z_0| < 1$ and $\varphi(z_0) = z_0$. Let g be the linear fractional transformation $g(z) = (z_0 - z)/(1 - \overline{z}_0 z)$ and T' the endomorphism of A induced by $\psi = g \circ \varphi \circ g$. Then $\psi(0) = 0$, $\psi'(0) = \varphi'(z_0)$ and $\sigma(T') = \sigma(T)$.

Proof. The map $\mathscr{U}: f(z) \to f((z_0 - z)/(1 - \overline{z}_0 z))$ is an isometry of A onto itself, and so $\sigma(T) = \sigma(\mathscr{U}^{-1}T\mathscr{U}) = \sigma(T')$. It is a routine verification that $\psi(0) = 0$ and $\varphi'(0) = \varphi'(z_0)$.

When T is an automorphism (a 1-1 onto endomorphism) of A, then the spectrum of T is easy to determine. Indeed, if φ is the function which induces T, then φ is a schlicht mapping of the disc onto itself. By Lemma 1 we may assume that $\varphi(0) = 0$. Then φ has the form $\varphi(z) = cz$ for some c, |c| = 1. Now, for each positive integer k, c^k is an eigenvalue of T because $T(z^k) = (\varphi(z))^k = (cz)^k = c^k z^k$, and so either $c^n = 1$ for some positive integer n in which case $T^n = I$ and $\sigma(T) = \{1, c, \dots, c^{n-1}\}$ or else, if c is not a root of unity, then $\sigma(T) = \{\lambda \mid |\lambda| = 1\}$.

DEFINITION. Let $\varphi \in A$ with $||\varphi|| \leq 1$. If k is a nonnegative integer, we denote the k^{th} iterate of φ by φ_k . That is, $\varphi_0(z) = z$ and $\varphi_k(z) = \varphi(\varphi_{k-1}(z)), |z| \leq 1$. Furthermore, we will call \bigcap_k range (φ_k) the fixed set of φ .

Straightforward topological arguments show that the fixed set of φ is a compact, connected subset of the unit disc and that φ maps its fixed set onto itself.

For the endomorphisms we are considering, the spectra will depend on the fixed set of the inducing maps.

LEMMA 2. Let $\varphi \in A$, $||\varphi|| \leq 1$, $\varphi(0) = 0$ and T be the endomorphism of A induced by φ . Then $\sigma(T) \supset \{(\varphi'(0))^n \mid n \text{ is a positive integer}\}.$

Proof. The assertion is clearly true if T is an automorphism or if $\mathcal{P}'(0) = 0$. For the case $0 < |\mathcal{P}'(0)| < 1$, we show that for each positive integer k, $((\mathcal{P}'(0)^k - T)f \neq z^k$ for all $f \in A$.

For, suppose $f \in A$ and $(\mathcal{P}'(0))^k f - f(\mathcal{P}) = z^k$. Then

$$(*) \qquad (\varphi'(0))^k f'(z) - f'(\varphi(z))\varphi'(z) = k z^{k-1}$$

At z = 0, (*) becomes $(\varphi'(0))^k f'(0) - f'(0)\varphi'(0) = 0$, or f'(0) = 0. Further,

$$(**) \quad (arphi'(0))^k f''(z) \, - \, f''(arphi(z))(arphi'(z))^2 \, - \, f'(arphi(z))arphi''(z) \, = \, k(k \, - \, 1) z^{k-2} \; .$$

At z = 0, $(^{**})$ becomes $(\mathcal{P}'(0))^k f''(0) - f''(0)(\mathcal{P}'(0))^2 - f'(0)\mathcal{P}''(0) = 0$, and since we already have f'(0) = 0, we obtain $(\mathcal{P}'(0))^k f''(0) - f''(0)(\mathcal{P}'(0))^2 = 0$, or f''(0) = 0.

Continuing, we obtain for j < k,

$$\begin{aligned} (\mathscr{P}'(0))^k f^{(j)}(z) &- f^{(j)}(z) (\mathscr{P}'(z))^j \\ (***) &- (\text{terms of derivatives of } f \text{ of degree} < j) \\ &= k(k-1) \cdots (k-j+1) z^{k-j} , \end{aligned}$$

so that at z = 0, $(\mathcal{P}'(0))^k f^{(j)}(0) - f^{(j)}(0)(\mathcal{P}'(0))^j = 0$, or $f^{(j)}(0) = 0$. For j = k,

$$(****) \quad rac{(arphi'(0))^k f^{(k)}(0) - f^{(k)}(z)(arphi'(z))^k}{- (ext{terms of derivatives of } f ext{ of degree} < k) = k!} \;.$$

The left side of (****) equals 0 at z = 0, while the right side equals k!. Thus $z^k \notin \text{range}((\mathcal{P}'(0))^k - T)$ so that $(\mathcal{P}'(0))^k \in \sigma(T)$ for all positive integers k.

