

# Pacific Journal of Mathematics

**ON A PROBLEM OF COMPLETION IN BORNOLOGY**

V. B. MOSCATELLI

# ON A PROBLEM OF COMPLETION IN BORNOLGY

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**In this note an example is given to show that the bornological completion of a polar space need not be polar. Also, a theorem of Grothendieck's type is proved, from which necessary and sufficient conditions for the completion of a polar space to be again polar are derived.**

1. Notation and terminology are as in [4]. In particular, b.c.s. means a locally convex, bornological linear space over the scalar field of real or complex numbers.

In [4, 5, p. 160] Hogbe-Nlend lists, among unsolved problems in bornology, the following one, which was first raised by Buchwalter in his thesis [1, Remarque, p. 26]:

*Is the bornological completion of a polar b.c.s. again polar?*

The purpose of this note is to exhibit an example that answers this question in the negative. We also prove a theorem of Grothendieck's type for regular b.c.s. with weakly concordant norms, which enables us to give necessary and sufficient conditions for the completion of a polar b.c.s. to be polar.

2. For each  $n$  let the double sequence  $\alpha^n = (\alpha_{ij}^n)$  be defined by  $\alpha_{ij}^n = j$  for  $i \leq n$  and all  $j$ ,  $\alpha_{ij}^n = 1$  for  $i > n$  and all  $j$ , and denote by  $E_n$  the normed space of scalar-valued double sequences  $(x_{ij})$  with only finitely many nonzero terms, under the norm

$$(1) \quad \|(x_{ij})\|_n = \sup_{i,j} \frac{|x_{ij}|}{\alpha_{ij}^n}.$$

Let  $E$  be the bornological inductive limit of the spaces  $E_n$ ; thus  $E = E_n$  algebraically, and a set  $B \subset E$  is bounded for the inductive limit bornology if and only if there exist positive integers  $n, k$  such that  $\|(x_{ij})\|_n \leq k$  for all  $(x_{ij}) \in B$ . It is easily seen that  $E$  is a polar b.c.s. whose dual  $E^\times$  consists of all scalar-valued double sequences  $(u_{ij})$  such that

$$\sum_{i,j=1}^{\infty} \alpha_{ij}^n |u_{ij}| < \infty \quad \text{for all } n.$$

By [1, Théorème (2.8.15)] the completion  $\hat{E}$  of  $E$  is given by  $\hat{E} = \lim_{\rightarrow} \hat{E}_n$  (bornological inductive limit), where  $\hat{E}_n$  is the completion of the normed space  $E_n$ , i.e., the Banach space of scalar-valued double sequences  $(x_{ij})$  such that  $\lim_{i,j \rightarrow \infty} x_{ij} / \alpha_{ij}^n = 0$  under the norm (1). It also

follows from [1, Théorème (2.8.15)] that  $\hat{E}^\times = E^\times$ . Thus, it remains to show that the b.c.s.  $\hat{E}$  is not polar with respect to the duality  $\langle \hat{E}, E^\times \rangle$ , i.e., that there is a bounded subset  $B$  of  $\hat{E}$  whose bipolar  $B^{00}$  is unbounded. In fact, the set

$$B = \left\{ (x_{ij}) \in \hat{E} : \sup_{i,j} |x_{ij}| \leq 1, \quad \lim_{i,j \rightarrow \infty} x_{ij} = 0 \right\}$$

is bounded in the Banach space  $\hat{E}_1$  and hence bounded in  $\hat{E}$ ; however, since

$$B^{00} = \left\{ (x_{ij}) \in \hat{E} : \sup_{i,j} |x_{ij}| \leq 1 \right\},$$

the sequence  $\{(x_{ij}^n)\}$  with  $x_{ij}^n = 0$  for  $i \neq n$  and all  $j$ ,  $x_{ij}^n = 1$  for  $i = n$  and all  $j$ , is contained in  $B^{00}$  and yet is unbounded, for

$$(x_{ij}^n) \in \hat{E}_n \sim \hat{E}_{n-1}.$$

Therefore,  $B^{00}$  is unbounded in  $\hat{E}$ .

3. Let  $E$  be a regular b.c.s. with dual  $E^\times$ . For a bounded, absolutely convex set  $B \subset E$  we set:

- $E_B$  = the normed space spanned by  $B$ ,
- $\hat{B}$  = the completion of  $B$  in the Banach space  $\hat{E}_B$ ,
- $E'_B$  = the dual of  $E_B$ ,
- $B'$  = the unit ball of  $E'_B$ ,
- $B^0$  = the polar of  $B$  in  $E^\times$ ,
- $B^{00}$  = the bipolar of  $B$  in  $\hat{E}$ ,
- $p_B$  = the gauge of  $B^0$  in  $E^\times$ ,
- $E^\times_B$  = the normed space  $E^\times/p_B^{-1}(0)$ .

Moreover, we denote by  $E^{*\times}$  the algebraic dual of  $E^\times$  and identify, as usual,  $E^\times_B$  with a  $\sigma(E'_B, E_B)$ -dense subspace of  $E'_B$ .

**THEOREM 1.** *Let  $E$  be a regular b.c.s. with weakly concordant norms. The completion  $\hat{E}$  of  $E$  consists, up to isomorphism, of all those linear functionals on  $E^\times$  whose restrictions to  $B^0$  are bounded and  $\sigma(E^\times, E_B)$ -continuous for some bounded, absolutely convex set  $B \subset E$ . Moreover, for every base  $\mathcal{B}$  of the bornology of  $E$ , the family  $\hat{\mathcal{B}} = \{\hat{B} : B \in \mathcal{B}\}$  is a base of the bornology of  $\hat{E}$  and we have*

$$(2) \quad \hat{B} = \{x \in B^{00} : x \text{ is } \sigma(E^\times, E_B)\text{-continuous on } B^0\}$$

for every  $\hat{B} \in \hat{\mathcal{B}}$ .

