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PROXIMITY CONVERGENCE STRUCTURES

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In this paper the notion of proximity convergence structures is introduced. These constitute a layer between Cauchy structures and uniform convergence structures (in the sense of Cook and Fischer [1]). They are a natural generalization of proximity structures. A study of the relations among these various structures constitutes §§2 and 3. In §4, compact extensions for a special class of proximity convergence spaces are constructed, and a characterization of these is obtained. They satisfy a mapping property with respect to compact T_2 proximity convergence spaces which satisfy a strong regularity condition. One problem left open is the obtaining of a more reasonable definition of regularity for these spaces.

1. Proximity convergence structures. A proximity convergence structure is the natural analogue, in the context of convergence spaces, of a proximity structure. Here convergence space is used in the sense of Fischer [3], and proximity in the sense of Efremovič and Smirnov. A proximity convergence structure is a filter of proximity-like orders on a set X , and satisfies a composition property. If the filter is principal then it corresponds to an ordinary proximity.

The notation used is largely that in Cook and Fischer [1]. By $\mathcal{O}(X)$ is meant the set of all symmetric topogenous orders on X .

So a relation $<$ on the subsets of X is in $\mathcal{O}(X)$ iff it satisfies the following:

- (ST 1) $\phi < A < X$ for $A \subseteq X$;
- (ST 2) $A < B \Rightarrow A \subseteq B$;
- (ST 3) if $A' \subseteq A < B \subseteq B'$ then $A' < B'$;
- (ST 4) if $A < C$ and $B < C$ then $A \cup B < C$; also if $C < A$ and $C < B$ then $C < A \cap B$;
- (ST 5) $A < B$ then $X \setminus B < X \setminus A$.

DEFINITION 1. A proximity convergence structure on a set X is a family $\mathcal{P} \subseteq \mathcal{O}(X)$ satisfying

- (P 1) if $<_1, <_2 \in \mathcal{P}$ then $<_1 \cap <_2 \in \mathcal{P}$;
- (P 2) if $< \in \mathcal{P}$ then $< \circ < \in \mathcal{P}$;
- (P 3) if $< \in \mathcal{P}$ and $< \subseteq <' \in \mathcal{O}(X)$ then $<' \in \mathcal{P}$.

We will call (X, \mathcal{P}) a *proximity convergence space*. Both concepts will be abbreviated by p.c.s.

REMARK AND DEFINITION 2. We say one p.c.s. on X is *less* than

another if it *contains* it. Under this ordering the set of all p.c.s.'s on X is a complete lattice. The largest member is $\{\subseteq\}$ and corresponds to the discrete topology on X . The smallest is $\mathcal{O}(X)$, which yields the indiscrete topology. (See Definition 30.) The intersection of any family of p.c.s.'s on X is also a p.c.s., so that suprema are easily described.

DEFINITION 3. If \mathcal{S} is a nonempty subset of $\mathcal{O}(X)$ then clearly there is a smallest p.c.s. $[\mathcal{S}]$ containing G . We will call \mathcal{S} a *base*, provided $[\mathcal{S}]$ consists of refinements of orders in \mathcal{S} . In case $[\mathcal{S}]$ consists of refinements of finite intersections of orders in \mathcal{S} , we call \mathcal{S} a *subbase* for $[\mathcal{S}]$.

As in the uniform case, ordinary proximity relations on X correspond to "principal" p.c.s.'s; i.e., those which have a single element as base.

THEOREM 4. Let $\ll \in \mathcal{O}(X)$. Then \ll is a proximity on X iff $\{\ll\}$ is a base for a p.c.s. on X .

Proof. Let \mathcal{S} denote the set of refinements (in $\mathcal{O}(X)$) of \ll . If \ll is a proximity relation then $\ll = \ll \circ \ll$ and hence \mathcal{S} satisfies (P 2). The other properties are clearly satisfied. Conversely, if \mathcal{S} is a p.c.s. then $\ll \circ \ll \in \mathcal{S}$ and so \ll is dense. Clearly then \ll is a proximity relation.

DEFINITION 5. If $\subset \subset$ is a proximity on X we will call $[\subset \subset]$ a *proximity structure*.

2. Relation with uniform convergence structure. As with ordinary proximities, each uniform convergence structure (abbreviated u.c.s.) gives rise to a p.c.s. This allows us to divide the uniform convergence structures into proximity classes. Each class contains a smallest member, which is *strongly bounded*. This last is a condition stronger than total boundedness, and more satisfying in that every proximity class contains a *unique* strongly bounded member. (A class can contain more than one totally bounded member.) Moreover if the p.c.s. is a *proximity structure* than the strongly bounded member in its class is a uniform structure; the other totally bounded uniform convergence structures in the class will not be uniform structures.

DEFINITION 6. A *standard filter* on $X \times X$ is a symmetric filter $\Phi \subseteq [A]$, the filter generated by the diagonal on X . For Φ a standard filter we define

$A <_{\phi} B$ iff $H(A) \subseteq B$ for some $H \in \Phi$.

This imitates the usual way a proximity is obtained from a uniformity. Notice that if Φ is standard, $<_{\phi} \in \mathcal{O}(X)$.

If \mathcal{J} is a uniform convergence structure (abbreviated u.c.s.) we define

$$\mathcal{B}_{\mathcal{J}} = \{<_{\phi} : \Phi \text{ is a standard filter in } \mathcal{J}\}.$$

It turns out that $\mathcal{B}_{\mathcal{J}}$ is a base for a p.c.s. $\mathcal{P}_{\mathcal{J}}$ on X .

LEMMA 7. Let Φ and Ψ be standard filters on $X \times X$.

- (i) If $\theta = \Phi \cap \Psi$ then $<_{\theta} = <_{\phi} \cap <_{\psi}$
- (ii) If $\mathcal{L} = \Phi \circ \Phi$ then $<_{\mathcal{L}} = <_{\phi} \circ <_{\phi}$.

Proof. Straightforward.

THEOREM 8. If \mathcal{J} is a u.c.s. on X then $\mathcal{B}_{\mathcal{J}}$ is a base for a p.c.s. $\mathcal{P}_{\mathcal{J}}$ on X . If \mathcal{J} is generated by a uniformity \mathcal{U} then $\mathcal{P}_{\mathcal{J}}$ is a proximity structure generated by $<_{\mathcal{U}}$.

Proof. From the preceding lemma it is clear that $\mathcal{B}_{\mathcal{J}}$ is a base for a p.c.s. on X . Suppose \mathcal{U} is a uniformity which generates \mathcal{J} . Then for $< \in \mathcal{P}_{\mathcal{J}}$ we have $<_{\mathcal{U}} \subseteq <$. Hence $\{<_{\mathcal{U}}\}$ is a base for $\mathcal{P}_{\mathcal{J}}$.

DEFINITION 9. If \mathcal{C} is a cover of X we define $H_{\mathcal{C}} = \bigcup \{C \times C : C \in \mathcal{C}\}$. If \mathcal{C} is finite then any entourage which contains $H_{\mathcal{C}}$ is said to be *strongly bounded*. A filter Φ on $X \times X$ is *strongly bounded* iff it consists of strongly bounded entourages. A u.c.s. is *strongly bounded* iff it has a base of strongly bounded filters.

REMARK 10. Notice that for uniform structures strongly bounded is equivalent to totally bounded. However, in the case of a u.c.s. total-boundedness is a weaker condition.