LEMMA 3. Let $\varphi \in A$, $||\varphi|| \leq 1$, $\varphi(0) = 0$ and T be the endomorphism of A induced by φ . Assume $\lambda \neq (\varphi'(0))^n$ for all positive integers n, and $\lambda \neq 0, 1$. If ν is a positive integer, f, $g \in A$ with $(\lambda - T)f = g$ and $g(0) = g'(0) = \cdots = g^{(\nu)}(0) = 0$, then $f(0) = f'(0) = \cdots = f^{(\nu)}(0) = 0$.

Proof. Assume $\lambda f - f(\varphi) = g$ and $g(0) = g'(0) = \cdots = g^{(\nu)}(0) = 0$. Evaluating at z = 0 gives $\lambda f(0) - f(0) = g(0) = 0$, so that f(0) = 0 since $\lambda \neq 1$.

Further, $\lambda f' - f'(\varphi)\varphi' = g'$. At z = 0, this becomes $\lambda f'(0) - f'(0)\varphi'(0) = g'(0) = 0$, so that f'(0) = 0 since $\lambda \neq \varphi'(0)$.

In general, for $k \leq \nu$,

$$\lambda f^{(k)} - f^{(k)}(arphi)(arphi')^k = g^{(k)} + (ext{terms in } f^{(j)}(arphi), j < k) \; .$$

Again evaluating at z = 0 gives $\lambda f^{(k)}(0) - f^{(k)}(0)(\mathcal{P}'(0))^k = 0$, so that $f^{(k)}(0) = 0$ for $k = 0, 1, \dots, \nu$, since $\lambda \neq (\mathcal{P}'(0))^k$, for all positive integers k.

When g = 0, Lemma 3 can be restated as

COROLLARY 4. If $\lambda \neq (\mathcal{P}'(0))^n$ for all positive integers n and $\lambda \neq 0, 1$, then λ is not an eigenvalue.

LEMMA 5. Let $\varphi \in A$, $||\varphi|| \leq 1$ and T be the endomorphism of A induced by φ . If $g \in A$ and $(\lambda - T)f = g$, then

(')
$$\lambda^n f = f(\varphi_n) + \lambda^{n-1}g + \lambda^{n-2}g(\varphi) + \cdots + \lambda g(\varphi_{n-2}) + g(\varphi_{n-1})$$
.

Proof. By induction on n. (') is true for n = 1. Assume (') is true for n. Then

$$\lambda^n f = f(\varphi_n) + \lambda^{n-1}g + \lambda^{n-2}g(\varphi) + \cdots + \lambda g(\varphi_{n-2}) + g(\varphi_{n-1})$$
.

Also $\lambda f(\varphi_n) = f(\varphi_{n+1}) + g(\varphi_n)$ by hypothesis. Hence

$$\lambda^{n+1}f = f(arphi_{n+1}) + \lambda^n g + \lambda^{n-1}g(arphi) + \cdots + \lambda g(arphi_{n-1}) + g(arphi_n)$$

as needed.

LEMMA 6. Let $\varphi \in A$, $\varphi(0) = 0$ and $||\varphi|| \leq 1$. If |z| < 1 (or, in fact, if $|\varphi_j(z)| < 1$ for some positive integer j), then $\overline{\lim}_k |\varphi_k(z)|^{1/k} \leq |\varphi'(0)|$. Furthermore, (1) if $\varphi'(0) = 0$, then given $\varepsilon > 0$, and $r \in [0, 1)$, there exists B > 0 so that for each positive integer m, $|\varphi_m(z)| \leq B\varepsilon^m$ for all $z, |z| \leq r$; (2) if $0 < |\varphi'(0)| < 1$, then given $\varepsilon > 0$ and $r \in [0, 1)$,

there exists B > 0 so that for each positive integer $m, |\varphi_m(z)| \leq B((1 + \varepsilon) |\varphi'(0)|)^m$ for all $z, |z| \leq r$.

