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ON A PROBLEM OF COMPLETION IN BORNOLOGY

V. B. MOSCATELLI

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In this note an example is given to show that the bornological completion of a polar space need not be polar. Also, a theorem of Grothendieck's type is proved, from which necessary and sufficient conditions for the completion of a polar space to be again polar are derived.

1. Notation and terminology are as in [4]. In particular, b.c.s. means a locally convex, bornological linear space over the scalar field of real or complex numbers.

In [4, 5, p. 160] Hogbe-Nlend lists, among unsolved problems in bornology, the following one, which was first raised by Buchwalter in his thesis [1, Remarque, p. 26]:

Is the bornological completion of a polar b.c.s. again polar?

The purpose of this note is to exhibit an example that answers this question in the negative. We also prove a theorem of Grothendieck's type for regular b.c.s. with weakly concordant norms, which enables us to give necessary and sufficient conditions for the completion of a polar b.c.s. to be polar.

2. For each n let the double sequence $a^n = (a_{ij}^n)$ be defined by $a_{ij}^n = j$ for $i \leq n$ and all j , $a_{ij}^n = 1$ for $i > n$ and all j , and denote by E_n the normed space of scalar-valued double sequences (x_{ij}) with only finitely many nonzero terms, under the norm

$$(1) \quad \|(x_{ij})\|_n = \sup_{i,j} \frac{|x_{ij}|}{a_{ij}^n}.$$

Let E be the bornological inductive limit of the spaces E_n ; thus $E = E_n$ algebraically, and a set $B \subset E$ is bounded for the inductive limit bornology if and only if there exist positive integers n, k such that $\|(x_{ij})\|_n \leq k$ for all $(x_{ij}) \in B$. It is easily seen that E is a polar b.c.s. whose dual E^\times consists of all scalar-valued double sequences (u_{ij}) such that

$$\sum_{i,j=1}^{\infty} a_{ij}^n |u_{ij}| < \infty \quad \text{for all } n.$$

By [1, Théorème (2.8.15)] the completion \hat{E} of E is given by $\hat{E} = \lim_{\rightarrow} \hat{E}_n$ (bornological inductive limit), where \hat{E}_n is the completion of the normed space E_n , i.e., the Banach space of scalar-valued double sequences (x_{ij}) such that $\lim_{i,j \rightarrow \infty} x_{ij}/a_{ij}^n = 0$ under the norm (1). It also

follows from [1, Théorème (2.8.15)] that $\hat{E}^\times = E^\times$. Thus, it remains to show that the b.c.s. \hat{E} is not polar with respect to the duality $\langle \hat{E}, E^\times \rangle$, i.e., that there is a bounded subset B of \hat{E} whose bipolar B^{00} is unbounded. In fact, the set

$$B = \left\{ (x_{ij}) \in \hat{E} : \sup_{i,j} |x_{ij}| \leq 1, \quad \lim_{i,j \rightarrow \infty} x_{ij} = 0 \right\}$$

is bounded in the Banach space \hat{E}_1 and hence bounded in \hat{E} ; however, since

$$B^{00} = \left\{ (x_{ij}) \in \hat{E} : \sup_{i,j} |x_{ij}| \leq 1 \right\},$$

the sequence $\{(x_{ij}^n)\}$ with $x_{ij}^n = 0$ for $i \neq n$ and all j , $x_{ij}^n = 1$ for $i = n$ and all j , is contained in B^{00} and yet is unbounded, for

$$(x_{ij}^n) \in \hat{E}_n \sim \hat{E}_{n-1}.$$

Therefore, B^{00} is unbounded in \hat{E} .

3. Let E be a regular b.c.s. with dual E^\times . For a bounded, absolutely convex set $B \subset E$ we set:

E_B = the normed space spanned by B ,

\hat{B} = the completion of B in the Banach space \hat{E}_B ,

E'_B = the dual of E_B ,

B' = the unit ball of E'_B ,

B^0 = the polar of B in E^\times ,

B^{00} = the bipolar of B in \hat{E} ,

p_B = the gauge of B^0 in E^\times ,

E_B^\times = the normed space $E^\times / p_B^{-1}(0)$.

Moreover, we denote by E^{**} the algebraic dual of E^\times and identify, as usual, E_B^\times with a $\sigma(E'_B, E_B)$ -dense subspace of E'_B .

THEOREM 1. *Let E be a regular b.c.s. with weakly concordant norms. The completion \hat{E} of E consists, up to isomorphism, of all those linear functionals on E^\times whose restrictions to B^0 are bounded and $\sigma(E^\times, E_B)$ -continuous for some bounded, absolutely convex set $B \subset E$. Moreover, for every base \mathcal{B} of the bornology of E , the family $\hat{\mathcal{B}} = \{\hat{B} : B \in \mathcal{B}\}$ is a base of the bornology of \hat{E} and we have*

$$(2) \quad \hat{B} = \{x \in B^{00} : x \text{ is } \sigma(E^\times, E_B)\text{-continuous on } B^0\}$$

for every $\hat{B} \in \hat{\mathcal{B}}$.