THEOREM 11. Every strongly bounded u.c.s. is totally bounded.

Proof. Let \mathcal{J} be a strongly bounded u.c.s. on X , and let \mathcal{U} be an ultrafilter on X . Let Φ be any strongly bounded filter in \mathcal{J} . We claim that $\Phi \subseteq \mathcal{U} \times \mathcal{U}$.

Let $H \in \Phi$, and let \mathcal{C} be a finite cover of X such that $H_{\mathcal{C}} \subseteq H$. Since \mathcal{U} is an ultrafilter, $\mathcal{U} \cap \mathcal{C} \neq \emptyset$. But if $C \in \mathcal{U} \cap \mathcal{C}$ then $H \supseteq C \times C \in \mathcal{U} \times \mathcal{U}$.

THEOREM 12. Let \mathcal{J} be a u.c.s. on X . The following conditions

are equivalent:

- (i) \mathcal{J} is strongly bounded;
- (ii) \mathcal{J} has at least one strongly bounded member;
- (iii) The filter $[A]^*$ of all strongly bounded entourages is in \mathcal{J} .

Proof. Since $\mathcal{J} \neq \emptyset$, clearly (i) \Rightarrow (ii). Now suppose (ii) holds. Note $[A]^*$ is a filter. If Φ is any strongly bounded filter in \mathcal{J} then $\Phi \subseteq [A]^*$.

Finally, assume $[A]^* \in \mathcal{J}$. If $\Phi \in \mathcal{J}$ then $\Phi \cap [A]^*$ is a strongly bounded filter in \mathcal{J} , and is contained in Φ .

LEMMA 13. A strongly bounded u.c.s. is the smallest member of its proximity class.

Proof. Let \mathcal{J} and \mathcal{K} be u.c.s.'s on X and suppose \mathcal{J} is strongly bounded, with $\mathcal{P}_{\mathcal{J}} = \mathcal{P}_{\mathcal{K}}$. We wish to show $\mathcal{K} \subseteq \mathcal{J}$. Let $\Phi \in \mathcal{K}$ and let $\Psi = \Phi \cap \Phi^{-1} \cap [A]$. Then $<_{\Psi} \in \mathcal{P}_{\mathcal{K}} = \mathcal{P}_{\mathcal{J}}$, so we can choose $\theta \in \mathcal{J}$ so that $<_{\theta} \subseteq <_{\Psi}$. Let $\theta^* = \theta \cap [A]^*$. We claim that $\theta^* \circ \theta^* \subseteq \Phi$.

Let $H \in \theta^*$, and let \mathcal{C} be a finite cover of X with $H_{\mathcal{C}} \subseteq H$. Then for $C \in \mathcal{C}$ we have $C <_{\theta} H(C)$. Let $K_C \in \Psi$ such that $K_C(C) \subseteq H(C)$, and define K to be the intersection of the K_C 's. Then $K \in \Phi$. We claim $K \subseteq H \circ H$.

Let $(x, y) \in K$. Choose $C \in \mathcal{C}$ so $x \in C$. Then $y \in K_C(C) \subseteq H(C)$. Set $c \in C$ with $(c, y) \in H$. Then $(x, c) \in C \times C \subseteq H$. Hence $(x, y) \in H \circ H$.

THEOREM 14. Let \mathcal{P} be a p.c.s. on X , and define

$$\mathcal{B}_{\mathcal{P}} = \{\Phi: \Phi \text{ is standard and } <_{\Phi} \in \mathcal{P}\}.$$

Then $\mathcal{B}_{\mathcal{P}}$ is a base for a strongly bounded u.c.s. $\mathcal{J}_{\mathcal{P}}$ in the proximity class of \mathcal{P} .

Proof. If $\Phi = [A]$ then $<_{\Phi} = \subseteq$, so $[A] \in \mathcal{B}_{\mathcal{P}}$. From Lemma 7 it is clear that $\mathcal{B}_{\mathcal{P}}$ is a base for a u.c.s. $\mathcal{J}_{\mathcal{P}}$.

- (1) $\mathcal{J}_{\mathcal{P}}$ is strongly bounded.

Let $\theta = [A]^*$. We will show $<_{\theta} = \subseteq$, so that $\theta \in \mathcal{B}_{\mathcal{P}}$. Let $A \subseteq B$, and define $\mathcal{C} = \{B, X \setminus A\}$. Then $H_{\mathcal{C}} \in [A]^*$ and $H_{\mathcal{C}}(A) \subseteq B$. Thus $A <_{\theta} B$.

- (2) $\mathcal{J}_{\mathcal{P}}$ is in the proximity class of \mathcal{P} .

Clearly the p.c.s. determined by $\mathcal{J}_{\mathcal{P}}$ is contained in \mathcal{P} . Now let $< \in \mathcal{P}$. We define

$$\mathcal{A} = \{H \subseteq X \times X: A < H(A) \text{ if } A \subseteq X\}$$

$$\mathcal{B} = \{H_{\mathcal{C}}: \exists A, B \subseteq X \text{ with } A <^2 B \text{ and } \mathcal{C} = \{B, X \setminus A\}\}.$$

Notice $\mathcal{B} \subseteq [A]$, so \mathcal{B} is a subbase for a proper filter Φ on $X \times X$. Since each member of \mathcal{B} is symmetric, clearly Φ is symmetric; hence Φ is standard. We will show $<^2 \subseteq <_\circ \subseteq <$. If this holds, then $\Phi \in \mathcal{J}_\circ$ and hence $<$ is in the p.c.s. induced by \mathcal{J}_\circ .

If $A <^2 B$ we define $\mathcal{C} = \{B, X \setminus A\}$. Then $H_\mathcal{C} \in \Phi$, and $H_\mathcal{C}(A) \subseteq B$. Thus $<^2 \subseteq <_\circ$. To show that $<_\circ \subseteq <$ it is sufficient to establish that $\Phi \subseteq A$.

Let $H_i \in \mathcal{B}$ for $1 \leq i \leq n$ and suppose $\bigcap_i H_i \subseteq H$. For each i , let $A_i <^2 B_i$ such that $H_i = H_{\mathcal{C}_i}$, where $\mathcal{C}_i = \{B_i, X \setminus A_i\}$. Choose D_i so $A_i < D_i < B_i$, and define $\mathcal{D}_i = \{D_i, X \setminus D_i\}$. Set $\mathcal{K} = \prod_i \mathcal{D}_i$, and for $k \in \mathcal{K}$ let $C_k = \bigcap_i k(i)$. Note the C_k 's cover X .

Now let $E \subseteq X$. We must show $E < H(E)$. This holds, provided $E \cap C_k < H(E)$ for $k \in \mathcal{K}$. We will actually show that if $E \cap C_k \neq \emptyset$ then $C_k < H(E)$.

Let $k \in \mathcal{K}$, with $E \cap C_k \neq \emptyset$. Define $h(i)$ to be B_i if $k(i) = D_i$, and $X \setminus A_i$ otherwise. Then $k(i) < h(i)$ for $1 \leq i \leq n$, and so $C_k < \bigcap_i h(i)$. We claim $\bigcap_i h(i) \subseteq H(E)$.