Proof.

(i) $\varphi'(0) = 0$.

By the definition of derivative, given $\varepsilon > 0$, there exists $\delta > 0$ so that $|\varphi(w)| \leq \varepsilon |w|$ for $|w| < \delta$. Using the fact that Schwarz's lemma implies $|\varphi(w)| < \delta$ when $|w| < \delta$, an induction argument shows that $|\varphi_n(w)| \leq \varepsilon^n |w|$ for all positive integers *n* and all *w*, $|w| < \delta$.

Now if $r \in [0, 1)$, there is a positive integer N with $|\varphi_N(z)| < \delta$ for all $z, |z| \leq r$. Thus $|\varphi_{n+N}(z)| = |\varphi_n(\varphi_N(z))| \leq \varepsilon^n |\varphi_N(z)|$ for $|z| \leq r$, and so $|\varphi_m(z)| \leq \varepsilon^m |\varphi_N(z)\varepsilon^{-N}|, m \geq N$ when $|z| \leq r$. Letting $B = (r/\varepsilon)^N$ proves (1).

Furthermore, since $|\varphi_m(z)|^{1/m} \leq \varepsilon B^{1/m}$, we find $\overline{\lim}_m |\varphi_m(z)|^{1/m} \leq \varepsilon$. Since ε is an arbitrary positive number, we conclude that $\lim_m |\varphi_m(z)|^{1/m} = 0$.

(ii) $0 < |\varphi'(0)| < 1.$

Given $\varepsilon > 0$, there exists $\delta > 0$ so that $|\varphi(w)| \leq (1 + \varepsilon) |\varphi'(0)| |w|$ for $|w| < \delta$. Again using Schwarz's lemma to show that $|\varphi(w)| < \delta$ if $|w| < \delta$, we can show by induction that $|\varphi_n(w)| \leq ((1 + \varepsilon) |\varphi'(0)|)^n |w|$ for all positive integers n and all w, $|w| < \delta$.

As before, if $r \in [0, 1)$, there exists a positive integer N for which $|\varphi_N(z)| < \delta$ for all $z, |z| \leq r$. Thus $|\varphi_{n+N}(z)| \leq ((1 + \varepsilon) |\varphi'(0)|)^n |\varphi_N(z)|$ for $|z| \leq r$, so that $|\varphi_m(z)| \leq ((1 + \varepsilon) |\varphi'(0)|)^m |\varphi_N(z)| ((1 + \varepsilon) |\varphi'(0)|)^{-N}$ for $m \geq N, |z| \leq r$. Letting $B = ((1 + \varepsilon) |\varphi'(0)|)^{-N}$ proves (2).

Also, since $|\mathscr{P}_m(z)|^{1/m} \leq B^{1/m}(1+\varepsilon) |\mathscr{P}'(0)|$, we find that $\overline{\lim_m} |\mathscr{P}_m(z)|^{1/m} \leq (1+\varepsilon) |\mathscr{P}'(0)|$. Since $\varepsilon > 0$ is arbitrary, we have $\overline{\lim_m} |\mathscr{P}_m(z)|^{1/m} \leq |\mathscr{P}'(0)|$.

(iii) If $|\varphi'(0)| = 1$, then $\varphi(z) = cz$ for some c, |c| = 1, and for $z \neq 0$, $|\varphi_k(z)| = |z| \neq 0$, for all positive integers k. Clearly, $\lim_k |\varphi_k(z)|^{1/k} = 1 = |\varphi'(0)|, |z| \neq 0$.

THEOREM 7. Let $\varphi \in A$, $||\varphi|| \leq 1$ and T be the endomorphism of A induced by φ . Suppose φ has a fixed point in the open unit disc and that the fixed set of φ is infinite. If T is not an automorphism, then $\sigma(T) = \{\lambda \mid |\lambda| \leq 1\}$.

Proof. We may assume that 0 is the fixed point of φ and since T is not an automorphism we have $|\varphi'(0)| < 1$.

Now fix a positive integer ν . We show that $\sigma(T) \supset \{\lambda | \varphi'(0) | \nu < |\lambda| < 1\}$.

