ON A PROBLEM OF COMPLETION IN BORNOLOGY

V. B. MOSCATELLI
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In this note an example is given to show that the bornological completion of a polar space need not be polar. Also, a theorem of Grothendieck's type is proved, from which necessary and sufficient conditions for the completion of a polar space to be again polar are derived.

1. Notation and terminology are as in [4]. In particular, b.c.s. means a locally convex, bornological linear space over the scalar field of real or complex numbers.

In [4, 5. p. 160] Hogbe-Nlend lists, among unsolved problems in bornology, the following one, which was first raised by Buchwalter in his thesis [1, Remarque, p. 26]:

Is the bornological completion of a polar b.c.s. again polar?

The purpose of this note is to exhibit an example that answers this question in the negative. We also prove a theorem of Grothendieck's type for regular b.c.s. with weakly concordant norms, which enables us to give necessary and sufficient conditions for the completion of a polar b.c.s. to be polar.

2. For each $n$ let the double sequence $\alpha^* = (\alpha^*_{ij})$ be defined by $\alpha^*_{ij} = j$ for $i \leq n$ and all $j$, $\alpha^*_{ij} = 1$ for $i > n$ and all $j$, and denote by $E_n$ the normed space of scalar-valued double sequences $(x_{ij})$ with only finitely many nonzero terms, under the norm

\[ \| (x_{ij}) \|_n = \sup_{i,j} \frac{|x_{ij}|}{\alpha^*_{ij}}. \]

Let $E$ be the bornological inductive limit of the spaces $E_n$; thus $E = E^*_n$ algebraically, and a set $B \subseteq E$ is bounded for the inductive limit bornology if and only if there exist positive integers $n, k$ such that $\| (x_{ij}) \|_n \leq k$ for all $(x_{ij}) \in B$. It is easily seen that $E$ is a polar b.c.s. whose dual $E^*$ consists of all scalar-valued double sequences $(u_{ij})$ such that

\[ \sum_{i,j=1}^{\infty} \alpha^*_{ij} |u_{ij}| < \infty \]

for all $n$.

By [1, Théorème (2.8.15)] the completion $\hat{E}$ of $E$ is given by $\hat{E} = \lim_{\rightarrow} E_n$ (bornological inductive limit), where $E_n$ is the completion of the normed space $E^*_n$, i.e., the Banach space of scalar-valued double sequences $(x_{ij})$ such that $\lim_{i,j \to \infty} x_{ij}/\alpha^*_{ij} = 0$ under the norm (1). It also
follows from [1, Théorème (2.8.15)] that \( \hat{E}^\times = E^\times \). Thus, it remains to show that the b.c.s. \( \hat{E} \) is not polar with respect to the duality \( \langle \hat{E}, E^\times \rangle \), i.e., that there is a bounded subset \( B \) of \( \hat{E} \) whose bipolar \( B^{oo} \) is unbounded. In fact, the set

\[
B = \left\{ (x_{ij}) \in \hat{E}: \sup_{i,j} |x_{ij}| \leq 1, \lim_{i,j \to \infty} x_{ij} = 0 \right\}
\]

is bounded in the Banach space \( \hat{E} \), and hence bounded in \( \hat{E} \); however, since

\[
B^{oo} = \left\{ (x_{ij}) \in \hat{E}: \sup_{i,j} |x_{ij}| \leq 1 \right\},
\]

the sequence \( \{(x^n_{ij})\} \) with \( x^n_{ij} = 0 \) for \( i \neq n \) and all \( j \), \( x^n_{ij} = 1 \) for \( i = n \) and all \( j \), is contained in \( B^{oo} \) and yet is unbounded, for

\[
(x^n_{ij}) \in \hat{E}_n \sim \hat{E}_{n-1}.
\]

Therefore, \( B^{oo} \) is unbounded in \( \hat{E} \).

3. Let \( E \) be a regular b.c.s. with dual \( E^\times \). For a bounded, absolutely convex set \( B \subset E \) we set:

\[
E_B = \text{the normed space spanned by } B,
\]

\[
\hat{B} = \text{the completion of } B \text{ in the Banach space } \hat{E}_B,
\]

\[
E_B' = \text{the dual of } E_B,
\]

\[
B' = \text{the unit ball of } E_B',
\]

\[
B^0 = \text{the polar of } B \text{ in } E^\times,
\]

\[
B^{oo} = \text{the bipolar of } B \text{ in } \hat{E},
\]

\[
p_B = \text{the gauge of } B^{oo} \text{ in } E^\times,
\]

\[
E^\times_B = \text{the normed space } E^\times/p_B(0).
\]

Moreover, we denote by \( E_B^* \) the algebraic dual of \( E^\times \) and identify, as usual, \( E_B^\sim \) with a \( \sigma(E_B^*, E_B) \)-dense subspace of \( E_B' \).