Let $x \in \bigcap_i h(i)$, and pick $x_0 \in E \cap C_k$. We will show $(x_0, x) \in H$. Choose i , and suppose $k(i) = D_i$. Then $h(i) = B_i$, and so $(x_0, x) \in D_i \times B_i \subseteq H_i$. Similarly if $k(i) = X \setminus D_i$ then x_0 and x are both in $X \setminus A_i$, and hence $(x_0, x) \in H$.

THEOREM 15. *If \mathcal{P} is a proximity structure then \mathcal{J}_\circ is a uniform structure.*

Proof. Suppose \ll generates \mathcal{P} . Let $\Phi \in \mathcal{J}_\circ$ so $<_\circ \subseteq \ll$ and Φ is strongly bounded. We claim Φ^2 generates \mathcal{J}_\circ .

Let $\Psi \in \mathcal{J}_\circ$ and assume Ψ is standard. Then $<_\Psi \in \mathcal{P}$, so $<_\circ \subseteq <_\Psi$. Let $H \in \Phi$. Then we can choose \mathcal{C} a finite cover of X such that $H_\mathcal{C} \subseteq H$. For $C \in \mathcal{C}$ we have $C <_\Psi H(C)$. Pick $K \in \Psi$ so $K(C) \subseteq H(C)$ for all C in \mathcal{C} . Then $K \subseteq H^2$, so $H^2 \in \Psi$. This establishes that $\Phi^2 \subseteq \Psi$.

EXAMPLE 16. We conclude this section with an example to show that a totally bounded u.c.s. need not be strongly bounded. Let τ be a compact T_2 convergence structure on a set X , and suppose that every finite intersection of convergent filters has a member with an infinite complement. For example, we would let τ be the usual topology on the closed unit interval. Let \mathcal{J} be the u.c.s. generated by $\{\mathcal{F} \times \mathcal{F} : \mathcal{F} \text{ is convergent}\}$. Clearly \mathcal{J} is totally bounded. We claim it is not strongly bounded.

Let $\Phi \in \mathcal{J}$. We will exhibit a member of Φ which is not strongly bounded. Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be convergent filters with $(\bigcap_i \mathcal{F}_i \times \mathcal{F}_i) \cap [A] \subseteq \Phi$. Pick $F \in \bigcap_i \mathcal{F}_i$ so that $X \setminus F$ is infinite. Define $H = (F \times$

$F) \cup \Delta$. Note $H \in \Phi$.

Now let \mathcal{C} be any cover X with $H_{\mathcal{C}} \subseteq H$. For $x \in X \setminus F$ let $C_x \in \mathcal{C}$ such that $x \in C_x$. Since $C_x \times C_x \subseteq H$, clearly $C_x = \{x\}$ for $x \notin F$. Thus \mathcal{C} is infinite, and H is not strongly bounded.

3. Relation with Cauchy structures. In contrast to the classical case, a totally bounded Cauchy structure \mathcal{C} can be induced by several different p.c.s.'s. However there always exist a smallest and a largest p.c.s. which induce \mathcal{C} . If \mathcal{C} is uniform, the smallest p.c.s. associated with it is a proximity structure, but the largest need not be. We call the smallest p.c.s. yielding \mathcal{C} a *saturated* p.c.s.

DEFINITION 17. A Cauchy structure on X is a family \mathcal{C} of proper filters on X such that

- (C1) if $x \in X$ then $\dot{x} \in \mathcal{C}$;
- (C2) if \mathcal{F} is a proper filter which contains a member of \mathcal{C} then $\mathcal{F} \in \mathcal{C}$;
- (C3) if $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ with $\mathcal{F} \vee \mathcal{G} \neq [\emptyset]$ then $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$.

Keller [4] has shown that \mathcal{C} is a Cauchy structure on X iff it is the set of Cauchy filters for some u.c.s. on X . If \mathcal{C} is induced by a *uniformity* we call \mathcal{C} a *uniform* Cauchy structure. We say \mathcal{C} is totally bounded iff every ultrafilter on X is in \mathcal{C} .

DEFINITION 18. For \mathcal{F} a filter on X we define a relation $<_{\mathcal{F}}$ on X by $A <_{\mathcal{F}} B$ iff $A \subseteq B$ and B or $X \setminus A$ is in \mathcal{F} .

REMARK 19. Notice that $<_{\mathcal{F}}$ is in $\mathcal{O}(X)$. Also if $\Phi = (\mathcal{F} \times \mathcal{F}) \cap [\Delta]$ then $<_{\Phi} = <_{\mathcal{F}}$.

THEOREM 20. Let $\mathcal{C}_{\mathcal{P}} = \{\mathcal{F} : <_{\mathcal{F}} \in \mathcal{P}\}$, where \mathcal{P} is a p.c.s. on X . If \mathcal{J} is any totally bounded u.c.s. in the proximity class of \mathcal{P} then $\mathcal{C}_{\mathcal{P}}$ is the set of \mathcal{J} -Cauchy filters.

Proof. Let \mathcal{F} be a filter on X and define $\Phi = (\mathcal{F} \times \mathcal{F}) \cap [\Delta]$. If \mathcal{F} is \mathcal{J} -Cauchy then $\Phi \in \mathcal{J}$, and so $<_{\mathcal{F}} = <_{\Phi} \in \mathcal{P}$. Hence $\mathcal{F} \in \mathcal{C}_{\mathcal{P}}$.

Conversely, suppose $\mathcal{F} \in \mathcal{C}_{\mathcal{P}}$. Then $<_{\Phi} = <_{\mathcal{F}} \in \mathcal{P} = \mathcal{P}_{\mathcal{J}}$ and so we can choose $\Psi \in \mathcal{J}$ with $<_{\Psi} \subseteq <_{\Phi}$. Let \mathcal{U} be an ultrafilter containing \mathcal{F} . Then \mathcal{U} is \mathcal{J} -Cauchy, and therefore $\Psi(\mathcal{U})$ is also \mathcal{J} -Cauchy. (By $\Psi(\mathcal{U})$ is meant the filter generated by all sets of the form $H(U)$, where $H \in \Psi$ and $U \in \mathcal{U}$. It is easy to check that $[\Psi \cap (\mathcal{U} \times \mathcal{U})]^{\circ} \subseteq \Psi(\mathcal{U}) \times \Psi(\mathcal{U})$.)

We claim that $\Psi(\mathcal{U}) \subseteq \mathcal{F}$. Let $H \in \Psi$ and $U \in \mathcal{U}$. Then $U <_{\Psi} H(U)$,

and since $<_{\mathcal{U}} \subseteq <_{\mathcal{O}}$ we can choose $K \in \mathcal{O}$ with $K(U) \subseteq H(U)$. Now pick $F \in \mathcal{F}$ so $F \times F \subseteq K$. Then since $\mathcal{F} \subseteq \mathcal{U}$ we have $F \cap U \neq \emptyset$, and so $F \subseteq K(U)$. This establishes that $H(U) \in \mathcal{F}$, as desired.

REMARK 21. This theorem tells us that the totally bounded u.c.s.'s in the same proximity class all induce the same Cauchy structure.

DEFINITION 22. A p.c.s. \mathcal{P} is compatible with a totally bounded Cauchy structure \mathcal{C} iff $\mathcal{C} = \mathcal{C}_{\mathcal{P}}$.