*Proof.* If  $x \in \hat{E}$ , then by [3, Théorème 2, p. 221] there exists a bounded, absolutely convex subset  $B$  of  $E$  such that  $x \in \hat{E}_B$ ; hence there is a sequence  $\{x_n\} \subset E_B$  which converges to  $x$  in the Banach

space  $\hat{E}_B$ . It is easily seen that  $\{x_n\}$  converges to an element  $y \in E^{\times*}$  for the topology  $\sigma(E^{\times*}, E^\times)$  and, therefore,  $y = x$ . Since  $\{x_n\}$  is a bounded sequence in  $E_B$ , there is a positive number  $M$  such that  $|\langle x_n, u \rangle| \leq M$  for all  $n$  and all  $u \in B^0$ . It follows that  $|\langle x, u \rangle| \leq M$  for all  $u \in B^0$ . It remains to show that the restriction of  $x$  to  $B^0$  is  $\sigma(E^\times, E_B)$ -continuous. By Grothendieck's theorem  $x$  is  $\sigma(E'_B, E_B)$ -continuous on  $B'$ ; hence  $x$  determines a unique bounded linear functional  $z$  on  $E_B^\times$  whose restriction to the unit ball of  $E_B^\times$  is  $\sigma(E_B^\times, E_B)$ -continuous. Let  $\phi$  be the canonical map  $E^\times \rightarrow E_B^\times$ . Since  $p_B^{-1}(0) = (E_B)^0$ ,  $\phi$  is continuous from  $(E^\times, \sigma(E^\times, E_B))$  to  $(E_B^\times, \sigma(E_B^\times, E_B))$  and, therefore, the restriction of  $x = z \circ \phi$  to  $B^0$  is  $\sigma(E^\times, E_B)$ -continuous.

We have also proved that

$$(3) \quad \hat{B} \subset \{x \in B^{00} : x \text{ is } \sigma(E^\times, E_B)\text{-continuous on } B^0\}.$$

Conversely, let  $x \in E^{\times*}$  and suppose that, for some bounded, absolutely convex subset  $B$  of  $E$ , the restriction of  $x$  to  $B^0$  is  $\sigma(E^\times, E_B)$ -continuous and satisfies

$$(4) \quad |\langle x, u \rangle| \leq M \quad \text{for all } u \in B^0,$$

with  $M > 0$ . By going through the mapping  $\phi$  introduced above we see that  $x$  determines a unique bounded linear functional  $z$  on  $E_B^\times$  ( $z \circ \phi = x$ ) whose restriction to the unit ball  $B^0/p_B^{-1}(0)$  of  $E_B^\times$  is  $\sigma(E_B^\times, E_B)$ -continuous. Now  $\sigma(E_B^\times, E_B)$  is the topology induced by  $\sigma(E'_B, E_B)$  on  $E_B^\times$ ,  $B^0/p_B^{-1}(0)$  is a  $\sigma(E'_B, E_B)$ -dense subset of  $B'$  and  $B'$  is a complete uniform space for the uniformity induced by that of  $(E'_B, \sigma(E'_B, E_B))$ . It follows that  $z$ , being uniformly  $\sigma(E'_B, E_B)$ -continuous on  $B^0/p_B^{-1}(0)$ , has a unique extension  $y \in (E'_B)^*$  which is uniformly  $\sigma(E'_B, E_B)$ -continuous on  $B'$ . By Grothendieck's theorem  $y \in \hat{E}_B$  and, by (4),

$$|\langle y, u \rangle| \leq M \quad \text{for all } u \in B'.$$

This essentially proves the converse implication of (3). Thus (2) holds and the proof is complete, in virtue of the fact that if  $\mathcal{B}$  is a base of the bornology of  $E$ , then  $\hat{\mathcal{B}} = \{\hat{B} : B \in \mathcal{B}\}$  is a base of the bornology of  $\hat{E}$  by [3, Théorème 2, p. 221].

**COROLLARY.** *Let  $E$  be a regular b.c.s. with weakly concordant norms. Then  $E$  is complete if and only if every linear functional on  $E^\times$  which is bounded and  $\sigma(E^\times, E_B)$ -continuous on  $B^0$  for some bounded, absolutely convex subset  $B$  of  $E$ , is  $\sigma(E^\times, E)$ -continuous on  $E^\times$ .*

The referee has informed us of a Note [2] where Theorem 1 and

its Corollary for polar b.c.s. are arrived at independently, and where counter examples to the same effect as that given in Section 2 are to be found. As every polar b.c.s. has weakly concordant norms (the converse being clearly false), the results in [2] are a particular case of the ones given here.

An immediate consequence of Theorem 1 is the following criterion for the completion of a polar b.c.s. to be again polar.

**THEOREM 2.** *Let  $E$  be a polar b.c.s. The completion  $\hat{E}$  of  $E$  is polar if and only if every bounded subset  $B$  of  $E$  is contained in a bounded, absolutely convex set  $C \subset E$  such that the restriction of every  $x \in B^{00}$  to  $C^0$  is  $\sigma(E^\times, E_C)$ -continuous.*

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