**Theorem 1.** Let \( E \) be a regular b.c.s. with weakly concordant norms. The completion \( \hat{E} \) of \( E \) consists, up to isomorphism, of all those linear functionals on \( E^\times \) whose restrictions to \( B^0 \) are bounded and \( \sigma(E^\times, E_B) \)-continuous for some bounded, absolutely convex set \( B \subset E \). Moreover, for every base \( \mathcal{B} \) of the bornology of \( E \), the family

\[
\hat{\mathcal{B}} = \{ \hat{B}: B \in \mathcal{B} \}
\]

is a base of the bornology of \( \hat{E} \) and we have

\[
(2) \quad \hat{B} = \{ x \in B^{oo}: x \text{ is } \sigma(E^\times, E_B) \text{-continuous on } B^0 \}
\]

for every \( \hat{B} \in \hat{\mathcal{B}} \).

**Proof.** If \( x \in \hat{E} \), then by [3, Théorème 2, p. 221] there exists a bounded, absolutely convex subset \( B \) of \( E \) such that \( x \in \hat{E}_B \); hence there is a sequence \( \{x_n\} \subset E_B \) which converges to \( x \) in the Banach
space $\hat{E}_B$. It is easily seen that $\{x_n\}$ converges to an element $y \in E^{x*}$ for the topology $\sigma(E^{x*}, E^{*})$ and, therefore, $y = x$. Since $\{x_n\}$ is a bounded sequence in $E_B$, there is a positive number $M$ such that $|\langle x_n, u \rangle| \leq M$ for all $n$ and all $u \in B^0$. It follows that $|\langle x, u \rangle| \leq M$ for all $u \in B^0$. It remains to show that the restriction of $x$ to $B^0$ is $\sigma(E^{x}, E_B)$-continuous. By Grothendieck's theorem $x$ is $\sigma(E'_b, E_B)$-continuous on $B'$; hence $x$ determines a unique bounded linear functional $z$ on $E''_b$ whose restriction to the unit ball of $E''_b$ is $\sigma(E''_b, E_b)$-continuous. Let $\phi$ be the canonical map $E^{*} \to E''_b$. Since $p''_b(0) = (E''_b)^0$, $\phi$ is continuous from $(E^{*}, \sigma(E^{*}, E'_b))$ to $(E''_b, \sigma(E''_b, E_b))$ and, therefore, the restriction of $x = z \circ \phi$ to $B^0$ is $\sigma(E^{*}, E_b)$-continuous.

We have also proved that

$$(3) \quad \hat{B} \subset \{x \in B^{\infty} : x \text{ is } \sigma(E^{*}, E_B)\text{-continuous on } B'\}.
$$

Conversely, let $x \in E^{x*}$ and suppose that, for some bounded, absolutely convex subset $B$ of $E$, the restriction of $x$ to $B^0$ is $\sigma(E^{x}, E_B)$-continuous and satisfies

$$(4) \quad |\langle x, u \rangle| \leq M \quad \text{for all } u \in B^0,
$$

with $M > 0$. By going through the mapping $\phi$ introduced above we see that $x$ determines a unique bounded linear functional $z$ on $E''_b$ ($x \circ \phi = x$) whose restriction to the unit ball $B''/p''_b(0)$ of $E''_b$ is $\sigma(E''_b, E_b)$-continuous. Now $\sigma(E''_b, E_b)$ is the topology induced by $\sigma(E''_b, E_b)$ on $E''_b$, $B''/p''_b(0)$ is a $\sigma(E''_b, E_b)$-dense subset of $B''$ and $B''$ is a complete uniform space for the uniformity induced by that of $(E''_b, \sigma(E''_b, E_b))$. It follows that $z$, being uniformly $\sigma(E''_b, E_b)$-continuous on $B''/p''_b(0)$, has a unique extension $y \in (E''_b)^*$ which is uniformly $\sigma(E''_b, E_b)$-continuous on $B'$. By Grothendieck's theorem $y \in \hat{E}_B$ and, by (4),

$$|\langle y, u \rangle| \leq M \quad \text{for all } u \in B'.
$$

This essentially proves the converse implication of (3). Thus (2) holds and the proof is complete, in virtue of the fact that if $\mathfrak{B}$ is a base of the bornology of $E$, then $\hat{\mathfrak{B}} = \{\hat{B} : B \in \mathfrak{B}\}$ is a base of the bornology of $\hat{E}$ by [3, Théorème 2, p. 221].

**Corollary.** Let $E$ be a regular b.c.s. with weakly concordant norms. Then $E$ is complete if and only if every linear functional on $E^*$ which is bounded and $\sigma(E^*, E_b)$-continuous on $B^0$ for some bounded, absolutely convex subset $B$ of $E$, is $\sigma(E^*, E)$-continuous on $E^*$.

The referee has informed us of a Note [2] where Theorem 1 and
its Corollary for polar b.c.s. are arrived at independently, and where counter examples to the same effect as that given in Section 2 are to be found. As every polar b.c.s. has weakly concordant norms (the converse being clearly false), the results in [2] are a particular case of the ones given here.

An immediate consequence of Theorem 1 is the following criterion for the completion of a polar b.c.s. to be again polar.

**Theorem 2.** Let $E$ be a polar b.c.s. The completion $\hat{E}$ of $E$ is polar if and only if every bounded subset $B$ of $E$ is contained in a bounded, absolutely convex set $C \subseteq E$ such that the restriction of every $x \in B^o$ to $C^o$ is $\sigma(E^x, E^-)$-continuous.

**References**


Received April 25, 1972. This work was supported by an Italian National Research Council grant.

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