NOTATION 23. Let \mathcal{F} be a filter on X and let $< \in \mathcal{O}(X)$. Then

$$r_{<}(\mathcal{F}) = \{A: F < A \text{ for some } F \in \mathcal{F}\}.$$

Notice $r_{<}(\mathcal{F})$ is a filter contained in \mathcal{F} .

DEFINITION 24. Let \mathcal{C} be a totally bounded Cauchy structure on C .

- (1) $\mathcal{P}_L(\mathcal{C}) = [\{<_{\mathcal{F}}: \mathcal{F} \in \mathcal{C}\}]$;
- (2) $\mathcal{P}_S(\mathcal{C}) = \{< \in \mathcal{O}(X): \mathcal{F} \in \mathcal{C} \implies r_{<}(\mathcal{F}) \in \mathcal{C}\}.$

THEOREM 25. If \mathcal{C} is a totally bounded Cauchy structure on X then $\mathcal{P}_L(\mathcal{C})$ is the largest p.c.s on X compatible with \mathcal{C} , and $\mathcal{P}_S(\mathcal{C})$ is the smallest. Moreover, $\mathcal{S} = \{<_{\mathcal{F}}: \mathcal{F} \in \mathcal{C}\}$ is a subbase for $\mathcal{P}_L(\mathcal{C})$.

Proof.

- (1) \mathcal{S} is a subbase for $\mathcal{P}_L(\mathcal{C})$.

Let \mathcal{R} be the set of refinements of finite intersections of orders in \mathcal{S} . We need $\mathcal{R} = \mathcal{P}_L(\mathcal{C})$. It is sufficient to show that \mathcal{R} is a p.c.s. Clearly \mathcal{R} satisfies (P 1) and (P 3).

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathcal{C}$ with $<_i = <_{\mathcal{F}_i}$. Suppose $\bigcap_i <_i \subseteq < \in \mathcal{O}(X)$. We wish to show $< \circ < \in \mathcal{R}$. We may assume the \mathcal{F}_i 's are pairwise disjoint; i.e., $\mathcal{F}_i \vee \mathcal{F}_j = [\emptyset]$ for $i \neq j$. This follows by induction from (C 3), since if $\mathcal{F}_i \vee \mathcal{F}_j \neq [\emptyset]$ we replace $<_i \cap <_j$ by $<_{\mathcal{F}}$, where $\mathcal{F} = \mathcal{F}_i \cap \mathcal{F}_j$. Choose $F_i \in \mathcal{F}_i$ so that the F_i 's are pairwise disjoint.

Suppose now that $A <_i B$ for $1 \leq i \leq n$. We will show $A <^2 B$. For each i , define

$$D_i = \begin{cases} F_i \cap B & \text{if } B \in \mathcal{F}_i \\ F_i \setminus A & \text{if } B \notin \mathcal{F}_i. \end{cases}$$

Note $D_i \in \mathcal{F}_i$ for all i . Let $H = (\bigcup_i D_i \times D_i) \cup \Delta$. We claim $A < H(A) < B$.

Clearly $A \subseteq H(A)$. To see that $H(A) \subseteq B$, let $a \in A$ with $(a, x) \in H$. If $x = a$ then $x \in B$. If $x \neq a$ then for some i , a and x are both in D_i . Since $a \notin \mathcal{F}_i \setminus A$, clearly $D_i = F_i \cap B$, and so $x \in B$.

Now fix i . We wish to show $A <_i H(A) <_i B$. It is sufficient to show that either $H(A)$ or $X \setminus H(A)$ or $B \setminus A$ is in \mathcal{F}_i . If $D_i = F_i \setminus A$ it is not difficult to prove that $D_i \cap H(A) = \emptyset$, so that $X \setminus H(A) \in \mathcal{F}_i$. (Recall the D_j 's are pairwise disjoint.) If $D_i = F_i \cap B$ and $D_i \cap A = \emptyset$ then clearly $B \setminus A \in \mathcal{F}_i$. If $D_i \cap A \neq \emptyset$ then $D_i \subseteq H(A)$ and so $H(A) \in \mathcal{F}_i$.

(2) $\mathcal{P}_s(\mathcal{C})$ is a p.c.s.

If $< = <_1 \cap <_2$ and $\mathcal{F} \in \mathcal{C}$ then $r_<(F) = r_{<_1}(\mathcal{F}) \cap r_{<_2}(\mathcal{F})$. Using this and (C3), we conclude $\mathcal{P}_s(\mathcal{C})$ is closed under finite intersections. Similarly $r_{<^2}(\mathcal{F}) = r_<(r_<(\mathcal{F}))$, so $\mathcal{P}_s(\mathcal{C})$ is closed under "squaring". Since $r_<(\mathcal{F}) \subseteq r_{<'}(\mathcal{F})$ whenever $< \subseteq <'$ clearly (P3) holds.

(3) $\mathcal{P}_L(\mathcal{C}) \subseteq \mathcal{P}_s(\mathcal{C})$.

It is sufficient to show that $<_{\mathcal{F}} \in \mathcal{P}_s(\mathcal{C})$ for $\mathcal{F} \in \mathcal{C}$. Let $< = <_{\mathcal{F}}$ and let $\mathcal{G} \in \mathcal{C}$. If $\mathcal{G} \vee \mathcal{F} = [\emptyset]$ then $r_<(\mathcal{G}) = \mathcal{G}$; and if $\mathcal{G} \vee \mathcal{F} \neq [\emptyset]$ then $r_<(\mathcal{G}) \supseteq \mathcal{F} \cap \mathcal{G}$. Thus in either case $r_<(\mathcal{G}) \in \mathcal{C}$.

(4) $\mathcal{P}_s(\mathcal{C})$ and $\mathcal{P}_L(\mathcal{C})$ are both compatible with \mathcal{C} . Let \mathcal{C}_s denote the Cauchy structure induced by $\mathcal{P}_s(\mathcal{C})$; and similarly for \mathcal{C}_L . Suppose $\mathcal{F} \in \mathcal{C}$. Then by definition of $\mathcal{P}_L(\mathcal{C})$ we have $<_{\mathcal{F}} \in \mathcal{P}_L(\mathcal{C})$ and hence $\mathcal{F} \in \mathcal{C}_L$. Therefore $\mathcal{C} \subseteq \mathcal{C}_L \subseteq \mathcal{C}_s$.

Now suppose $\mathcal{G} \in \mathcal{C}_s$. Then $<_{\mathcal{G}} \in \mathcal{P}_s(\mathcal{C})$. Let $< = <_{\mathcal{G}}$ and let \mathcal{U} be an ultrafilter containing \mathcal{G} . Then $\mathcal{U} \in \mathcal{C}$, and so by definition of $\mathcal{P}_s(\mathcal{C})$ we have $r_<(\mathcal{U}) \in \mathcal{C}$. But $r_<(\mathcal{U}) \subseteq \mathcal{G}$, and so $\mathcal{C}_s \subseteq \mathcal{C}$.

(5) If \mathcal{P} is a p.c.s. compatible with \mathcal{C} then $\mathcal{P}_s(\mathcal{C}) \subseteq \mathcal{P} \subseteq \mathcal{P}_L(\mathcal{C})$.