Assume there exists $\lambda_0 \notin \sigma(T)$ with $|\varphi'(0)|^{\flat} < |\lambda_0| < 1$. Then $(\lambda - T)^{-1}$ exists for λ in a neighborhood of λ_0 which we assume small enough so that each λ in this neighborhood satisfies $|\varphi'(0)|^{\flat} < |\lambda| < 1$.

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Let $g(z) = z^{\nu}$ and let $f = (\lambda - T)^{-1}g$. By Lemma 5, for each positive integer n, we have

$$(*) \qquad f(z) = f(\varphi_n(z))\lambda^{-n} + \lambda^{-1}\sum_{k=0}^{n-1} g(\varphi_k(z))\lambda^{-k}$$
.

Since $g(0) = g'(0) = \cdots = g^{(\nu-1)}(0) = 0$, Lemma 3 implies that $f(0) = f'(0) = \cdots = f^{(\nu-1)}(0) = 0$, and so $|f(\mathcal{P}_n(z))| \leq ||f|| |\mathcal{P}_n(z)|^{\nu}$ for all positive integers *n*. Of course, $|g(\mathcal{P}_k(z))| = |\mathcal{P}_k(z)|^{\nu}$ for all positive integers *k*.

Lemma 6 asserts that $\overline{\lim}_n |\varphi_n(z)|^{1/n} \leq |\varphi'(0)|$ for all z, |z| < 1, so that for such z,

$$\overline{\lim}_{n} |f(\varphi_{n}(z))\lambda^{-}|^{1/n} \leq \overline{\lim}_{n} (||f|| |\varphi_{n}(z)^{\nu}\lambda^{-n}|)^{1/n} |(\varphi'(0))^{\nu}\lambda^{-1}| < 1.$$

Hence the first term of the right hand side of (*) approaches 0 as $n \rightarrow \infty$.

Furthermore, $\overline{\lim}_{k} |g(\varphi_{k}(z))\lambda^{-k}|^{1/k} = \overline{\lim}_{k} |\varphi_{k}(z)^{\nu}\lambda^{-k}|^{1/k} \leq |\varphi'(0)|^{\nu} |\lambda|^{-1} < 1$ so that $\sum_{k=0}^{\infty} g(\varphi_{k}(z))\lambda^{-k}$ converges for all z, |z| < 1. Thus for λ in some neighborhood of λ_{0} with $|\varphi'(0)|^{\nu} < |\lambda| < 1$,

$$f(z) = (\lambda - T)^{-1}g(z) = \lambda^{-1}\sum_{k=0}^{\infty} g(\varphi_k(z))\lambda^{-k}$$
 for all $z, |z| < 1$.

Now let S be the fixed set of φ . Since φ is analytic on the open unit disc, $|\varphi'(0)| < 1$ and φ maps S onto itself, we can construct a sequence $\{x_n\}_{-\infty}^{\infty}$ in S satisfying

(i) $0 < |x_0| < 1$, (ii) $\varphi(x_n) = x_{n+1}$, and (iii) the x_n 's are distinct. If x_0 is fixed, then $x_n = \varphi_n(x_0)$ are uniquely determined for n > 0, but unless φ is 1 - 1 on S, there may be many choices for x_{-1}, x_{-2}, \cdots .

Let B be the Banach algebra of bounded functions on $\{x_n\}$ with component-wise addition and multiplication and sup-norm. The map φ induces an isometric automorphism \tilde{T} , say, on B, by $\tilde{T}h(x_n) = h(\varphi(x_n)) =$ $h(x_{n+1})$, for $h \in B$. For convenience, define φ_{-k} on $\{x_n\}$ by $\varphi_{-k}(x_n) =$ x_{n-k} .

Now, $\sigma(\widetilde{T}) = \{\lambda | |\lambda| = 1\}$ so that if $|\lambda| < 1$, then $(\lambda - \widetilde{T})^{-1}$ exists on *B* and $F = (\lambda - \widetilde{T})^{-1}g$ (on $\{x_n\}$) satisfies

$$egin{aligned} F(x_{0}) &= - ~\widetilde{T}^{-1}[(I-\lambda\widetilde{T}^{-1})^{-1}g](x_{0}) = - ~\widetilde{T}^{-1}\sum\limits_{k=0}^{\infty}\lambda^{k}\widetilde{T}^{-k}g(x_{0}) = - \sum\limits_{k=0}^{\infty}\lambda^{k}\widetilde{T}^{-(k+1)}g(x_{0}) \ &= -\sum\limits_{k=0}^{\infty}\lambda^{k}g(arphi_{-(k+1)}(x_{0}) = - \lambda^{-1}\sum\limits_{k=1}^{\infty}\lambda^{k}g(arphi_{-k}(x_{0})) \ &= -\lambda^{-1}\sum\limits_{k=-\infty}^{-1}g(arphi_{k}(x))\lambda^{-k} ~. \end{aligned}$$