Proof. If $x \in \hat{E}$, then by [3, Théorème 2, p. 221] there exists a bounded, absolutely convex subset B of E such that $x \in \hat{E}_B$; hence there is a sequence $\{x_n\} \subset E_B$ which converges to x in the Banach

space \widehat{E}_B . It is easily seen that $\{x_n\}$ converges to an element $y \in E^{\times*}$ for the topology $\sigma(E^{\times*}, E^\times)$ and, therefore, $y = x$. Since $\{x_n\}$ is a bounded sequence in E_B , there is a positive number M such that $|\langle x_n, u \rangle| \leq M$ for all n and all $u \in B^0$. It follows that $|\langle x, u \rangle| \leq M$ for all $u \in B^0$. It remains to show that the restriction of x to B^0 is $\sigma(E^\times, E_B)$ -continuous. By Grothendieck's theorem x is $\sigma(E'_B, E_B)$ -continuous on B' ; hence x determines a unique bounded linear functional z on E_B^\times whose restriction to the unit ball of E_B^\times is $\sigma(E_B^\times, E_B)$ -continuous. Let ϕ be the canonical map $E^\times \rightarrow E_B^\times$. Since $p_B^{-1}(0) = (E_B)^0$, ϕ is continuous from $(E^\times, \sigma(E^\times, E_B))$ to $(E_B^\times, \sigma(E_B^\times, E_B))$ and, therefore, the restriction of $x = z \circ \phi$ to B^0 is $\sigma(E^\times, E_B)$ -continuous.

We have also proved that

$$(3) \quad \widehat{B} \subset \{x \in B^{00} : x \text{ is } \sigma(E^\times, E_B)\text{-continuous on } B^0\}.$$

Conversely, let $x \in E^{\times*}$ and suppose that, for some bounded, absolutely convex subset B of E , the restriction of x to B^0 is $\sigma(E^\times, E_B)$ -continuous and satisfies

$$(4) \quad |\langle x, u \rangle| \leq M \quad \text{for all } u \in B^0,$$

with $M > 0$. By going through the mapping ϕ introduced above we see that x determines a unique bounded linear functional z on E_B^\times ($z \circ \phi = x$) whose restriction to the unit ball $B^0/p_B^{-1}(0)$ of E_B^\times is $\sigma(E_B^\times, E_B)$ -continuous. Now $\sigma(E_B^\times, E_B)$ is the topology induced by $\sigma(E'_B, E_B)$ on E_B^\times , $B^0/p_B^{-1}(0)$ is a $\sigma(E'_B, E_B)$ -dense subset of B' and B' is a complete uniform space for the uniformity induced by that of $(E'_B, \sigma(E'_B, E_B))$. It follows that z , being uniformly $\sigma(E'_B, E_B)$ -continuous on $B^0/p_B^{-1}(0)$, has a unique extension $y \in (E'_B)^*$ which is uniformly $\sigma(E'_B, E_B)$ -continuous on B' . By Grothendieck's theorem $y \in \widehat{E}_B$ and, by (4),

$$|\langle y, u \rangle| \leq M \quad \text{for all } u \in B'.$$

This essentially proves the converse implication of (3). Thus (2) holds and the proof is complete, in virtue of the fact that if \mathcal{B} is a base of the bornology of E , then $\widehat{\mathcal{B}} = \{\widehat{B} : B \in \mathcal{B}\}$ is a base of the bornology of \widehat{E} by [3, Théorème 2, p. 221].

COROLLARY. *Let E be a regular b.c.s. with weakly concordant norms. Then E is complete if and only if every linear functional on E^\times which is bounded and $\sigma(E^\times, E_B)$ -continuous on B^0 for some bounded, absolutely convex subset B of E , is $\sigma(E^\times, E)$ -continuous on E^\times .*

The referee has informed us of a Note [2] where Theorem 1 and

its Corollary for polar b.c.s. are arrived at independently, and where counter examples to the same effect as that given in Section 2 are to be found. As every polar b.c.s. has weakly concordant norms (the converse being clearly false), the results in [2] are a particular case of the ones given here.

An immediate consequence of Theorem 1 is the following criterion for the completion of a polar b.c.s. to be again polar.

THEOREM 2. *Let E be a polar b.c.s. The completion \hat{E} of E is polar if and only if every bounded subset B of E is contained in a bounded, absolutely convex set $C \subset E$ such that the restriction of every $x \in B^{00}$ to C^0 is $\sigma(E^\times, E_C)$ -continuous.*

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