For $\mathcal{F} \in \mathcal{C} = \mathcal{C}_{\mathcal{P}}$ we have $<_{\mathcal{F}} \in \mathcal{P}$. Thus $\mathcal{P}_L(\mathcal{C}) \subseteq \mathcal{P}$. Now let $< \in \mathcal{P}$ and choose $\mathcal{F} \in \mathcal{C}$. Let $\mathcal{G} = r_<(\mathcal{F})$. We must show $\mathcal{G} \in \mathcal{C}$; i.e., $<_{\mathcal{G}} \in \mathcal{P}$. It is straightforward to check that $(<_{\mathcal{F}} \cap <)^3 \subseteq <_{\mathcal{G}}$.

REMARK 26. This theorem tells us that each totally bounded Cauchy structure has a largest and smallest p.c.s. compatible with it. Since an intersection of proximity convergence structures is also a p.c.s., we see that the set of proximity convergence structures compatible with a given totally bounded Cauchy structure is a complete lattice.

THEOREM 27. If \mathcal{C} is a totally bounded Cauchy structure and \mathcal{P} is a proximity structure compatible with \mathcal{C} then $\mathcal{P} = \mathcal{P}_s(\mathcal{C})$.

Proof. Let \mathcal{P} be a p.c.s. compatible with \mathcal{C} and suppose $\{\ll\}$ is a base for \mathcal{P} . We will show $\mathcal{P}_s(\mathcal{C}) \subseteq \mathcal{P}$.

Let $< \in \mathcal{P}_s(\mathcal{C})$ and suppose $A \not< B$. We wish to show $A < / < B$. For this it is sufficient to produce a filter \mathcal{F} in \mathcal{C} with $A \not<_{\mathcal{F}} B$. (Recall if $\mathcal{F} \in \mathcal{C}$ then $<_{\mathcal{F}} \in \mathcal{P}$ and so $\ll \subseteq <_{\mathcal{F}}$.)

Set $\mathcal{S} = \{D: A < D\} \cup \{X \setminus E: E < B\}$. Then since $A \not< B$, \mathcal{S} has the finite intersection property. Let \mathcal{U} be an ultrafilter containing \mathcal{S} . Then $\mathcal{U} \in \mathcal{C}$. Since $< \varepsilon_{\mathcal{P}_s(\mathcal{C})}$ we have $r_{<}(\mathcal{U}) \in \mathcal{C}$. Clearly neither B nor $X \setminus A$ is in $r_{<}(\mathcal{U})$.

REMARK AND DEFINITION 28. From this theorem it follows easily that if \mathcal{C} is uniform (and totally bounded) then $\mathcal{P}_s(\mathcal{C})$ is the unique proximity structure compatible with \mathcal{C} . We will call $\mathcal{P}_s(\mathcal{C})$ a *saturated* p.c.s. (whether or not \mathcal{C} is uniform). Obviously then every proximity structure is saturated.

EXAMPLE 29. Even if \mathcal{C} is uniform, $\mathcal{P}_L(\mathcal{C})$ need not be a proximity structure. For example let \mathcal{X} be a totally bounded uniformity with Cauchy family \mathcal{C} . Assume that no finite intersection of Cauchy filters equals $\{X\}$. This is the case as long as $\mathcal{X} \neq \{X \times X\}$, but the proof is somewhat involved and will not be given. Certainly it is true for the usual uniformity on the closed unit interval. Assume also that if $A <_{\mathcal{X}} A$ then $A = \emptyset$ or X . This is true if the associated topology is connected, for example.

Suppose $<_{\mathcal{X}} \in \mathcal{P}_L(\mathcal{C})$. By Theorem 25, there are Cauchy filters $\mathcal{F}_1, \dots, \mathcal{F}_n$ such that $\bigcap_i <_{\mathcal{F}_i} \subseteq <_{\mathcal{X}}$. Therefore if $F \in \bigcap_i \mathcal{F}_i$ then $F <_{\mathcal{X}} F$, and so $F = X$. Hence $\bigcap_i \mathcal{F}_i = \{X\}$, which is impossible. Therefore $<_{\mathcal{X}} \notin \mathcal{P}_L(\mathcal{C})$, and so $\mathcal{P}_L(\mathcal{C}) \neq \mathcal{P}_s(\mathcal{C})$. By Theorem 27, $\mathcal{P}_L(\mathcal{C})$ is not a proximity structure.

4. The Σ -compactification. A p.c.s. is compact, provided the associated convergence structure is compact. A *compactification* of p.c.s. is a compact p.c.s. in which the given space can be densely embedded. In general a p.c.s. has many compactifications. We will confine ourselves to one, called the Σ -compactification. This works at least for relatively round spaces, and has a nice characterization. Using it we can obtain a generalization of the classical one-to-one correspondence between proximity structures and T_2 compactifications of a given topological space.

Continuous maps to compact T_2 spaces can be extended to this compactification, provided the range spaces satisfy a strong regularity condition. We leave open the problem of obtaining the “right” definition of regularity for a p.c.s.

DEFINITION 30. Let \mathcal{P} be a p.c.s. on X . For $x \in X$ we define $\tau_{\mathcal{P}}(x)$ to be the intersection ideal generated by the filters of the form $r_{<}(\dot{x})$, where $< \in \mathcal{P}$.

THEOREM 31. If \mathcal{J} is in the proximity class of \mathcal{P} then $\tau_{\mathcal{J}} = \tau_{\mathcal{P}}$.

Proof. Notice that $\{r_{<}(\dot{x}): < \in \mathcal{P}\}$ is a base for $\tau_{\mathcal{P}}(x)$. Thus if $\mathcal{F} \in \tau_{\mathcal{P}}(x)$ then for some $< \in \mathcal{P}$ we have $r_{<}(\dot{x}) \subseteq \mathcal{F}$. Let $\Psi \in \mathcal{J}$ with $<_{\Psi} \subseteq <$. Now, $\Psi \subseteq \dot{x} \times \Psi(\dot{x})$, so $\Psi(\dot{x}) \in \tau_{\mathcal{J}}(x)$. But $\Psi(\dot{x}) \subseteq r_{<}(\dot{x})$, since for $H \in \Psi$ we have $\{x\} <_{\Psi} H(x)$.

Now suppose $\mathcal{F} \in \tau_{\mathcal{J}}(x)$. Let $\mathcal{G} = \mathcal{F} \cap \dot{x}$ and let $\Phi = \mathcal{G} \times \mathcal{G} \cap [A]$. Then $\Phi \in \mathcal{J}$ and so $<_{\Phi} \in \mathcal{P}$. Set $< = <_{\Phi} = <_{\mathcal{F}}$. Then $r_{<}(\dot{x}) \subseteq \mathcal{F}$.

REMARK 32. We can also describe $\tau_{\mathcal{P}}$ as follows: $\mathcal{F} \in \tau_{\mathcal{P}}(x)$ iff for some $\mathcal{G} \in \mathcal{F} \cap \dot{x}$ we have $<_{\mathcal{G}} \in \mathcal{P}$.

Next we will describe the construction of the Σ -extension of a p.c.s.

DEFINITION 33. Let \mathcal{C} be a Cauchy structure on X . Two filters in \mathcal{C} are *equivalent* iff their intersection is in \mathcal{C} . We denote the associated partition by $X^*(\mathcal{C})$, or just X^* . The map which assigns to a point x in X the equivalence class of \dot{x} is denoted by j . If (X, \mathcal{C}) is T_2 then j is an injection of X into X^* .