Therefore, for each λ in some ball about λ_0 , with $|\varphi'(0)|^{\nu} < |\lambda| < 1$, we have

$$(**) \qquad \qquad \lambda^{-1}\sum_{k=0}^{\infty}g(arphi_k(x_0))\lambda^{-k} = - \ \lambda^{-1}\sum_{k=-\infty}^{-1}g(arphi_k(x_0))\lambda^{-k} \;,$$

since both expressions represent $((\lambda - \widetilde{T})^{-1}g)(x_0)$.

On the other hand, $\sum_{k=-\infty}^{\infty} g(\varphi_k(x_0)) w^{-k}$ is the Laurent expansion of a function analytic in the annulus $\{w \mid |\varphi'(0)|^{\nu} < |w| < 1\}$ since $\overline{\lim}_k |g(\varphi_k(x_0))|^{1/k} = \overline{\lim}_k |(\varphi_k(x_0))^{\nu}|^{1/k} \leq |\varphi'(0)|^{\nu}$ and $\overline{\lim}_k |g(\varphi_{-k}(x_0))|^{1/k} \leq 1$. But (**) implies that $\sum_{k=-\infty}^{\infty} g(\varphi_k(x_0))\lambda^{-k} = 0$ in a ball about λ_0 and so the analytic function $\sum_{k=-\infty}^{\infty} g(\varphi_k(x_0))w^{-k}$ is identically zero in the entire annulus $\{w \mid |\varphi'(0)|^{\nu} < |w| < 1\}$. Thus $g(\varphi_k(x_0)) = 0$ for all integers k. Since $\{\varphi_k(x_0)\}$ is infinite and $\varphi_k(x_0) \to 0$ as $k \to \infty$, the analytic function g vanishes on an infinite set with 0 as a limit point. Hence g = 0. But this contradicts the assumption that $g(z) = z^{\nu}$.

Therefore, the assumption that there exists $\lambda_0 \in \sigma(T)$ with $|\varphi'(0)|^{\nu} < |\lambda_0| < 1$ is false. Hence $\sigma(T) \supset \{\lambda \mid |\varphi'(0)|^{\nu} < |\lambda| < 1\}$. Since ν is arbitrary, $\sigma(T) = \{\lambda \mid |\lambda| \leq 1\}$.

LEMMA 8. Let $\varphi \in A$, $||\varphi|| \leq 1$, $\varphi(0) = 0$ and T be the endomorphism of A induced by φ . Let ν be a positive integer. Suppose every function in A with a zero of order at least $(\nu + 1)$ at 0 is in the range of $(\lambda - T)$, where $\lambda \neq 0, 1, (\varphi'(0))^n$, n a positive integer. Then $1, z, z^2, \dots, z^{\nu}$ are in the range of $\lambda - T$.

Proof. Let g be defined on the unit disc by $g(z) = (\varphi(z))^{\nu} - (\varphi'(0))^{\nu}z^{\nu}$. Then $g \in A$ and has a zero of order at least $(\nu + 1)$ at 0. By hypothesis we can find $h \in A$ with $(\lambda - T)h = g$. Let $f = (\lambda - (\varphi'(0))^{\nu})^{-1}(h + z^{\nu})$. Then

$$\begin{split} (\lambda - T)f &= (\lambda - (\mathscr{P}'(0))^{\nu})^{-1} [(\lambda - T)h + (\lambda - T)z^{\nu}] \\ &= (\lambda - (\mathscr{P}'(0))^{\nu})^{-1} [g + \lambda z^{\nu} - (\mathscr{P}(z))^{\nu}] \\ &= (\lambda - (\mathscr{P}'(0))^{\nu})^{-1} [(\mathscr{P}(z))^{\nu} - (\mathscr{P}'(0))^{\nu} z^{\nu} + \lambda z^{\nu} - (\mathscr{P}(z))^{\nu}] = z^{\nu} \,. \end{split}$$

Thus if range $(\lambda - T)$ contains all functions with a zero of order at least $(\nu + 1)$ at 0, and if $\lambda \neq 0, 1, (\mathcal{P}'(0))^n$ for all positive integers *n*, then $z^{\nu} \in \text{range } (\lambda - T)$.