We define Σ to be the set of all maps σ which assign to each equivalence class p in X^* a filter in p ; we further require for $x \in X$ and $\sigma \in \Sigma$ that $\sigma(j(x)) = \dot{x}$.

For each σ in Σ we obtain a map from $\mathcal{P}(X)$ to $\mathcal{P}(X^*)$; namely,

$$A^{\sigma} = \{p \in X^*: A \in \sigma(p)\}.$$

This allows us to define a map from $\mathcal{O}(X)$ to the set of relations on X^* . For $< \in \mathcal{O}(X)$ we define $A <^{\sigma} B$ iff there are subsets C and D of X with $A \subseteq C^{\sigma}$, $D^{\sigma} \subseteq B$, and $C < D$.

Now suppose \mathcal{C} is totally bounded, and let \mathcal{P} be a compatible p.c.s. We define $\mathcal{P}_x = \{<' \in \mathcal{O}(X^*): \text{for } \sigma \in \Sigma, \exists < \in \mathcal{P} \text{ with } <' \subseteq <^{\sigma}\}$. It is easy to check that \mathcal{P}_x is a p.c.s. on X . We will call $(j, (X^*, \mathcal{P}_x))$ the Σ -extension of (X, \mathcal{P}) . It is closely related to the Kowalsky completion of (X, \mathcal{C}) , described in [5] and in [7].

DEFINITION 34. Let $k: (X, \mathcal{P}) \rightarrow (Y, \mathcal{Q})$. For $< \in \mathcal{O}(X)$ we define $k(<) \in \mathcal{O}(Y)$ by $A k(<) B$ iff $A \subseteq B$ and $k^{-1}(A) < k^{-1}(B)$. We say k is a *dense embedding* of (X, \mathcal{P}) into (Y, \mathcal{Q}) , provided k is one-to-one and for $< \in \mathcal{O}(X)$ we have $< \in \mathcal{P}$ iff $k(<) \in \mathcal{Q}$.

Next we will establish that j is a dense embedding of (X, \mathcal{P}) into (X^*, \mathcal{P}_x) .

LEMMA 35. Let (X, \mathcal{P}) be T_2 and let τ' denote the convergence structure induced by \mathcal{P}_x .

- (i) If $p \in X^*$ and $\mathcal{F} \in p$ then $j(\mathcal{F}) \in \tau'(p)$.
- (ii) If $\mathcal{G} \in \tau'(p)$ and $\sigma \in \Sigma$ then the filter $\mathcal{G}_{\sigma} = \{A: A^{\sigma} \in \mathcal{G}\}$ is in p .

Proof. Suppose $\mathcal{F} \in p$ and define $\mathcal{G} = j(\mathcal{F}) \cap \dot{p}$. To show $j(\mathcal{F}) \rightarrow p$ it is sufficient to establish $<_{\mathcal{G}}$ is in \mathcal{P}_x .

Pick $\sigma \in \Sigma$ and set $\mathcal{H} = \mathcal{F} \cap \sigma(p)$. Now \mathcal{H} is Cauchy, and so $<_{\mathcal{H}} \in \mathcal{P}$. Observe that $\mathcal{H} \subseteq \mathcal{G}_\sigma$, so that $<_{\mathcal{H}} \subseteq <_{\mathcal{G}}$.

Now assume $\mathcal{U} \in \tau'(p)$, and let $\sigma \in \Sigma$. Pick $< \in \mathcal{P}_x$ with $r_{<}(\dot{p}) \subseteq \mathcal{U}$, and choose $<_1 \in \mathcal{P}$ so that $<_1 \subseteq <$. Then $r_{<_1}(\sigma(p)) \subseteq \mathcal{U}_\sigma$. For if $A \in \sigma(p)$ and $A <_1 B$ then $A^\sigma <_1^\sigma B^\sigma$ and hence $A^\sigma < B^\sigma$. Since $p \in A^\sigma$ we have $B^\sigma \in r_{<}(\dot{p}) \subseteq \mathcal{U}$.

Now $\sigma(p) \in p$ and $<_1 \in \mathcal{P}$. Therefore $r_{<_1}(\sigma(p)) \in p$. (Use Theorem 25 and (C 3)).

THEOREM 36. *Let (X, \mathcal{P}) be T_2 . Then (X^*, \mathcal{P}_x) is T_2 and j is a dense embedding of (X, \mathcal{P}) into (X^*, \mathcal{P}_x) .*

Proof. Suppose \mathcal{G} converges to both p and q . Let $\sigma \in \Sigma$. By the preceding lemma $\mathcal{G}_\sigma \in p \cap q$. Thus $p = q$, and \mathcal{P}_x is T_2 .

Notice that for $\sigma \in \Sigma$ and $A \subseteq X$ we have $j^{-1}(A^\sigma) = A$. Here strong use is made of the fact that $\sigma(j(x)) = \dot{x}$ for $x \in X$. From this it is easy to see that for $< \in \mathcal{P}$ and $\sigma \in \Sigma$ we have $<^\sigma \subseteq j(<)$. Thus $j(<) \in \mathcal{P}_x$.

Now suppose $< \in \mathcal{O}(X)$ and $j(<) \in \mathcal{P}_x$. Let $\sigma \in \Sigma$ and choose $<_1 \in \mathcal{P}$ with $<_1^\sigma \subseteq j(<)$. Using the same fact as before, we see that $<_1 \subseteq <$. This establishes that j is an embedding.

It is easy to check that $j(X)$ is dense in X^* , since for $\mathcal{F} \in p$ we have $j(\mathcal{F}) \rightarrow p$. (Lemma 35).

Next we will give conditions under which the Σ -extension is actually a compactification.

DEFINITION 37. Let (X, \mathcal{P}) be a p.c.s. For $\sigma \in \Sigma$ we define

$$<_\sigma = \bigcap \{ <_{\mathcal{F}} : \mathcal{F} = \sigma(p) \text{ for some } p \in X^* \}.$$

Then \mathcal{P} is *relatively round* iff each $<_\sigma$ is in \mathcal{P} .

Notice that every proximity structure is relatively round. In fact if $\subset \subset$ is a proximity on X then $\subset \subset = \bigcap \{ <_{\mathcal{F}} : \mathcal{F} \in \mathcal{C}(\subset \subset) \}$.

THEOREM 38. *If (X, \mathcal{P}) is relatively round and T_2 then $(j, (X^*, \mathcal{P}_x))$ is a compactification of (X, \mathcal{P}) .*

Proof. In view of Theorem 36, we need only establish that \mathcal{P}_x is compact. Let \mathcal{U} be an ultrafilter on X^* .

Notice that for $\sigma \in \Sigma$, if $A <_\sigma B$ then $(X^* \setminus B^\sigma) \subseteq (X \setminus A)^\sigma$; thus either B^σ or $(X \setminus A)^\sigma$ is in \mathcal{U} . This yields $<_\sigma \subseteq <_{\mathcal{U}_\sigma}$. Since \mathcal{P} is relatively round, we conclude \mathcal{U}_σ is Cauchy for $\sigma \in \Sigma$.

Moreover, the \mathcal{U}_σ 's are all in the same equivalence class. To see

this, suppose σ and μ are in Σ and let $\eta(p) = \sigma(p) \cap \mu(p)$ for $p \in X^*$. Then $\eta \in \Sigma$, and also $\mathcal{U}_\eta \subseteq \mathcal{U}_\sigma \cap \mathcal{U}_\mu$. Thus \mathcal{U}_σ and \mathcal{U}_μ are equivalent.

Let q be the equivalence class of the \mathcal{U}_σ 's. We claim $\mathcal{U} \rightarrow q$. Let $\sigma \in \Sigma$ and define $\mathcal{F} = \mathcal{U}_\sigma \cap \sigma(q)$. Then $\mathcal{F} \in q$, so $<_\sigma \in \mathcal{P}$. Let $\mathcal{V} = \mathcal{U} \cap q$. Then it is simple to check that $<_\sigma \subseteq <_\mathcal{V}$.

Next we wish to characterize the Σ -compactification of (X, \mathcal{P}) as its unique relatively round T_2 compactification. This will be done by using the corresponding fact for uniform convergence spaces, established in [7].

DEFINITION 39. Let $f: (X, \mathcal{P}) \rightarrow (Y, \mathcal{Q})$. Then f is *p-continuous* iff $f(<) \in \mathcal{Q}$ whenever $< \in \mathcal{P}$.

LEMMA 40. Let $f: (X, \mathcal{P}) \rightarrow (Y, \mathcal{Q})$

(i) f is *p-continuous* iff it is uniformly continuous with respect to $\mathcal{J}_\mathcal{P}$ and $\mathcal{J}_\mathcal{Q}$.

(ii) f is an embedding of (X, \mathcal{P}) into (Y, \mathcal{Q}) iff it embeds $(X, \mathcal{J}_\mathcal{P})$ in $(Y, \mathcal{J}_\mathcal{Q})$.

Proof. Notice that if Φ is a standard filter on $X \times X$ and $\Psi = (f \times f)(\Phi) \cap [4]$ then $<_\Psi = f(<_\Phi)$. Clearly then (i) holds. Also if $\Psi \in \mathcal{J}_\mathcal{Q}$ and f is a p -embedding then $\Phi \in \mathcal{J}_\mathcal{P}$. Therefore every p -embedding is a uniform embedding.

Now assume f is a uniform embedding. Suppose $< \in \mathcal{O}(X)$ with $f(<) \in \mathcal{Q}$. Pick $\theta \in \mathcal{J}_\mathcal{Q}$ with $<_\theta \subseteq f(<)$. Set $\theta_1 = (f \times f)^{-1}(\theta)$. We claim $\theta_1 \in \mathcal{J}_\mathcal{P}$ and $<_{\theta_1} \subseteq <$.

Since $<_\theta$ is defined, θ is standard; therefore θ_1 is standard, and in particular it is proper. Note $\theta \subseteq (f \times f)(\theta_1)$, so that $\theta_1 \in \mathcal{J}_\mathcal{P}$. Now if $A <_{\theta_1} B$ then $f(A) <_\theta Y \setminus f(X \setminus B)$. Since $<_\theta \subseteq f(<)$ we conclude $A < B$.

DEFINITION 41. Let $f: (X, \mathcal{P}) \rightarrow (Y, \mathcal{Q})$. By $\Sigma(f)$ we mean the set of all maps σ which assign to each point y in Y a filter converging to y . We further require that for $y \in f(X)$ and $\sigma \in \Sigma(f)$ we have $\sigma(y) = \dot{y}$.

We define $(f, (Y, \mathcal{Q}))$ to be *relatively round* provided $<_\sigma \in \mathcal{Q}$ for each σ in $\Sigma(f)$. We say $(f, (Y, \mathcal{J}_\mathcal{Q}))$ is *relatively round* iff for $\sigma \in \Sigma(f)$ the filter $\bigcap \{\sigma(y) \times \sigma(y) : y \in Y\}$ is in $\mathcal{J}_\mathcal{Q}$.

LEMMA 42. If $(k, (Y, \mathcal{Q}))$ is a relatively round compactification of (X, \mathcal{P}) then $(k, (Y, \mathcal{J}_\mathcal{Q}))$ is a relatively round completion of $(X, \mathcal{J}_\mathcal{P})$.

Proof. From the preceding lemma we know that k is an embedding of $(X, \mathcal{J}_\mathcal{P})$ into $(Y, \mathcal{J}_\mathcal{Q})$. Since $\mathcal{J}_\mathcal{Q}$ and \mathcal{Q} induce the same

convergence structure τ' , clearly this embedding is dense. Since τ' is compact, \mathcal{F}_σ is complete.

Now let $\sigma \in \Sigma(f)$. Then $<_\sigma \in \mathcal{Q}$. Set $\theta = \bigcap \{\sigma(y) \times \sigma(y) : y \in Y\}$. We claim $<_\theta = <_\sigma$, so that $\theta \in \mathcal{F}_\sigma$. To see that $<_\sigma \subseteq <_\theta$ notice that if $A <_\sigma B$ then $(B \times B) \cap (X \setminus A \times X \setminus A) \in \theta$.

THEOREM 43. *If (X, \mathcal{P}) is relatively round and T_2 then $(j, (X^*, \mathcal{P}_\Sigma))$ is the unique relatively round T_2 compactification of (X, \mathcal{P}) .*

Proof. In [7], Theorem 19, it was shown that any two relatively round T_2 completions of a u.c.s. are equivalent. From this, and from the two preceding lemmas, it follows that (X, \mathcal{P}) can have at most one relatively round T_2 compactification.

By Theorem 38 we know $(j, (X^*, \mathcal{P}_\Sigma))$ is a compactification of (X, \mathcal{P}) . To see that it is relatively round pick $\sigma \in \Sigma(j)$ and let $\mu \in \Sigma$. Set $\eta(p) = \sigma(p)_\mu$ for $p \in X^*$. By Lemma 35, $\eta(p) \in p$ for $p \in X^*$. It is easy to check that if $p = j(x)$ then $(\dot{p})_\mu = \dot{x}$. Thus $\eta \in \Sigma$, and $<_\eta \in \mathcal{P}$. Notice that $<_\eta^\mu \subseteq <_\sigma$, so that $<_\sigma \in \mathcal{P}_\Sigma$.

THEOREM 44. *If (X, \mathcal{P}) is a relatively round saturated T_2 p.c.s. then $(X^*, \mathcal{P}_\Sigma)$ is saturated.*

Proof. Suppose $<' \in \mathcal{O}(X^*)$, and $r_{<' }(\mathcal{F})$ is Cauchy whenever \mathcal{F} is. Let $\sigma \in \Sigma$ and define

$$A < B \text{ iff } X^* \setminus (X \setminus A)^\sigma <' B^\sigma.$$

Then $< \in \mathcal{O}(X)$ and $<^\sigma \subseteq <'$. We claim $< \in \mathcal{P}$.

Let $\mathcal{F} \in \mathcal{C}_\sigma$, and let p be its equivalence class. Then $j(\mathcal{F}) \rightarrow p$ (Lemma 35). Define $\mu \in \Sigma(j)$ by $q \rightarrow j(\sigma(q)) \cap \dot{q}$. Since \mathcal{P} is relatively round, so is $(j, (X^*, \mathcal{P}_\Sigma))$ (Theorem 43). Thus $<_\mu \in \mathcal{P}_\Sigma$, and $\mathcal{G}_1 = r_{<_\mu}(j(\mathcal{F}))$ converges to p . Let $\mathcal{G} = r'_{<}(\mathcal{G}_1)$. Then $\mathcal{G} \rightarrow p$, and so $\mathcal{G}_\sigma \in p$.