In the same way we can conclude, successively, that $z^{\nu-1}$, $z^{\nu-2}$, ..., $z \in \text{range } (\lambda - T)$. Also $(\lambda - T)(\lambda - 1)^{-1} = 1$ showing that the constants are in range $(\lambda - T)$.

THEOREM 9. Let $\varphi \in A$, $||\varphi|| \leq 1$ and T be the endomorphism of A induced by φ . Let z_0 be a fixed point of φ in the open unit disc and suppose $\{z_0\}$ is the fixed set of φ . Then $\sigma(T) = \{(\varphi'(z_0))^n \mid n \text{ is a positive integer}\} \cup \{0, 1\}.$

Proof. By Lemma 1 we may assume that $z_0 = 0$ and Lemma 2

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implies that $\sigma(T) \supset \{(\mathcal{P}'(0))^n \mid n \text{ is a positive integer}\}$. Certainly 0 and 1 are in $\sigma(T)$.

We prove that $\sigma(T) = \{(\varphi'(0))^n \mid n \text{ is a positive integer}\} \cup \{0, 1\}$ for the case $0 < |\varphi'(0)| < 1$. The case $\varphi'(0) = 0$ is entirely similar.

Since the fixed set of φ is $\{0\}$, given $r \in (0, 1)$, there exists a positive integer m with $|\varphi_m(z)| < r$ for all $z, |z| \leq 1$. Choose $\varepsilon > 0$ so that $(1 + \varepsilon) |\varphi'(0)| < 1$. Let ν be an arbitrary positive integer and consider λ satisfying $((1 + \varepsilon) |\varphi'(0)|)^{\nu+1} < |\lambda|$.

By Lemma 6, there exists $B_1 > 0$ so that

 $|\varphi_k(\varphi_m(z))| < B_1((1 + \varepsilon) |\varphi'(0)|)^k$ for all $z, |z| \leq 1$, and all positive integers k. Hence

 $|\varphi_k(z)| < B((1+\varepsilon) |\varphi'(0)|)^k$ for all $z, |z| \leq 1$, and all positive integers k, where $B = B_1 ((1+\varepsilon) |\varphi'(0)|)^{-m}$.

Now let $g \in A$ with $g(0) = g'(0) = \cdots = g^{(\nu)}(0) = 0$. We claim that $g \in \text{range } (\lambda - T)$. To see this, we observe first that $\sum_{k=0}^{\infty} g(\varphi_k(z))\lambda^{-k}$ converges uniformly in z. Indeed, $|g(z)| \leq ||g|| |z|^{\nu+1}$ and

Since $((1 + \varepsilon) | \varphi'(0) |)^{\nu+1} < |\lambda|$, the right most term of (*) goes to 0 as $N, M \to \infty$.

Define f on the closed unit disc by $f(z) = \lambda^{-1} \sum_{k=0}^{\infty} g(\varphi_k(z)) \lambda^{-k}$. Then $f \in A$ and $\lambda f(z) - f(\varphi(z)) = \sum_{k=0}^{\infty} g(\varphi_k(z)) \lambda^{-k} - \lambda^{-1} \sum_{k=0}^{\infty} g(z_{k+1}(z)) \lambda^{-k} = g(z)$.

Hence, if $((1 + \varepsilon) | \mathscr{P}'(0) |)^{\nu+1} < |\lambda| < 1$ and g has a zero of order at least $(\nu + 1)$ at 0, then $g \in \text{range} (\lambda - T)$. By the preceding lemma, if λ also is not equal to 0, 1, $(\mathscr{P}'(0))^n$ for positive integers n, then 1, z, \cdots , z^{ν} also belong to range $(\lambda - T)$.