It is not difficult to check that $\mathcal{G}_\sigma \subseteq r_{<}(\mathcal{F})$ so that $r_{<}(\mathcal{F})$ is Cauchy. Since \mathcal{P} is saturated we conclude $< \in \mathcal{P}$, and $<' \in \mathcal{P}_\Sigma$.

REMARK 45. There is a one-to-one correspondence between certain T_2 compactifications of a given T_2 convergence space (X, τ) and certain of its compatible p.c.s.'s. If \mathcal{P} is relatively round then $(j, (X^*, \tau(\mathcal{P}_\Sigma)))$ is a T_2 compactification of (X, τ_σ) . It is also a *relatively round* compactification meaning that if $\mathcal{F} \rightarrow p$ and $\sigma \in \Sigma(j)$ then $r_{<_\sigma}(\mathcal{F}) \rightarrow p$. Thus the map $\mathcal{P} \rightarrow (j, (X^*, \tau(\mathcal{P}_\Sigma)))$ takes relatively round p.c.s.'s on (X, τ) to relatively round T_2 compactifications of (X, τ) .

This map is one-to-one, provided we limit ourselves to *saturated* structures. This follows from the preceding theorem and from the

fact that a homeomorphism is p -continuous with respect to the largest compatible saturated structures.

The above map is also a surjection. Given a relatively round T_2 compactification $(k, (Y, \tau))$ we define \mathcal{P}' to be the (unique) compatible saturated p.c.s. Set $\mathcal{P} = \{< : k(<) \in \mathcal{P}'\}$. Then \mathcal{P} is relatively round, saturated and compatible with τ . Moreover, $(k, (Y, \mathcal{P}'))$ is a compactification of (X, \mathcal{P}) . Using Theorem 43, we can establish that the given compactification is equivalent to $(j, (X^*, \tau(\mathcal{P}_s)))$.

If $\mathcal{P}_1 \geq \mathcal{P}_2$ then $\kappa_1 \geq \kappa_2$. (κ_i is the compactification associated with \mathcal{P}_i .) However it is not clear the converse holds.

In the final part of this section we will show that a certain class of p -continuous functions on (X, \mathcal{P}) extend to its Σ -compactification.

DEFINITION 46. For any convergence space (X, τ) we define an order $<^\circ$ on X by $A <^\circ B$ iff $\bar{A} \subseteq B^i$. A compatible p.c.s. \mathcal{P} is *c-regular* iff $<^\circ \in \mathcal{P}$. A compatible u.c.s. \mathcal{L} is *c-regular* iff it is regular in the sense of Pervin and Biesterfeldt [6]. In their notation, this means if $\Phi \in \mathcal{L}$ then $\Phi^\circ \in \mathcal{L}$.

REMARK 47. Both these definitions of regularity seem too strong. If \mathcal{P} is *c-regular* then $\tau_\mathcal{P}$ is a *regular topological* structure. The same is true if \mathcal{L} is *c-regular* and strongly bounded. Finding a better definition of regularity has proved unexpectedly difficult.

THEOREM 48. Let \mathcal{L} be the strongly bounded u.c.s. in the proximity class of \mathcal{P} . Then \mathcal{P} is *c-regular* iff \mathcal{L} is *c-regular*.

Proof. Let Φ be a standard, strongly bounded member of \mathcal{L} , and set $\Psi = \Phi^\circ \cap (\Phi^\circ)^{-1}$. We will establish that $<^\circ \in \mathcal{P}$ iff $\Psi \in \mathcal{L}$. Since the standard strongly bounded members of \mathcal{L} are a base for \mathcal{L} this is sufficient to establish the desired equivalence.

$$(1) \quad (<^\circ \cap <_\Phi)^2 \subseteq <_\Psi.$$

This is established by the following observations.

(i) If $H \subseteq X \times X$ then $H^\circ(A) \subseteq H(A)^-$ for $A \subseteq X$.

If $(a, x) \in H^\circ$ with $a \in A$ then $x \in H(a)^- \subseteq H(A)^-$.

(ii) If $H = H^{-1}$ then $(H^\circ)^{-1}(A^i) \subseteq H(A)$. Let $a \in A^i$ with $(x, a) \in H^\circ$. Then $a \in H(x)^-$ and so $A \cap H(x) \neq \emptyset$. For $z \in A \cap H(x)$ we have $x \in H(z) \subseteq H(A)$.

$$(2) \quad <_\Psi^2 \subseteq <^\circ.$$

We will show first that if K is strongly bounded then $A^- \subseteq K^\circ(A)$ for $A \subseteq X$. Let \mathcal{C} be a finite cover of X such that $H_\mathcal{C} \subseteq K$. Pick $x \in A^-$. Then there is a set C in \mathcal{C} with $x \in C^-$ and $C \cap A \neq \emptyset$. To see this, let $\mathcal{F} \rightarrow x$ such that $A \in \mathcal{F}$, and let \mathcal{U} be an ultrafilter

containing \mathcal{F} . Then $\mathcal{U} \cap \mathcal{C} \neq \emptyset$, and for C in $\mathcal{U} \cap \mathcal{C}$ the desired conditions hold.

Now pick $u \in C \cap A$. Then $C \subseteq K(u)$ and so $x \in K(u)^-$. This means $(u, x) \in K^\circ$ and thus $x \in K^\circ(A)$.

From this it follows that if $A <_p B$ then $A^- \subseteq B$. Moreover, $A \subseteq B^i$; note $X \setminus B <_p X \setminus A$, so that $(X \setminus B)^- \subseteq X \setminus A$. Therefore if $A <_p^2 B$ then $A^- \subseteq B^i$.

THEOREM 49. *Let (X, \mathcal{P}) be T_2 . Every p -continuous function from (X, \mathcal{P}) to a c -regular compact T_2 p.c.s. has a unique extension to (X^*, \mathcal{P}_s) .*

Proof. Let f be a p -continuous function from (X, \mathcal{P}) to a c -regular compact T_2 space (Y, \mathcal{Q}) . It is easy to check that f is Cauchy-continuous. Since Y is compact and T_2 , the image of a filter in $\mathcal{C}_\mathcal{Q}$ has a unique limit in Y . Moreover, the images of equivalent filters have the same limit. This defines a map $h: X^* \rightarrow Y$; namely, $h(p)$ is the limit of the f -image of any filter in p . Notice $h \circ j = f$. We need to establish that h is p -continuous. This is where c -regularity is used.

Let $< \in \mathcal{P}_s$ and select $\sigma \in \Sigma$. Choose $<_1 \in \mathcal{P}$ so $<_1^\sigma \subseteq <$ and set $<_2 = f(<_1) \cap <^\circ$. We claim $<_2^\sigma \subseteq h(<)$. This is based on the following observations.

- (i) If $A \subseteq B^i$ then $h^{-1}(A) \subseteq f^{-1}(B)^\sigma$.
- (ii) If $C^- \subseteq D$ then $f^{-1}(C)^\sigma \subseteq h^{-1}(D)$.
- (iii) If $B f(<_1) C$ then $f^{-1}(B)^\sigma < f^{-1}(C)^\sigma$.

Note h is unique, since every continuous extension of f must agree with h on the dense subset $j(X)$.

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