Now, every $h \in A$ may be written as

$$h(z) = \left(h(0) + h'(0)z + \cdots + \frac{h^{(\nu)}(0)}{\nu!}z^{\nu}\right) + g(z)$$

where

$$g(z) = \left(h(z) - h(0) - h'(0)z - \cdots - \frac{h^{(\nu)}(0)}{\nu!}z^{\nu}\right).$$

Clearly, $g(0) = g'(0) = \cdots = g^{(\nu)}(0)$. As we have shown, $g \in \text{range}(\lambda - T)$ when $|\lambda| > ((1 + \varepsilon) | \varphi'(0) |)^{\nu+1}$. Also, if $\lambda \neq 0, 1, (\varphi'(0))^n$, *n* a positive integer, then $1, z, \cdots, z^{\nu} \in \text{range}(\lambda - T)$. Thus, for these λ , every *h* in *A* is in the range of $(\lambda - T)$, so $(\lambda - T)$ is onto. Also, by Corollary 4, $(\lambda - T)$ is 1 - 1 if $\lambda \neq 0, 1, (\varphi'(0))^n$, *n* a positive integer.

Hence, $(\lambda - T)^{-1}$ exists for all λ , $|\lambda| > ((1 + \varepsilon) | \mathscr{P}'(0) |)^{\nu+1}$ and $\lambda \neq 0, 1$, $(\mathscr{P}'(0))^n$, *n* a positive integer. Since ν is arbitrary, we conclude that if $\lambda \neq 0, 1$, $(\mathscr{P}'(0))^n$, *n* a positive integer, then $\lambda \notin \sigma(T)$.

As we noted, Lemma 2 shows that for each positive integer n, $(\mathcal{P}'(0))^n$ is in $\sigma(T)$. Since 0 and 1 are in $\sigma(T)$, $\sigma(T) = \{\mathcal{P}'(0)\}^n \mid n$ is a positive integer $\} \cup \{0, 1\}$.

The case when $\varphi'(0) = 0$ is similar. We just replace $(1 + \varepsilon) |\varphi'(0)|$ by ε .

To summarize, we have shown that if T is the endomorphism of A induced by $\varphi \in A$, $||\varphi|| \leq 1$, and if there is a fixed point z_0 of φ in the open unit disc, then the spectrum of T is determined as follows.

(1) If φ is schlicht and onto, then T is an automorphisms and $\sigma(T)$ is the closure of $\{(\varphi'(z_0))^n \mid n \text{ is a positive integer}\}$. We have seen that $\sigma(T)$ is contained in the unit circle and that $\sigma(T)$ may be finite.

(2) If T is not an automorphism, but the fixed set of φ is infinite, then Theorem 7 shows that $\sigma(T) = \{\lambda \mid |\lambda| \leq 1\}$.

(3) If the fixed set of φ consists of the single point z_0 in the open unit disc, then Theorem 9 shows that $\sigma(\sigma) = \{(\varphi'(z_0))^n \mid n \text{ is a positive integer}\} \cup \{0, 1\}.$

Some simple examples of the various types of endomorphisms we have discussed are (i) T is induced by a linear fractional transformation φ of the unit disc onto itself. Then T is an automorphism and $\sigma(T) = \text{closure } \{(\varphi'(z_0))^n \mid n \text{ is a positive integer}\}$ where $|z_0| < 1$ and $\varphi(z_0) = z_0$. If φ is normalized to have $z_0 = 0$, then φ has the form $\varphi(z) = cz$, |c| = 1. Here, $\sigma(T) = \{\lambda \mid |\lambda| = 1\}$ if c is not a root of unity; otherwise, if $c^n = 1$, $\sigma(T) = \{1, c, \dots c^{n-1}\}$.

(ii) T is induced by $\varphi(z) = (z + z^2)/2$. Here one can show that the fixed set is infinite and hence $\sigma(T) = \{\lambda \mid |\lambda| \leq 1\}$.

(iii) T is induced by $\varphi(z) = (z + z^2)/4$. Here the fixed set of φ is $\{0\}, \varphi'(0) = 1/4$, and so $\sigma(T) = \{4^{-n} \mid n \text{ is a nonnegative integer}\} \cup \{0\}.$

(iv) T is induced by $\varphi(z) = cz^k$, k a positve integer > 1. If |c| = 1, then the fixed set of φ is the entire disc and $\sigma(T) = \{\lambda \mid |\lambda| \leq 1\}$, while if |c| < 1, then the fixed set of φ is $\{0\}$ and $\sigma(T) = \{0, 1\}$.

As a final remark, the question of determining the spectra of endomorphisms induced by $\varphi \in A$ with fixed points only on the unit circle is still open. Again the spectra seem to depend on the fixed set of φ , but only partial results have been obtained.

Received September 15, 1971. Partially supported by NSF grant GP 11281.

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The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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